Pramāṇa, Vol. 13, No. 2, August 1979, pp. 97–104, © printed in India

On the Maxwell-Boltzmann statistical distribution function of an ideal gaseous assembly in mass motion

T M KARADE and K G PANGARKAR*
Department of Mathematics, Nagpur University, Nagpur 440 010
*Department of Statistics, Institute of Science, Nagpur 440 001

MS received 20 October 1978; revised 30 April 1979

Abstract. The invariance of the Maxwell-Boltzmann statistical distribution function has been established using Hsu's space-light transformations. The transformations of temperature and heat turn out to be the same as given by Ott, contradicting Planck-Einstein's views regarding their transformation. Incidently the invariance of entropy is obtained.

Keywords. Maxwell-Boltzmann distribution function; transformation; gaseous assembly; pressure; entropy.

1. Introduction

Introducing the conservation of the net linear momentum of the assembly as an additional constraint, Pathria (1955) established the invariance of the statistical distribution function of an ideal gaseous assembly in motion via the relativistic Lorentz transformations. Recently, the non-relativistic ideas of Newton have again started gaining interest. In his new scheme based on the non-universality of the speed of light and the absoluteness concept of time, Hsu (1976) proposed the following space-light transformations,

$$\left.\begin{aligned} x' &= (x - vt)\,q, \\ y' &= y,\ z' = z, \\ c't &= \left(ct - \frac{v}{c}x\right) q, \end{aligned}\right\} \tag{1}$$

where $\quad q = (1 - v^2/c^2)^{-1/2}.$

He argues that the principle of relativity nowhere claims the universality of the speed of light, nor does it prevent one from assuming time to be universal and 'absolute' in the sense that its properties are independent of the inertial frames. Therefore sticking to the principle of relativity, he considers the speed of light to be frame-dependent and the time to be frame-independent. One easily finds that the new theory of Hsu (1976) and the special theory of relativity are based on fundamentally different concepts of time and the speed of light. However, Hsu's new theory accounts for the life-time dilatation of unstable particles and is also consistent with the other known experiments such as Michelson-Morley experiment, the Kennedy-Thorndike experiment and so on. It is noteworthy that Hsu predicts a new law of Doppler frequency shift, which differs from the usual one in the order $(v/c)^2$.

In this paper we study the consequences of Hsu's new ideas with special reference to the Maxwell-Boltzmann (M-B) distribution function of an ideal gaseous assembly in mass motion. We establish the invariance of the M-B distribution function under Hsu's space-light transformations. This invariance could be established only if we postulate the transformation of temperature $T=qT'$ given by Ott and not the transformation $T=q^{-1}T'$, suggested by Planck-Einstein (Tolman 1934). Moreover the entropy is invariant under the space-light transformations. This result agrees with the result under Lorentz transformations. Our investigation also suggests a transformation for h, the Planck's constant. It is to be noted here that in Hsu's theory it is not h, but $\bar{h}=h/c$ which is a universal constant.

2. Distribution function

Consider a frame K' which is moving uniformly with a velocity v with respect to a frame K, the motion being along their common x axis. (The primed quantities in (1) refer to the K' system). The gaseous assembly is at rest in K'. On the introduction of the conservation of the net linear momentum of the assembly, the restrictive conditions, controlling the M-B distribution, with respect to the K frame become

$$\left.\begin{aligned} \sum_j \vec{N}_j &= N, \\ \sum_j N_j \epsilon_j &= E, \\ \sum_j N_j \mathbf{p}_j &= \mathbf{P}, \end{aligned}\right\} \tag{2}$$

where N_j is the number of distinguishable particles in the g_j-fold degenerate energy level and the momentum $\mathbf{p}_j$ has components p_x, p_y, p_z as measured in K. N, E and $\mathbf{P}$ stand for the total number of (distinguishable) particles in the assembly, total energy and total momentum, respectively. The probability $W(N_1, N_2, \ldots, N_j \ldots)$ that there are N_j particles in the g_j-fold degenerate energy level will then be given by

$$\log W = \sum_j N_j \log (g_j/N_j). \tag{3}$$

Maximising the probability of the distribution under the conservation constraints (2) and using α, β and $\mathbf{b}$ as Lagrange's undetermined multipliers, we get

$$N_j = g_j/\exp\left[-(\alpha + \beta\,\epsilon_j + \mathbf{b}\cdot\mathbf{p}_j)\right], \tag{4}$$

as the distribution law under M-B statistics.

