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# Uncertainty relation for successive measurements

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**Abstract.** It is noted that the Heisenberg uncertainty relations set a lower bound on the product of variances of two observables $A$, $B$ when they are separately measured on two distinct, but identically prepared ensembles. A new uncertainty relation is derived for the product of the variances of the two observables $A$, $B$ when they are measured sequentially on a single ensemble of systems. It is shown that the two uncertainty relations differ significantly whenever $A$ and $B$ are not compatible.

**Keywords.** Uncertainty relations; collapse postulate; successive measurements; quantum interference of probabilities.

## 1. Introduction

Ever since their discovery by Heisenberg (1927) the uncertainty relations (UR) have played a very important role in our understanding of quantum theory (Jammer 1974). In this paper we derive a new UR which pertains to a situation very different from that encompassed by the conventional UR, in that the proposed relation involves the variances in the outcomes of successive observations performed on the same ensemble of systems.

We shall first briefly summarise the essential content of the Heisenberg UR as would follow directly from the basic principles of quantum theory. If $A$ is the self adjoint operator associated with an observable and $\rho$ is the density operator associated with the state of an ensemble, then the dispersion or standard deviation $\sigma^{\rho}(A)$ and the variance $\mathrm{Var}^{\rho}(A)$ of the outcomes of an experiment to measure $A$ on the ensemble of systems in state $\rho$ is given by the equation

$$\mathrm{Var}^{\rho}(A) = [\sigma^{\rho}(A)]^2 = \langle A^2\rangle_{\rho} - \langle A\rangle_{\rho}^2, \tag{1}$$

where we have adopted the notation $\langle A\rangle_{\rho} = \mathrm{Tr}\, \rho A$. The above expression for the variance $\mathrm{Var}^{\rho}(A)$ follows directly from (i) the well-known definition of the variance of a random variable in classical probability theory (Parthasarathy 1977), and (ii) the basic statistical prescription of quantum theory that the probability $\mathrm{Pr}_A^{\rho}(\triangle)$ of observing the value of $A$ in state $\rho$ to lie in the subset $\triangle \subset R$ is given by

$$\mathrm{Pr}_A^{\rho}(\triangle) = \mathrm{Tr}\,(\rho P^A(\triangle)), \tag{2}$$

where $\triangle \to P^A(\triangle)$ is the spectral measure associated with $A$. Similarly, if $B$ is another observable (self adjoint operator), then the variance $\mathrm{Var}^\rho(B)$ is given by

$$\mathrm{Var}^\rho(B) = (\sigma^\rho(B))^2 = \langle B^2\rangle_\rho - \langle B\rangle_\rho^2. \tag{3}$$

Based on the positivity properties of the density operator $\rho$, and the self adjointness of the operators $A$, $B$ it has now become a routine exercise to derive* the following (Schrödinger form of the) Heisenberg UR (Robertson 1929; Schrödinger 1930):

$$[\sigma^\rho(A)\sigma^\rho(B)]^2 \geqslant \left\{\frac{\langle AB+BA\rangle_\rho}{2} - \langle A\rangle_\rho \langle B\rangle_\rho\right\}^2 + \left\{\frac{\langle AB-BA\rangle_\rho}{2i}\right\}^2. \tag{4}$$

In order to clearly appreciate the meaning of the UR(4) it is very essential to realise that $\sigma^\rho(A)$ and $\sigma^\rho(B)$ as defined by (1), (3) refer to the following experimental situations: $\sigma^\rho(A)$ (respectively. $\sigma^\rho(B)$) is the dispersion in the outcome of an experiment to measure $A(B)$ on an ensemble of systems prepared in state $\rho$, *when no other measurements are carried out prior to* the observation of $A(B)$. The italicised clause 'when no... of $A(B)$' in the previous statement follows from the fact that in defining the variances (1), (3), no use is made of the so-called 'collapse postulate' which is essential for any discussion of the state of the system (and also of subsequent observations performed on it), after any given observation has been carried out. In fact it is noteworthy that the entire analysis leading upto the UR(4) can be carried out without any reference being made to the collapse postulate.

