Pramāṇa, Vol. 12, No. 5, May 1979, pp. 447–464, © printed in India

Long range interactions between some charged solitons

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MS received 23 October 1978

Abstract. A procedure is offered for evaluating the forces between classical, charged solitons at large distances. This is employed for the solitons of a complex, scalar two-dimensional field theory with a U(1) symmetry, that leads to a conserved charge Q. These forces are the analogues of the strong interaction forces. The potential, $U(Q, R)$, is found to be attractive, of long range, and strong when the coupling constants in the theory are small. The dependence of $U(Q, R)$ on Q, the sum of the charges of the two interacting solitons (Q will refer to isospin in the SU(2) generalisation of the U(1) symmetric theory) is of importance in the theory of strong interactions; group theoretical considerations do not give such information. The interaction obtained here will be the leading term in the corresponding quantum field theory when the coupling-constants are small.

Keywords. Solitons; intersoliton potential; strong interactions,

1. Introduction

With the discovery of stable, finite-energy, localized solutions of some classical field theories (see, for instance, Scott *et al* 1973), the traditional description of the quantum theory of fields, which was based on fluctuations about vacuum solutions alone, has had to be enlarged (Dashen *et al* 1974; Goldstone and Jackiw 1975; Christ and Lee 1975). When these solutions are included, new particle states—the extended states—are exposed in the quantal Hilbert space, in addition to the usual meson states (for reviews, see Rajaraman 1975 and Jackiw 1977).

Amongst the various interesting properties of the soliton solutions, is one which is the topic of study here. This is the interaction between solitons. This aspect is very important in particle physics, because it is believed that many of the particles are solitons. It is of great relevance in solid state physics too, where the vortex solutions in liquid helium III and type II superconductors are classical soliton solutions.

In the absence of an equivalent LSZ formalism for solitons, conventional methods for computing forces between the ordinary quanta of the theory, cannot be blindly used here for solitons. However, semiclassical, non-perturbative methods can be employed—the extended particles themselves have been obtained only by such methods.

Intersoliton potentials for some real, scalar field theories have been studied (Rajaraman 1977; Joseph and Shenoi 1978). In this paper, interactions between classical charged solitons of a non-linear, complex, scalar field theory are investigated for large inter-soliton distances R. The charge is the conserved physical quantity due to the

U(1) symmetry of the Lagrangian of this system (higher symmetry groups can also be considered). In the limit of small coupling constants, the classical charge of the classical soliton corresponds to the internal quantum number of the corresponding quantum soliton; in the same limit, the classical energy of the soliton provides the leading contribution to the quantum energy (Rajaraman and Weinberg 1975). Then, the results obtained in this paper may be carried over directly to yield the dominant contribution in the corresponding quantum case. (We shall hereafter refer to the classical charge Q as either charge or internal quantum number for convenience).

Though Q may be the electric charge, the forces do not refer to the electromagnetic forces because there are no photon fields (which are vector fields) in our theory. Rather, the forces evaluated here are the analogues of the strong interaction forces.

The potential between charged solitons depends on Q as well as R. This is unlike the case of the solitons of real field theories, where the potential could depend only on R. Since the potential depends only on Q and R, it may hope to represent the correct dynamics only for large R, when the two solitons are almost undisturbed. For small R, when there is distortion of the solitons, we would have to consider the plane wave solutions besides the soliton solutions, because of the finite amplitude for the dissociation of the solitons into the mesons of the theory. The method employed here can deal only with the case when the two interacting solitons have the same charge Q, but this charge can take any value. The dependence of the potential on the charge gives much more information than can be got by symmetry considerations from group theory. In the SU(2) generalisation of this U(1) symmetric theory, when Q refers to the isospin I, the potential can be obtained as a function of I by our method; group theoretical considerations would have given relations between the potentials for values of I_z, the third component of I, corresponding to the same value of I only (i.e., only within a multiplet of I).

In this paper, we concern ourselves only with the simplest system that can support charged soliton solutions, for calculational case. Amongst the systems for which single, charged soliton solutions have been obtained (Lee 1976; Friedberg *et al* 1976; Rajaraman and Weinberg 1975; Montonen 1976), the simplest one is the nonlinear, complex, scalar field theory in one space and one time dimension, governed by the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi^+)(\partial^\mu \phi) - m^2\phi^+\phi + 4b(\phi^+\phi)^2 - 8c(\phi^+\phi)^3. \tag{1}$$

This has a global U(1) symmetry leading to a conserved Abelian charge Q.