Since the velocity of motion is along the x axis, from (4) we get

$$N_j = g_j/\exp\,[-(\alpha + \beta\,\epsilon_j + bp_x)], \tag{5}$$

whence

$$N = \frac{V}{h^3}\int_{-\infty}^{\infty} \frac{d^3p}{\exp\,[-(\alpha + \beta\,\epsilon + bp_x)]}, \tag{6}$$

with $$\sum_j g_j = g = \frac{V}{h^3}\int_{-\infty}^{\infty} d^3p, \tag{7}$$

and ϵ_j is replaced by the mean value ϵ. The quantity V is the volume of the gaseous assembly. The entropy of the assembly in its equilibrium state is statistically defined as $S=k \log W_{\max}$ and as such we have

$$S/k = -\alpha N - \beta E - bP. \tag{8}$$

The first law of thermodynamics for reversible processes, including adiabatic changes in the imposed momentum would then be given by

$$dE = TdS - \pi\, dV + \mu\, dN + v\, dP, \tag{9}$$

where v is the velocity of mass motion. On comparing $(\partial S/\partial P)_{V,N,E}$ and $(\partial S/\partial E)_{V,N,P}$ derived from (9) with the corresponding results derived from (8) we get

$$\beta = -1/kT \text{ and } b = v/kT. \tag{10}$$

To evaluate b as a function of the macroscopic properties of the assembly we evaluate v. This is nothing but $\bar{u}_x$, the x-component of the particle velocity averaged over the whole assembly in its equilibrium state.

Now $$u_x = \frac{\partial \epsilon}{\partial p_x} = \frac{c^2 p_x}{\epsilon} \tag{11}$$

and hence

$$v = \int_{-\infty}^{\infty} \frac{\{(d^3p)\,[c^2(p_x/\epsilon)]\}/\exp\,[-\alpha + (\epsilon/kT) - bp_x]}{(d^3p)/\exp\,[-\alpha + (\epsilon/kT) - bp_x]}. \tag{12}$$

To solve the integrals we make use of the space-light transformations (1) whereby we get

$$p_x = q\frac{c}{c'}\left(p'_x + \frac{v}{c}\,\epsilon'/c'\right) = q\frac{c}{c'}\left(p'_x + \frac{bkT}{c}\cdot\frac{\epsilon'}{c'}\right) \text{ [because of 10]},$$

$$p_y = p'_y\frac{c}{c'},\ p_z = p'_z\frac{c}{c'},\ \epsilon_j^2 = c^2\,(p_j^2 + m^2c^2), \tag{13}$$

and $$\epsilon = q\frac{c^2}{c'}\left(\frac{\epsilon'}{c'} + \frac{v}{c}p'_x\right) = q\frac{c^2}{c'}\left(\frac{\epsilon'}{c'} + \frac{bkT}{c} \cdot p'_x\right).$$

Now from (13) we obtain

$$c'^2\,(p'^2_x + p'^2_y + p'^2_z) - \epsilon'^2 = -m^2c'^4, \tag{14}$$

$$d^3p = d^3p'\,\frac{\partial(p_x, p_y, p_z)}{\partial(p'_x, p_y, p_z)} = d^3p'\,q\left(\frac{c}{c'}\right)^3\left(1 + \frac{bkT}{c}\,c'\frac{p'_x}{\epsilon'}\right), \tag{15}$$

and $$\frac{\epsilon}{kT} - bp_x = \frac{q}{kT}\left[\frac{c^2}{c'^2}\,\epsilon'\left(1 - \frac{b^2k^2T^2}{c^2}\right)\right]. \tag{16}$$

Considering (14) whereby ϵ' is an even function of p'_x and using (13), (15) and (16) in (12) we get

$$v = bkT, \tag{17}$$

and hence $$b = v/kT, \tag{18}$$

as indicated in (10).
Now the M-B distribution law (5) in K will read as

$$N_j = g_j/\exp\,[-\alpha + (\epsilon - vp_x/kT)]. \tag{19}$$

Using (13) and (18) we get

$$(\epsilon/kT) - bp_x = (\epsilon - vp_x/kT) = (c^2/c'^2)\,(\epsilon'/kT)\,q^{-1}. \tag{20}$$

