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Exact solutions of Dirac equation

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Abstract. In this paper Dirac equation for two electromagnetic potentials viz vector potential and scalar potential have been solved. These solutions of the Dirac equation are written in terms of known solutions of the Schrödinger equation. The presentation is within the two-component relativistic description. Mainly the bound state solutions have been obtained.

Keywords. Two component equation; bound state solution; Kratzer's potential; Dirac equation.

1. Introduction

Recent investigation on the intense electron beams (Rander *et al* 1970) and the presence of other configurations of parallel electric and magnetic fields (Occhionero and Demianski 1969) in astrophysical problems where relativistic particles are likely to exist (Datlowe and Meyer 1970) have led to the necessity for a clear understanding of a relativistic quantum particle in external electromagnetic fields. This can be readily accomplished if we know the exact solutions. A careful literature survey reveals that very few solvable configurations have been found since the formulation of the Dirac equation. The important amongst them are: Coulomb potential (Dirac 1928), a constant magnetic field (Rabi 1928), a constant electric field (Sauter 1931), the field of a plane wave (Volkov 1935), the field of a plane wave with a constant magnetic field parallel to the direction of propagation of plane wave (Redmond 1965), four cases in which the electromagnetic potentials assume particular functional dependence on the space coordinates (Stanciu 1966) and the one where the electric and magnetic fields are crossed (Lam 1970). Recently a few more exact solutions of the Dirac equations have been found. Franklin and Marburger (1975) have derived a new class of exact solutions which is applicable to variety of fields, including electromagnetic waves for which the length of the wave four-vector is non-zero. Critchfield (1976) has generalised the Dirac equation in a central field to include the scalar potential proportional to r and r^{-1}. Barut and Kraus (1976) have solved the Dirac equation with the Coulomb potential plus the additional interaction due to the anomalous magnetic moment of the electron in the Coulomb field.

The aim of this paper is to present the exact solutions for the electromagnetic potentials which assume particular functional dependance on the space co-ordinate.

Uncoupling of the four equations implied by the Dirac equation is effected through the introduction of a two component spinor. The Dirac equation in the Pauli representation is

$$[\gamma_j(\mathbf{P}+e\mathbf{A})_j + i\gamma_4(W+eV)-im]\,\Psi=0, \tag{1}$$

where $\gamma_j=\rho_2\sigma,\ \gamma_4=\rho_3,$ ρ's and σ's

have their usual meanings.

The relativistically invariant two-component equation for an electron in the external electromagnetic field is written as (Feynman and Gell-Mann 1958)

$$[(\mathbf{P}+e\mathbf{A})^2 + m^2 + e^2\,\sigma\cdot(\mathbf{H}+i\mathbf{E})]\,\Psi = (W+ev)^2\,\Psi. \tag{2}$$

It is sufficient to solve this two component equation. The four component spinors which are solutions of the Dirac equation (1) are generated from the solutions of (2) by

$$\psi_D = \begin{pmatrix} [\sigma\cdot(\mathbf{P}+e\mathbf{A}) + (W+ev)+m]\,\Psi \\ [\sigma\cdot(\mathbf{P}+e\mathbf{A}) + (W+ev)-m]\,\Psi \end{pmatrix}. \tag{3}$$

Sections 2 and 3 deal with the solution of two component equation (2) for the particular configurations of the vector and scalar potentials.

2. Magnetic field solution

It is observed that the two component equation (2) becomes very simple when the scalar field V is zero and if we choose the magnetic field to have only a z-component depending only on one co-ordinate, say y. This external field along with the asymmetric choice of the gauge, $A_y=A_z=0$, transforms (2) into

$$[P_y^2+P_z^2+[(P_x+eA_x(y)]^2+m^2+esH_z(y)]\,\Psi_s X_s=W^2\Psi X_s. \tag{4}$$

The spin index s assumes the values $+1$ and -1 corresponding to the spinors

$$X_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \text{ and } X_{-1} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \text{ respectively.}$$

In (4), the variables P_x and P_z are constants of motion and hence can be taken as constants. After suppressing plane wave dependance on x and z, ψ_s is only a function of y and then (4) is in the canonical form of one-dimensional Schrödinger equation. With the proper choice of $A_x(y)$, (4) can be made equivalent to Schrödinger equation with the solvable potential.