Another important feature of the UR(4), which has been emphasized by Levy-Leblond (1972) is that in (4) there is no restriction whatsoever that the measurements of $A$ and $B$ must be carried out at the same time. In fact we shall hereafter adopt the Heisenberg picture of evolution and replace $A$, $B$ in (1)–(4) by $A(t_1)$ and $B(t_2)$ where, for the time being, the times $t_1$, $t_2$ are arbitrary. Thus the UR(4) are as much valid for nonsimultaneous measurements as for simultaneous measurements. Moreover, for the case of nonsimultaneous measurements UR(4) are the same irrespective of which of the observations ($A(t_1)$ or $B(t_2)$) is being carried out at an earlier time. This is because, even in the case of nonsimultaneous measurements, it should be clearly understood that $\sigma^\rho(A(t_1))$ (respectively $\sigma^\rho(B(t_2))$) is the dispersion in the outcome of an experiment to measure $A(t_1)$ ($B(t_2)$) on an ensemble of systems prepared in state $\rho$, with the stipulation that no other measurements are carried out prior to the observation of $A(t_1)$ ($B(t_2)$) at time $t_1$ ($t_2$). In other words, while computing $\sigma^\rho(A(t_1))$ and $\sigma^\rho(B(t_2))$ according to equations (1) and (3) we consider a situation in which two

*It should be mentioned that in eq. (1), (3) and in the usual derivation of eq, (4), there is always the implicit assumption that the operators A, B are bounded. For unbounded observables (like position, momentum, etc.) the derivation of (4) should be based on a careful consideration of the various domains of definition in each individual case, and it will be seen that in general the eq. (4) will make sense for a restricted class of states only (see for example Gesztesy and Pittner 1978).

*different* ensembles are prepared *identically* (in state $\rho$), and each of them is *separately* subjected to a different experiment—in one case to measure $A(t_1)$ and in the other case to measure $B(t_2)$. We shall therefore refer to the Heisenberg uncertainty relation (4) as the ' *uncertainty relation for distinct measurements* ' (URDM).

## 2. Uncertainty relation for successive measurements

In this section we shall consider an altogether different situation in which an ensemble of systems is prepared in state $\rho$ and is first subjected to a measurement of $A(t_1)$ at time $t_1$. The same ensemble is later subjected to a measurement of $B(t_2)$ at time $t_2>t_1$. Let

$$\sigma^{\rho}_{A(t_1),\, B(t_2)}(A(t_1)) \text{ (respectively } \sigma^{\rho}_{A(t_1)\, B(t_2)}(B(t_2)))$$

denote the dispersion in the resulting outcome for $(A(t_1))$ $(B(t_2))$ when the ensemble (originally prepared in state $\rho$) is subjected to the sequence of measurements $\{A(t_1), B(t_2)\}$ in that order.* In order to compute these dispersion, we will have to take recourse to the collapse postulate which fixes the state of the system after the first experiment to measure $A(t_1)$ has been carried out. For this purpose we shall restrict ourselves to the case where $A(t_1)$ and $B(t_2)$ are bounded observables with purely discrete spectra and the following spectral decompositions:

$$A\,(t_1) = \sum_i \lambda_i\, P_i\,(t_1), \tag{5}$$

$$B\,(t_2) = \sum_j \mu_j\, Q_j\,(t_2), \tag{6}$$

where $\lambda_i$, $\mu_j$ are eigenvalues and $P_i$, $Q_j$ are the associated eigenprojectors. For such observables, the collapse postulate may now be stated as follows (von Neumann 1955; Luders 1951; Furry 1966). If the value $\lambda_i$ is observed as a result of measuring $A(t_1)$ on a system originally in state $\rho$, then the state of the system after the measurement will be

$$\rho' = P_i\,(t_1)\,\rho\,P_i\,(t_1)\,/\,\mathrm{Tr}\,[P_i\,(t_1)\,\rho\,P_i\,(t_1)].$$

From the above equation it follows that the joint probability

$$\mathrm{Pr}^{\rho}_{A(t_1),\, B(t_2)}\ (\lambda_i,\, \mu_j)$$

that the values $\lambda_i$ and $\mu_j$ will result, when we observe $A(t_1)$, $B(t_2)$ (in that order) on an ensemble of systems in state $\rho$, is given by the equation (Wigner 1971)

$$\mathrm{Pr}^{\rho}_{A(t_1), B(t_2)}\ (\lambda_i,\, \mu_j) = \mathrm{Tr}\ [Q_j\,(t_2)\,P_i\,(t_1)\,\rho\,P_i\,(t_1)\,Q_j\,(t_2)]. \tag{7}$$

*If we were to follow the same notation, then the dispersions $\sigma^{\rho}\ (A(t_,))$ and $\sigma^{\rho}\ (B(t_2))$ defined in the last section can be more precisely written as $\sigma^{\rho}_{A(t_1)}\ (A(t_1))$ and $\sigma^{\rho}_{B(t_2)}\ (B(t_2))$ respectively.