Analytic charged soliton solutions are known for this system (Lee 1976). For the sake of completeness, we discuss these solutions and some of their properties in § 2. These are then used as candidates for the computation of the intersoliton potential for large separations R of charged solitons. A procedure for doing this is given in § 3 and the potential evaluated. The potential is seen to be attractive and strong in the weak coupling limit. It is an even function of the charge Q. The potential has an exponential form in R with an inverse range, $(m^2-\omega^2)^{1/2}$, where m is the mass of the free meson in the theory and ω is the frequency of rotation in internal space. So, this potenital may perhaps be regarded as being due to the exchange of bound mesons. This will be discussed further in the conclusion, where we also investigate the dependence of the potential on charge.

2. Single charged soliton solution

We study the charged soliton solutions of the system governed by the Lagrangian given in equation (1), and some of their properties. These, as mentioned, have been obtained (Lee 1976).

We will first show the conditions under which this system can have charged soliton solutions.

The potential term in (1) is given by

$$V = m^2\phi^+\phi - 4b\,(\phi^+\phi)^2 + 8c\,(\phi^+\phi)^3. \tag{2}$$

This is equal to zero for the value $\phi=0$. For the charge operator to be well-defined, the vacuum must be symmetric with respect to U(1). So, $\phi=0$ must be the unique vacuum. Since the potential vanishes for $\phi=0$, it must be positive for all other values of the field ϕ. This places a condition on the parameters, m, b, c, of the theory, namely

$$b^2 < 2m^2C. \tag{3}$$

The boundary conditions for a finite energy solution are that ϕ must approach its vacuum value as $x \to \pm\,\infty$, i.e.

$$\phi \to 0 \text{ as } x \to \pm\,\infty,$$

and also $\partial\phi/\partial x \to 0$ in the same limit.

Consider a static solution of the form

$$\phi = \frac{1}{\sqrt{2}}\,\rho\,(x)\,\exp\,[-i\theta(x)].$$

We shall show that such a solution is not possible. The Euler-Lagrange equation for ρ is

$$\ddot{\rho} - \rho'' = -\,(\partial V/\partial\rho) - \rho\theta'^2 - \rho\dot{\theta}^2,$$

where primes and dots denote differentiation with respect to x and t respectively. Since we are looking for a static solution, this equation becomes

$$\rho'' = (\partial V/\partial\rho) + \rho\theta'^2.$$

θ must be independent of x, and therefore a constant, for finite energy solutions (Friedberg *et al* 1976). ρ must then satisfy the equation

$$\rho'' = (\partial V/\partial\rho).$$

Integrating once with respect to ρ, we get

$$\tfrac{1}{2}\,\rho'^2 = V, \tag{4}$$

where the integration constant is zero because of the boundary conditions on ρ. This equation, however, will not yield a solution for ρ that vanishes at the two far ends of space, since a necessary condition for this is that ρ' must be zero for a non-zero ρ. In other words, V must have another zero at some finite, non-zero ρ, and we know that it does not.

This difficulty can be averted if time-dependent solutions are considered. It is known that, for a fixed, finite charge $Q\,(= i \int (\phi^+\dot{\phi}-\dot{\phi}^+\phi)\, dx)$, the lowest energy solution is of the form

$$\phi = \frac{1}{\sqrt{2}}\rho(x)\exp(-i\omega t),$$

so that ω is the frequency of rotation in internal space (Friedberg *et al* 1976; Montonen 1976). In this case, the equation for ρ becomes

$$\rho'' = (\partial V/\partial\rho) - \omega^2\rho.$$

On integration, this gives

$$\tfrac{1}{2}\rho'^2 = V - \tfrac{1}{2}\omega^2\rho^2$$

where the integration constant is zero for the same reasons as before. Due to the presence of the second term, $-\frac{1}{2}\omega^2\rho^2$ the effective potential, $V-\frac{1}{2}\omega^2\rho^2$ will be zero for some finite ρ, only if $\omega > \omega_{\min}$ where

$$2(m^2 - \omega_{\min}^2)\, c = b^2. \tag{5}$$

There is an upper limit to ω too. Since $V \to \frac{1}{2}(m^2\rho^2)$ as $x \to \pm\infty$, we have

$$\rho'' \simeq (m^2 - \omega^2)\rho \text{ as } x \to \pm\infty$$

giving $\quad \rho \simeq \exp[-(m^2 - \omega^2)^{1/2}|x|]$ in the same limit.