Following Hsu (1976) where $c^2/c'^2 = q^2$ we obtain

$$(\epsilon/kT) - bp_x = (\epsilon - vp_x)/kT = q\,\epsilon'/kT. \tag{21}$$

If we take $$T = qT', \tag{22}$$

then $$(\epsilon - vp_x)/kT = \epsilon'/kT'. \tag{23}$$

For the frame K' the form of the M-B distribution law obtained from (19) would be

$$N'_j = g'_j/\exp\,[-\alpha' + (\epsilon'/kT')]. \tag{24}$$

The expressions for N and N', the total number of particles of the assembly in K and K', respectively, are

$$N = (V/h^3)\int_{-\infty}^{\infty} d_p{}^3/\exp\,[-\alpha + (\epsilon - vp_x)/kT], \tag{25}$$

$$N' = (V'/h'^3)\int_{-\infty}^{\infty} (d^3p'/\exp\,(-\alpha' + \epsilon'/kT'). \tag{26}$$

Also from (1) we get $V = q^{-1} V'$. (27)

Employing (15), (23) and (27) in (25) we get

$$N = \frac{V'}{h^3} \int_{-\infty}^{\infty} \frac{d^3p' \, [1 + (bkT/c)c'p_x'/\epsilon']}{\exp\,(-\alpha + \epsilon'/kT')} (c/c')^3. \tag{28}$$

The second part in the integral vanishes because of (14) and then comparing the remaining part with (26) we find that $N \neq N'$ unless one postulates

(i) $$(c/c'h)^3 = (1/h'^3) \tag{29}$$

i.e. $$(h/c) = (h'/c'),$$

(which is indeed a consideration in Hsu's theory.)

and (ii) $$\alpha = \alpha'. \tag{30}$$

Following Hsu, whereby $(c^2/c'^2) = q^2$ and using (29) we get

$$h = qh'. \tag{31}$$

On physical grounds N must be equal to N', which is achieved through (29) and (30).

We see that (31) implies that h is not a universal constant, on the other hand, (29) implies that it is $\bar{h} = h/c$ which is a universal constant. This may sound to be an unusual feature but there is nothing non-physical in getting (31) (under Hsu's theory), because the operational definition of a physical quantity may be different in different frameworks. In the background of any theory it is the basic principle or axiom which determines the structure of the theory and its consequences. Hence, one need not stick to the opinion that the deductions of the two theories which are based on two different fundamental concepts would agree in toto. The second postulate of the theory of relativity (constancy of velocity of light) is violated in Hsu's theory and, moreover, he considers time to be independent of the frame of reference. Hsu himself has stated that in his new theory it is $\bar{h} = h/c$ which is a universal constant. It is noteworthy that Hsu's proposal about $\bar{h} = h/c$ being a universal constant, has been condoned by statistical consideration as well.

Since $$d^3p = d^3p' \left(1 + \frac{bkT}{c} c' \, p_x'/\epsilon'\right) (c/c')^3 \, q,$$

and $$g = \frac{V}{h^3} \int_{-\infty}^{\infty} d^3p \text{ and } g' = \frac{V'}{h'^3} \int_{-\infty}^{\infty} d^3p',$$

with the help of (14), (15), (27) and (31) we get $g_j = g_j'$. Hence using this result (23) and (30) in (19) we get

$$N_j = g_j' / [\exp\,(-\alpha' + \epsilon'/kT')] \tag{32}$$

From (24) and (32) we see that the invariance of the M-B distribution function is established. In getting (32) from (19), we made use of the result (23) which was obtained through (22), but (22) is nothing but Ott's (1963) transformation for temperature. Hence it is the Ott's transformation for temperature that brings about the invariance of the M-B distribution function under Hsu's space-light transformations.

3. Pressure, energy and momentum

Using the value of the pressure π of the assembly for the system K, given by Pathria (1957) as

$$\pi = \frac{1}{h^3} \int_{-\infty}^{\infty} \frac{d^3p\, p_x (u_x - v)}{\exp\,[-\alpha + (\epsilon - vp_x)/kT]} \tag{33}$$

the corresponding expression for the system K' becomes

$$\pi' = \frac{1}{h'^3} \int_{-\infty}^{\infty} \frac{d^3p'\, p'_x\, u'_x}{\exp\,[-\alpha' + (\epsilon'/kT')]}\,. \tag{34}$$

Employing (11), (13), (15), (23), (30) and (31) to simplify (33) we get

$$\pi = q^2\, \pi'\,. \tag{35}$$

Hence the pressure of the assembly is not invariant under the present transformations. From (34) we obtain

$$\pi'\, V' = \frac{V'}{h'^3} \int_{-\infty}^{\infty} \frac{d^3p'\, p'_x\, u'_x}{\exp\,[-\alpha' + (\epsilon'/kT')]}\,. \tag{36}$$