The dispersions

$$\sigma^{\rho}_{A(t_1),B(t_2)}\,[A(t_1)] \text{ and } \sigma^{\rho}_{A(t_1),B(t_2)}\,[B(t_2)]$$

now given by the equations

$$\left[\sigma^{\rho}_{A(t_1),\,B(t_2)}\,[A(t_1)]\right]^2 = \sum \lambda_i^2\,\mathrm{Pr}^{\rho}_{A(t_1),B(t_2)}(\lambda_i,\mu_j) - \Big\{\sum_{i,j} \lambda_i\,\mathrm{Pr}^{\rho}_{A(t_1),B(t_2)}(\lambda_i,\mu_j)\Big\}^2; \tag{8}$$

$$\left[\sigma^{\rho}_{A(t_1),\,B(t_2)}\,[B(t_2)]\right]^2 = \sum \mu_i^2\,\mathrm{Pr}^{\rho}_{A(t_1),\,B(t_2)}(\lambda_i,\mu_j) - \Big\{\sum_{i,j} \mu_j\,\mathrm{Pr}^{\rho}_{A(t_1),\,B(t_2)}(\lambda_i,\mu_j)\Big\}^2. \tag{9}$$

To start with we may note that we have the relation

$$\sigma^{\rho}_{A(t_1),\,B(t_2)}\,[A(t_1)] = \sigma^{\rho}\,[A(t_1)], \tag{10}$$

where the r.h.s. is the dispersion defined by equation (1) (with $A(t_1)$ replacing $A$). Equation (10) follows directly from (8), if we employ (5) and (7) together with the cyclic invariance of the trace and the standard properties of the eigenprojectors such as

$$P_i\,P_j = \delta_{ij}\,P_i,\ \sum_i P_i = I, \text{ etc.}$$

Equation (10) is just a statement of the principle of casuality that later measurements do not affect the statistics of the outcome of earlier experiments. We can also similarly reduce equation (9) with the help of (6) and (7) to the following form

$$\sigma^{\rho}_{A(t_1),\,B(t_2)}\,[B(t_2)] = \sigma^{\bar{\rho}}\,[B(t_2)], \tag{11}$$

where
$$\bar{\rho} = \sum_i P_i(t_1)\,\rho\,P_i(t_1)\,. \tag{12}$$

is a density operator which of course depends on $\rho$ and $A(t_1)$.

From equations (5), (11) and (12) we can easily show that the following relation holds:

$$\sigma^{\rho}_{A(t_1),\,B(t_2)}\,(A(t_1) = \sigma^{\bar{\rho}}\,(A(t_1)). \tag{13}$$

Hence, we can employ the relation (4) for the state $\bar{\rho}$ and conclude that

$$\left[\sigma^{\rho}_{A(t_1),\,B(t_2)}\,(A(t_1))\sigma^{\rho}_{A(t_1),\,B(t_2)}\,(B(t_2))\right]^2 = [\sigma^{\bar{\rho}}\,(A(t_1))\,\sigma^{\bar{\rho}}\,(B(t_2))]^2 \geqslant \left\{\frac{\langle A(t_1)\,B(t_2) + B(t_2)\,A(t_1)\rangle_{\bar{\rho}}}{2} - \langle A(t_1)\rangle_{\bar{\rho}}\,\langle B(t_2)\rangle_{\bar{\rho}}\right\}^2 + \left\{\frac{\langle A(t_1)\,B(t_2) - B(t_2)\,A(t_1)\rangle_{\bar{\rho}}}{2i}\right\}^2 \tag{14}$$

Finally, by employing equation (13) we can show that the second term in the above relation (14) vanishes, and the first term can be simplified to yield the UR.

$$\left[\sigma^{\rho}_{A(t_1),\, B(t_2)}\ (A(t_1))\ \sigma^{\rho}_{A(t_1),\, B(t_2)}\ (B(t_2))\right]^2$$

$$\geqslant [\langle A(t_1)\ \{E^{A(t_1)}\ B(t_2)\}\rangle_\rho\ \langle A(t_1)\rangle_\rho\ \langle E^{A(t_1)}\ B(t_2)\rangle_\rho]^2, \tag{15}$$

where $E^{A(t_1)}\ B(t_2)$ denotes the self adjoint operator

$$E^{A(t_1)}\ B(t_2) = \sum_i P_i(t_1)\ B(t_2)\ P_i(t_1). \tag{16}$$

Equation (15), (together with (16)) constitutes a new UR which, unlike (4), sets a lower bound on the product of variances of two observables $A(t_1)$, $B(t_2)$ when they are measured (in that order) on the same ensemble of systems originally prepared in state $\rho$. We shall therefore refer to the relation (15) as the '*uncertainty relation for successive measurements*' (URSM).