We make the following choice of the vector potential

$$A_x(y) = -\,H_0/y. \tag{5}$$

The corresponding magnetic field is given by

$$H_z(y) = H_0/y^2, \tag{6}$$

where H_0 is a positive quantity.

With this choice, (4) takes the form

$$\left[P_y^2 - \frac{2eH_0P_x}{y} + \frac{eH_0\,(eH_0+s)}{y^2}\right]\Psi_s X_s = (W^2 - P_x^2 - P_z^2 - m^2)\,\Psi_s X_s. \quad (7)$$

It is seen that (7) is formally equivalent to a one-dimensional Schrödinger equation with Kratzer (1920) potential. Ψ_s satisfies the differential equation

$$\left[\frac{d^2}{dy^2} - \epsilon^2 + \frac{2\gamma^2}{y} - \frac{\lambda(\lambda-1)}{y^2}\right]\Psi_s = 0, \quad (8)$$

with the notations:

$$\epsilon^2 = P_x^2 + P_z^2 + m^2 - W^2,$$

$$\gamma^2 = eH_0P_x, \quad (9)$$

and λ given by the equation

$$\lambda(\lambda-1) = eH_0(eH_0+s). \quad (10)$$

We take only the positive solution of this equation, i.e.

$$\lambda = \tfrac{1}{2} + [eH_0(eH_0+s) + \tfrac{1}{4}]^{1/2}, \quad (11)$$

and exclude the negative solution as it will lead to a physically unacceptable solution for (8). The differential equation (8) has an irregular singularity at $y=\infty$ where its normalisable solutions in bound state behaves as $\exp(-\epsilon y)$. It further has a singularity at $y=0$ where $\Psi_s \propto y^\lambda$. Therefore, it is reasonable to set

$$\Psi_s = y^\lambda(e^{-\epsilon y})\cdot f_s(y). \quad (12)$$

With this substitution (8) leads to

$$y f_s''(y) + (2\lambda - 2\epsilon y) f_s'(y) + (2\gamma^2 - 2\epsilon\lambda) f_s(y) = 0. \quad (13)$$

This can be further transformed into Kummer's equation with the change of variable $z=2\epsilon y$. The equation thus obtained is

$$z f''(z) + (2\lambda - z) f'(z) + \left(\frac{\gamma^2}{\epsilon} - \lambda\right) f(z) = 0. \quad (14)$$

The solution of (13) is given in terms of confluent hyper-geometric function i.e.,

$$f_s(y) = {}_1F_1\left(\lambda - \frac{\gamma^2}{\epsilon},\ 2\lambda;\ 2\epsilon y\right) \quad (15)$$

and thus, solution (12) becomes

$$\Psi_s = y^{\lambda}(e^{-\epsilon y})_1F_1\left(\lambda - \frac{\gamma^2}{\epsilon}, 2\lambda; 2\epsilon y\right). \tag{16}$$

The regularity conditions imply that bound states exist only if

$$\lambda - \frac{\gamma^2}{\epsilon} = -n; n=0, 1, 2, \ldots, \epsilon > 0 \tag{17}$$

The energy eigenvalues are obtained from condition (16) and are given by:

$$W^2_{n,s}=P_x^2+P_z^2+m^2-\left(\frac{\gamma^2}{n+\lambda}\right)^2, \tag{18}$$

where γ and λ are given by (9) and (11).
The scattering solutions are not of much interest. However, the solutions obtained for pure imaginary value of ϵ are in the non-relativistic form of the Coulomb scattering. To show this we first obtain the solutions in terms of Whittakar functions.