This clearly shows that ω must be less than m; otherwise, we would have plane wave solutions and these do not tend to zero as $x \to \pm\infty$.

We now obtain the soliton solution by quadrature, from

$$\tfrac{1}{2}\rho'^2 = V - \tfrac{1}{2}\omega^2\rho^2,$$

$$\equiv a\rho^2 - b\rho^4 + c\rho^6,$$

where $\quad a \equiv \frac{1}{2}(m^2 - \omega^2)$.

This gives $\int dx = (1/\sqrt{2}) \int d\rho/(a\rho^2 - b\rho^4 + c\rho^6)^{1/2}$.

Substituting $y = \rho^2$, we get

$$\int dx = (1/\sqrt{2}) \int dy/y(a - by + cy^2)^{1/2}.$$

The solution of this is

$$x = \pm (1/2\sqrt{2}\sqrt{a}) \ln [(2a - b\rho^2 \pm 2\sqrt{aX})/\rho^2] + d,$$

where $X = a - b\rho^2 + c\rho^4$ and d is the constant of integration (Gradshteyn and Ryzhik 1965).

Defining $r \equiv 2\sqrt{2}\sqrt{a}(x-d)$, we have

$$e^{\pm r} = (2a - b\rho^2 \pm 2\sqrt{aX})/\rho^2.$$

This can easily be inverted to give ρ as a function of x. We then have

$$\rho^2 = 4ae^{\pm r}/[(e^{\pm r} + b)^2 - 4ac], \tag{7}$$

and $\rho^2 = 0.$

The latter is, of course, the vacuum solution. ρ^2 as defined by (7) approaches zero as $x \to \pm \infty$, when either sign is considered in $\exp(\pm r)$. But it may diverge if the denominator goes to zero, i.e. if

$$[\exp(\pm r) + b]^2 = 4ac,$$

or $$\exp(\pm r) = -b - 2\sqrt{ac}, \; -b + 2\sqrt{ac}.$$

The first root is a perfectly harmless one. It is the second one that makes ρ^2 blow up. The only way to avoid this singularity is to have

$$b > 2\sqrt{ac}, \text{ i.e.}$$

$$b^2 > 2(m^2 - \omega^2)\, c.$$

This puts a lower limit on the allowed values of ω, which must, therefore, be such that ω^2 is greater than $[m^2 - (b^2/2c)]$. This is the same as the condition obtained in (5). Also, it is obvious from the expression for ρ^2, equation (7), that for a real ρ, ω^2 must be less than m^2—a result obtained earlier.

With the condition (3) on the coupling constants, and the restriction on the range of allowed frequencies, ρ^2 as given by (7) is a well-behaved, finite energy, soliton solution. But it is not symmetric in x. To achieve this, we must locate the position of the maximum of ρ^2 and shift it to the origin. This is easily done by rewriting ρ^2 as (with $\triangle \equiv b^2 - 4ac$)

$$\rho^2 = 4a/(e^r - e^{-r}\triangle + 2b),$$

and equating the x-derivative of the denominator to zero. The maximum is found to lie at $r = \ln \sqrt{\triangle}$ i.e.

$$2\sqrt{2}\sqrt{a}\,(x-d) = \ln \sqrt{\triangle}.$$

(Note that the position of the maximum is a function of ω in this case). Taking d to be equal to $-\ln\sqrt{\triangle}/(2\sqrt{2}\sqrt{a})$ does the required job.

Then $e^r = \exp(2\sqrt{2}\sqrt{a}x)\sqrt{\triangle}$,

and ρ^2 takes the form

$$\rho^2 = 2a/[b+\sqrt{\triangle}\cosh(2\sqrt{2}\sqrt{a}x)]. \tag{8}$$

In terms of ω, this is

$$\rho^2 = \frac{m^2-\omega^2}{b+[b^2-2(m^2-\omega^2)c]^{1/2}\cosh[2(m^2-\omega^2)^{1/2}x]}.$$

This is symmetric in x.

Figure 1 shows the profile of this solution. In the solution $\sqrt{\triangle}$ and $\sqrt{a}$ stand for the positive roots; $\sqrt{\triangle}$ must be positive so that ρ is real, i.e. the condition $\rho^2 \geqslant 0$ is satisfied. Changing $+\sqrt{a}$ to $-\sqrt{a}$ is equivalent to reversing the sign of x. But the solution is symmetric in x. So, either sign is perfectly legitimate. We use $+\sqrt{a}$ throughout.