As $u'_x = \partial\epsilon'/\partial p'_x$ using (14) we get

$$\pi'\, V' = \frac{V'}{h'^3} \int_{-\infty}^{\infty} \frac{d^3p'\, c'^2\, p'^2_x/\epsilon'}{\exp\,[-\alpha' + (\epsilon'/kT')]}\,. \tag{37}$$

Now for the system K the total energy and total momentum are given by

$$E = \frac{V}{h^3} \int_{-\infty}^{\infty} \epsilon\, \frac{d^3p}{\exp\,[-\alpha + (\epsilon - vp_x)/kT]}\,, \tag{38}$$

$$\mathbf{P} = \frac{V}{h^3} \int_{-\infty}^{\infty} p_x\, \frac{d^3p}{\exp\,[-\alpha + (\epsilon - vp_x)/kT)]}\,. \tag{39}$$

The corresponding expressions for the K' system are

$$E' = \frac{V'}{h'^3} \int_{-\infty}^{\infty} \epsilon' \frac{d^3p'}{\exp(-\alpha' + \epsilon'/kT')}, \tag{40}$$

$$\mathbf{P}' = \frac{V'}{h'^3} \int p'_x \frac{d^3p'}{\exp(-\alpha' + \epsilon'/kT')}. \tag{41}$$

Employing (13), (23), (15), (27), (30), (31) and (37), in (38) and (39), we get

$$E = \frac{c^2}{c'^2}\left[E' + \pi' V' \frac{v^2}{c^2}\right] q, \tag{42}$$

$$\mathbf{P} = \frac{c}{c'}[E' + \pi' V']\, q \frac{V}{cc'}. \tag{43}$$

The transformation equation (35) for pressure, combined with (27) and (42), gives

$$(E + \pi V) = \frac{c^2}{c'^2}(E' + \pi' V')\, q. \tag{44}$$

From (42) and (43) it is seen that the quantities $[\mathbf{P}/c,\ i\,(E/c^2)]$ do not form a four-vector but the combination of (43) and (44) suggests that the quantities $\{\mathbf{P}/c,\ i[(E+\pi V)/c^2]\}$ do transform like a four vector. The analogous four vector of Pathria (1955), obtained under Lorentz transformations was $[\mathbf{P},\ i/c\,(E + \pi V)]$. Similarly the conventional energy-momentum four vector under Lorentz transformations is $(\mathbf{P},\ iE/c)$, whereas the same four-vector under the Hsu space-light transformations is

$$\left(\frac{\mathbf{P}}{c},\ i\frac{E}{c}\right).$$

4. Entropy

From (42) and (43) we have

$$q^{-1}(E - v\cdot P) = E'. \tag{45}$$

Let us examine the transformation of entropy. The statistical definition of entropy of an assembly in its equilibrium state is $S = k \log W_{\max}$.

From (8) we have $S/k = -\alpha N - \beta E - b.P.$

Using (10) we get $S/k = -\alpha N + \dfrac{E - v\cdot P}{kT}$ (46)

The corresponding expression for the system K' would be

$$S'/k = -\alpha' N' + (E'/kT'). \tag{47}$$

Taking $N = N'$ and employing (22), (30) and (45) in (46) we get

$$S = S'. \tag{48}$$

One may note from (48) that the entropy is invariant under the space-light transformations. From the considerations of the second law of thermodynamics one can now get the transformation for heat as $dQ = q dQ'$.

5. Conclusion

The investigation for the transformation of the M-B distribution function, temperature, heat and entropy is carried out using the space-light transformations of Hsu. Our investigation supports Ott's (1963) views regarding the transformation of temperature and heat. Moreover, we obtain the invariance of entropy, a result obtained under Lorentz transformations as well.

Acknowledgement

The authors are grateful to the referees for their encouragement and helpful suggestions.

References

Hsu J P 1976 *Found. Phys.* **6** 317
Ott H 1963 *Z. Phys.* **175** 70
Pathria R K 1955 *Proc. Natl. Inst. Sci. (India)* **21** 331
Pathria R K 1957 *Proc. Natl. Inst. Sci. (India)* **23** 168
Tolman R C 1934 *Relativity thermodynamics and cosmology* (Oxford: Clarendon Press), p. 159