To transform (14) into Whittakar equation, we set

$$\begin{aligned} f_s(z) &= z^{-\lambda} \exp(z/2)\, F_s(z) \\ \lambda-\gamma^2/\epsilon &= \tfrac{1}{2}K+\mu \\ 2\lambda &= 1+2\mu \end{aligned} \tag{19}$$

and get $$F_s''(z) + \left(-\frac{1}{4} + \frac{K}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) F_s(z) = 0. \tag{20}$$

The general solution (12) then becomes

$$\Psi_s = (2\epsilon)^{-\lambda}\Big[P(2\epsilon y)^{\gamma^2/\epsilon} \cdot \exp(-\epsilon y)\, {}_2F_0\left(-\frac{\gamma^2}{\epsilon}+2\lambda, -\frac{\gamma^2}{\epsilon}+1;\ -\frac{1}{2\epsilon y}\right)$$

$$+Q(-2\epsilon y)^{-\gamma^2/\epsilon} \cdot \exp(\epsilon y)\, {}_2F_0\left(\frac{\gamma^2}{\epsilon}, 1-2\lambda+\frac{\gamma^2}{\epsilon}; \frac{1}{2\epsilon y}\right)\Big]. \tag{21}$$

For pure imaginary value of $\epsilon=-i\eta$, (21) takes the form:

$$\Psi_s=(-2i\eta)^{-\lambda}\Big[P(-2i\eta y)^{-\gamma^2/i\eta} \exp(+i\eta y)\, {}_2F_0\left(\frac{\gamma^2}{i\eta}+2\lambda, 1+\frac{\gamma^2}{i\eta}; \frac{1}{2i\eta y}\right)$$

$$+ Q\,(2i\eta y)^{+\gamma^2/i\eta} \exp(-i\eta y)\, {}_2F_0\left(\frac{-\gamma^2}{i\eta}, 1-2\lambda-\frac{\gamma^2}{i\eta}; \frac{-1}{2i\eta y}\right)\Big]. \tag{22}$$

For studying the scattering, we find the behaviour of Ψ_s at $y=\pm\infty$,

$$\Psi_s \underset{y\to\pm\infty}{=} (-2i\eta)^{-\lambda}[P(-2i\eta y)^{-\gamma^2/i\eta} \exp(+i\eta y)+Q(2i\eta y)^{\gamma^2/i\eta}$$

$$\exp(-i\eta y)]. \tag{23}$$

Setting $\quad \xi = y+ (\gamma^2/\eta^2)\log y \qquad (24)$

gives $\quad (y)^{\gamma^2/i\eta} = \exp[-i\eta(\xi-y)]$

and $\quad (y)^{-\gamma^2/i\eta} = \exp[(i\eta(\xi-y)]$

Thus (23) takes the form

$$\Psi_s \underset{y\to\pm\infty}{=} (-2i\eta)^{-\lambda}[P\exp(i\eta\xi)(-2i\eta)^{-\gamma^2/i\eta}$$

$$+ Q\exp(-i\eta\xi)(2i\eta)^{\gamma^2/i\eta}]. \tag{25}$$

The asymptotic solution (25) has evidently the form of Coulomb scattering solution.

3. The electric field solution

For the external field configuration and a scalar potential depending upon only one direction, say z, (2) can be written in the following form:

$$[P^2 + m^2 + ise\,E_z]\,\Psi_s\,\chi_s = (W + eV)^2\,\Psi_s\,\chi_s. \tag{26}$$

Comparison of (26) with (4) shows that they are equivalent. Taking the scalar potential in the form

$$V(z) = -E_0/z \tag{27}$$

(26) is transformed to

$$\left[P_z^2 + \frac{1}{z^2}(e^2E_0^2 - ise\,E_0) + \frac{1}{z}(2e\,W E_0)\right]\Psi_s\,\chi_s =$$

$$\cdot\,[W^2 - P_x^2 - P_y^2 - m^2]\,\Psi_s\,\chi_s. \tag{28}$$

After relabelling the variables and the constants, this equation becomes identical to (8). So the scalar potential (27) gives rise to an equation which is formally equivalent to a one-dimensional Schrödinger equation with Kratzer's potential.

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