The energy of the system is given by

$$E = \int \epsilon\, dx, \tag{9}$$

where $\epsilon = \dot{\phi}\dot{\phi}^{+} + \phi'\phi^{+\prime} + V(\phi),$

$$= \tfrac{1}{2}\omega^2\rho^2 + \tfrac{1}{2}\rho'^2 + V(\rho).$$

The classical energy, E_0, of our soliton solution is obtained by substituting for ρ^2 from (8) into (9). Note that, for the solution

$$\tfrac{1}{2}\rho'^2 = V(\rho) - \tfrac{1}{2}\omega^2\rho^2,$$

so that $E_0 = \int 2V dx = \int_{-\infty}^{+\infty} (m^2\rho^2 - 2b\rho^4 + 2c\rho^6) dx.$

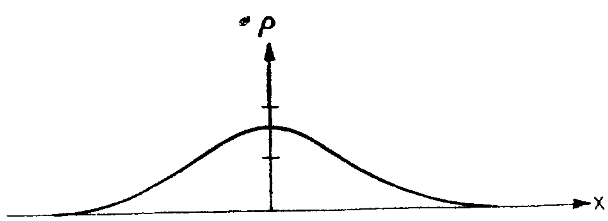

Figure 1. Plot of ρ versus x for a free charged soliton solution.

On evaluating the integrals, we get

$$E_0 = \frac{2m^2c+2\omega^2c-b^2}{8\sqrt{2c}\sqrt{c}} \ln\left(\frac{b+(4ac)^{1/2}}{b-(4ac)^{1/2}}\right)+\frac{\sqrt{ab}}{2\sqrt{2c}}. \tag{10}$$

The charge of the system is given by

$$Q = i\int(\phi^+\dot{\phi}-\dot{\phi}^+\phi)dx = \omega\int\rho^2dx.$$

For the soliton solution

$$Q_0 = \frac{\omega}{2\sqrt{2}\sqrt{c}} \ln\left(\frac{b+(4ac)^{1/2}}{b-(4ac)^{1/2}}\right). \tag{11}$$

Q_0 is positive for positive frequencies, and negative for negative frequencies. The expressions for the energy and charge show that

$$E_0,\ Q_0 \to 0 \text{ as } \omega^2 \to m^2 (\equiv \omega^2_{\max}),$$

and $$E_0,\ Q_0 \to \infty \text{ as } \omega^2 \to \omega^2_{\min}.$$

The soliton solution described by (8) is classically absolutely stable against complete dissociation into free mesons, because its energy E_0 is less than Q_0m for all frequencies, where Q_0m is the energy of the free meson solution (plane-wave solution) for a given charge Q_0. This is consistent with a theorem discussed by Lee (1976) which proves this sort of stability for a wider class of potentials.

3. The inter-soliton potential

We calculate the force exerted by charged extended particles on one another at large distances. Besides its dependence on the distance of separation, R, this force will be a function of Q, the internal quantum number characterising the interacting solitons.

We first outline the procedure employed for the real, scalar Higgs field theory (Rajaraman 1977). In this case, a static solution that was interpreted as a kink and an antikink held a distance R apart, was constructed analytically. This was done by introducing slope discontinuities at the centre of the kink and the antikink. These discontinuities contributed a δ-function term to the equation of motion of the field, and were therefore regarded as extra forces applied to the centre of the kink and the antikink to nail them down. The energy of the kink-antikink configuration was computed for large separations R. This energy, $E(R)$ is, of course, a function of R. The energy $E(\infty)$, of a free kink and an antikink was also calculated. The effective potential $U(R)$ was then defined as $U(R) = E(R)-E(\infty)$, the energy required to bring the pair from infinite separation to a separation R, and hold them fixed at that distance.

In extending this method to the case of charged solitons, we first note that the two soliton solutions (each soliton solution is of the form $(1/\sqrt{2})\,\rho\,(x)\exp(-i\omega t)$, where ω is independent of x) must have the same frequency of rotation ω in internal space. So the charge of the two interacting solitons must be the same. If this were not so, then the discontinuity in the ρ field at $x=0$ would make the energy of the two soliton configuration infinite.