At the outset it should be emphasised that all the crucial differences between the relations (4) and (15) arise because of the so-called 'quantum interference of probabilities' (Srinivas 1978 and references cited therein) which is essentially the basic non-classical feature of quantum theory that the statistics of the outcomes of an experiment to measure $B(t_2)$ is dependent on whether or not an experiment to measure $A(t_1)$ has been carried out on the same ensemble of systems earlier. It is this feature which, (via the collapse postulate), gives rise to the difference between the dispersions

$$\sigma^{\rho}\ [B(t_2)] \text{ and } \sigma^{\rho}_{A(t_1),\, B(t_2)}\ [B(t_2)].$$

However, if we consider two observables $A(t_1)$, $B(t_2)$ which are compatible, in the sense that

$$[A(t_1), B(t_2)] \equiv A(t_1)\ B(t_2) - B(t_2)\ A(t_1) = 0, \tag{17}$$

then we can show the following:

(i) $$\sigma^{\rho}_{A(t_1),\, B(t_2)}\ [B(t_2)] = \sigma^{\rho}\ [(B(t_2)], \tag{18}$$

where the r.h.s. is as defined in equation (3).

(ii) The relations URDM (4) and URSM (15) themselves become identical, and can be expressed in the following simple form:

$$\sigma^{\rho}\ (A(t_1))\ \sigma^{\rho}\ (B(t_2)) = \sigma^{\rho}_{A(t_1),\, B(t_2)}\ (A(t_1)\ \sigma^{\rho}_{A(t_1),\, B(t_2)}\ B(t_2))$$

$$\geqslant |\langle A(t_1)\ B(t_2)\rangle_\rho - \langle A(t_1)\rangle_\rho\ \langle B(t_2)\rangle_\rho|. \tag{19}$$

Thus, for the case of two compatible observables, there is just a single uncertainty relation (19), and it is also analogous to the well-known relation in classical probability theory (Parthasarathy 1977).

$$\sigma(X)\,\sigma(Y) \geqslant |\langle XY\rangle - \langle X\rangle\,\langle Y\rangle|, \tag{20}$$

which is valid for any two random variables $X$, $Y$. In fact, the relation (20) may therefore be called the '*Uncertainty relation in classical probability theory*'.

When the observables $A(t_1)$ and $B(t_2)$ are not compatible, the URSM(15) is totally different from the URDM (4). Also, the relation (15) itself is valid only when $t_1 < t_2$—i.e., the measurement of $A(t_1)$ is carried out prior to the measurement of $B(t_2)$. For times $t_1 > t_2$, we will have to employ a URSM in which $A(t_1)$ and $B(t_2)$ are interchanged in equation (15).

Another very important difference between the relations (4) and (15) arises from the fact that the operators

$$A(t_1) \text{ and } E^{A(t_1)}_{B(t_2)}$$

(as given by (16) commute with each other, irrespective of what particular observable $B(t_2)$ is chosen for the later measurement. Hence we have the rather surprising result that the lower bound on the product of the dispersions

$$\sigma^{\rho}_{A(t_1),\,B(t_2)}\,(A(t_1)) \text{ and } \sigma^{\rho}_{A(t_1),\,B(t_2)}\,(B(t_2))$$

as given by the URSM (15) is *always zero*. In fact this lower bound will be attained, (irrespective of whether or not the observables $A(t_1)$ and $B(t_2)$ commute with each other), for all those states which are obtained by mixing the various simultaneous eigenstates of the commuting operators

$$A(t_1) \text{ and } E^{A(t_1)}\,B\,(t_2).$$

We may note in passing that the URSM (15) is valid even when $B(t_2)$ is not restricted to be an observable with a purely discrete spectrum. However, in order to extend the URSM (15) for cases when $A(t_1)$ is arbitrary, we shall have to first extend the collapse postulate to observables with continuous spectra—a problem which has eluded a definitive solution so far (Davies 76).

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