Now, if slope discontinuities are applied to ρ, leaving the θ field untouched, then the charge $2Q$, which is

$$\omega \int_{-\infty}^{+\infty} \rho^2\, dx,$$

will be different from $2Q_0$, the charge of two free solitons (though ω remains the same, the value of the integral $\int \rho^2\, dx$ will be different). But this violates charge conservation. To regain the original charge, it is necessary to change ω as a function of R. This leads to changes in the shape of the soliton because ρ is a function of ω. So the whole procedure of maintaining the same charge by changing ω must be done self-consistently. Once having achieved this, we can define the potential $U(Q, R)$ as

$$U(Q, R) = E(Q, R) - E(Q, \infty),$$

where $E(Q, R)$ is the energy of the two soliton configuration (with total charge Q) with the two solitons held a distance R apart, and $E(Q, \infty)$ is the energy of two free solitons, each of charge $\frac{1}{2}Q$.

Before proceeding further, we rewrite the Lagrangian in (1) in terms of rescaled quantities $x_\mu \rightarrow (b/\sqrt{c})\, x_\mu$;

$$\rho \rightarrow \rho (c/b)^{1/2};\ (m^2, \omega^2, a) \rightarrow c/b^2\,(m^2, \omega^2, a),$$

so that $$\mathscr{L} = b^3/c^2\,[(\partial_\mu \phi^+)(\partial^\mu \phi) - m^2 \phi^+ \phi + 4(\phi^+ \phi)^2 + 8(\phi^+ \phi)^3],$$

where $$\phi = \frac{1}{\sqrt{2}}\rho \exp(-i\omega t).$$

This transformation is introduced purely for computational ease. The Euler-Lagrange equation of motion becomes

$$\rho'' = 2a\rho - 4\rho^3 + 6\rho^5.$$

Denoting the solution of this by ρ_0, and the frequency, charge, energy of this free soliton by ω_0, Q_0, E_0, we have

$$\rho_0^2 = 2a_0/[1 + \sqrt{\Delta_0}\cosh(2\sqrt{2}\sqrt{a_0}\, x)],$$

where $$2a_0 = m^2 - \omega^2;\ \Delta_0 = 1 - 4a_0,$$

$$Q_0 = \frac{\omega_0}{2\sqrt{2}} \ln \left(\frac{1 + \sqrt{4a_0}}{1 - \sqrt{4a_0}}\right), \tag{12}$$

$$E_0 = \frac{Q_0}{\omega_0} (\omega_0^2 + a_0 - \tfrac{1}{4}) + \frac{\sqrt{a_0}}{2\sqrt{2}}, \tag{13}$$

which is $\quad = \dfrac{2m^2+2\omega^2-1}{8\sqrt{2}} \ln \left(\dfrac{1+\sqrt{4a_0}}{1-\sqrt{4a_0}}\right) + \left(\dfrac{\sqrt{a_0}}{2\sqrt{2}}\right)$

We now construct analytically the two soliton solution, held a distance R apart. We do this by applying the discontinuity in slope at $x=0$. In other words, we hold the neighbouring ends of the two solitons at the point $x=0$. The point $x=0$ is preferred over other points for computational reasons. If the solitons are held near their centres, the algebra becomes messy because of the incomplete elliptic integrals encountered. This point will be elaborated on in the conclusion.

Let the two solitons held a distance R apart, be centred at $-\frac{1}{2}R$ and $+\frac{1}{2}R$. The soliton centred at $x=-\frac{1}{2}R$ is given by

$$\rho^2 = 2a/\{1+\sqrt{\triangle} \cosh [2\sqrt{2}\sqrt{a}\, (x+\tfrac{1}{2}R)]\}. \tag{14}$$

This is taken to be the profile of the two-soliton configuration for $x<0$.

At $x=0$,

$$\rho^2 = 2a/[1+\sqrt{\triangle} \cosh (\sqrt{2}\sqrt{a}R)],\ \rho'=0$$

This is a function of ω and R.
For $x>0$, ρ^2 is fixed by symmetry

$$\rho^2(x) = \rho^2(-x).$$

Therefore, $\rho^2 = 2a/\{1+\sqrt{\triangle} \cosh [2\sqrt{2}\sqrt{a}(x-\tfrac{1}{2}R)]\}$ (15)

This is centred at $x=R/2$. This two-soliton configuration rotates in internal space with a frequency ω.

The profile defined by (14) and (15) is clearly continuous at $x=0$ (both rotate with the same frequency ω) and has a finite discontinuity in the slope at that point. So the two solitons are held a distance R apart by a point force at the origin, $x=0$. The full-field configuration is shown in figure 2.

Since the charge (which arises from U(1) symmetry) must be conserved, the charge $2Q$ of the two-soliton solution must be the same as $2Q_0$, the charge of two free solitons.

$$2Q = \omega \int_{-\infty}^{+\infty} \rho^2_{\omega} dx,$$

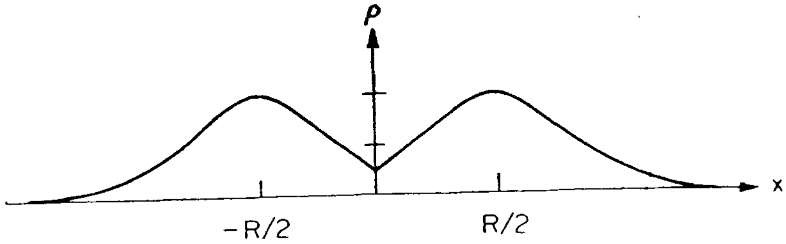

Figure 2. Plot of ρ versus x for the two-soliton configuration. The slope is discontinuous at $x=0$.

where the expressions for ρ^2 given by (14) and (15) are used for $x<0$ and $x>0$ respectively. Then

$$2Q = 2\omega \frac{2a}{2\sqrt{2}\sqrt{a}} \int_{-\infty}^{0} dy/(1+\sqrt{\triangle} \cosh y) \tag{16}$$

with $y = 2\sqrt{2}\sqrt{a}\,(x+\tfrac{1}{2}R)$

Equating this to $2Q_0$, where

$$Q_0 = \frac{\omega_0}{2\sqrt{2}} \ln \left(\frac{1+\sqrt{4a_0}}{1-\sqrt{4a_0}}\right)$$

we see that ω must be a function of R. We will now determine this function. On doing the integral in (16), we have

$$2Q = \frac{\omega}{\sqrt{2}}\left[\ln \left\{\frac{1+\sqrt{\triangle}+\sqrt{4a}\tanh(\sqrt{a}R/\sqrt{2})}{1+\sqrt{\triangle}-\sqrt{4a}\tanh(\sqrt{a}R/\sqrt{2})}\right\} + \tfrac{1}{2}\ln\left\{\frac{1+\sqrt{4a}}{1-\sqrt{4a}}\right\}\right]. \tag{17}$$

For large R (which is of interest to us since it is only in this limit that U(R) may hope to represent the correct dynamics), $\tanh(\sqrt{a}R/\sqrt{2})$ will have a value close to unity and may be written as $1-\eta$,

where $\eta = 2\exp(-\sqrt{2a}R)$,

$\eta \to 0$ as $R \to \infty$.

Using this in (17), we have

$$2Q \simeq \frac{\omega}{\sqrt{2}} \ln\left(\frac{1+\sqrt{4a}}{1-\sqrt{4a}}\right) - \frac{\omega\sqrt{4a}}{\sqrt{2\triangle}}\,\eta.$$

Note that in the limit of infinite separation, $2Q \to 2Q_0$ (because $\eta \to 0$). Making a Taylor expansion of this about $\omega = \omega_0$ and retaining terms of order $\epsilon (= \omega - \omega_0)$ only —for large R, ω will be close to ω_0—it is found that

$$2Q \simeq 2Q_0 - \frac{\omega_0 \sqrt{4a_0}}{\sqrt{2\triangle_0}} \eta + \frac{\epsilon}{\sqrt{2}} \left[\ln \left(\frac{1+\sqrt{4a_0}}{1-\sqrt{4a_0}} \right) - \frac{2\omega_0^2}{\sqrt{a_0}\triangle_0} \right].$$

Enforcing the charge conservation law, $2Q = 2Q_0$,

$$\epsilon = \frac{2\omega_0 \sqrt{a_0}\ \eta}{\sqrt{\triangle_0} \left[\ln \left(\frac{1+\sqrt{4a_0}}{1-\sqrt{4a_0}} \right) - \frac{2\omega_0^2}{(\sqrt{a_0})\triangle_0} \right]}. \tag{18}$$

This is a function of ω_0 and η, and therefore, it gives us ω as a function of ω_0 and R. From (18), it is clear that $\omega \to \omega_0$ as $R \to \infty$.

We now examine (18) to see whether ϵ is negative or positive. The denominator

$$D \equiv \ln \left(\frac{1+\sqrt{4a_0}}{1-\sqrt{4a_0}} \right) - \frac{2\omega_0^2}{\sqrt{a_0}\triangle_0}.$$

is found to be negative for frequencies close to the two extremeties, namely for $\omega \sim m$ (which is the small a limit) and $\omega \sim \omega_{\min}$ (which is the small $\triangle$ limit). Further, $\partial D/\partial \omega_0 \neq 0$ for any ω_0 in the allowed range, thus showing that D is always negative. Therefore ϵ is negative for positive ω_0 and positive for negative ω_0.

Therefore, $\omega < \omega_0$ for ω, ω_0 positive,

and $\qquad \omega > \omega_0$ for ω, ω_0 negative.

This shows for positive frequencies, as the two solitons are brought closer (of course, keeping R large all the while), the frequency of rotation must decrease.

The classical energy of the soliton-soliton configuration may be evaluated as a function of R and Q. It is given by $2E$ where

$$E = \int_{-\infty}^{0} (\tfrac{1}{2}\rho'^2 + V(\rho) + \tfrac{1}{2}\omega^2\rho^2)\, dx,$$

with the ρ^2 given by (14). The energy E can be written as (shown earlier in § 2)

$$E = \int_{-\infty}^{0} 2V\, dx,$$

$$= \int_{-\infty}^{0} (m^2\rho^2 - 2\rho^4 + 2\rho^6)\, dx. \tag{19}$$

Substituting for ρ^2 from (14) in this expression, the three terms in (19) become

$$\int \rho^2 \, dx = \frac{Q}{\omega},$$

$$\int \rho^4 \, dx = -\frac{\sqrt{a\triangle}}{2\sqrt{2}} \cdot \frac{\sinh(\sqrt{2a}\,R)}{[1+\sqrt{\triangle}\cosh(\sqrt{2a}\,R)]} - \frac{\sqrt{a}}{2\sqrt{2}} + \frac{1}{2}\frac{Q}{\omega},$$

$$\int \rho^6 \, dx = -\frac{a\sqrt{a}\sqrt{\triangle}}{2\sqrt{2}} \frac{\sin h(\sqrt{2a}\,R)}{[1+\sqrt{\triangle}\cosh(\sqrt{2a}\,R)]^2} - \frac{3\sqrt{a\triangle}}{8\sqrt{2}} \frac{\sinh(\sqrt{2a}\,R)}{[1+\sqrt{\triangle}\cosh(\sqrt{2a}\,R)]} - \frac{3\sqrt{a}}{8\sqrt{2}} + \frac{3Q}{8\omega} - \frac{aQ}{2\omega},$$

and therefore

$$E = \frac{Q}{\omega}(\omega^2 + a - \tfrac{1}{4}) + \frac{\sqrt{a}}{4\sqrt{2}} + \frac{\sqrt{a\triangle}}{4\sqrt{2}} \frac{\sinh(\sqrt{2a}\,R)}{[1+\sqrt{\triangle}\cosh(\sqrt{2a}\,R)]} - \frac{a\sqrt{a}\sqrt{\triangle}}{\sqrt{2}} \frac{\sinh(\sqrt{2a}\,R)}{[1+\sqrt{\triangle}\cosh(\sqrt{2a}\,R)]^2}. \tag{20}$$

Letting $R \to \infty$, we do get back E_0 (equation 13), the classical energy of a free soliton. Using $\omega = \omega_0 + \epsilon$ in (20) we express E as a function of ω_0 and R. Note that in the large R limit, both ϵ and η are of the same order and they tend to zero as $R \to \infty$. So we retain only terms of $O(\epsilon)$ and $O(\eta)$ while working in this limit. Then, the functions of ω that occur in the expression for E are expressible as follows

$$a \sim a_0 - \epsilon\omega_0,$$

$$\triangle \sim \triangle_0 + 4\epsilon\omega_0,$$

$$\sqrt{\triangle} \sim \sqrt{\triangle_0} + (2\epsilon\omega_0/\sqrt{\triangle_0}).$$

Using these in (20), we get

$$E \simeq E_0 - \frac{a_0 Q}{\omega_0^2}\epsilon + \frac{Q}{4\omega_0^2}\epsilon - \frac{\omega_0}{4\sqrt{2}\sqrt{a_0}}\epsilon - \frac{\sqrt{a_0}}{4\sqrt{2}\sqrt{\triangle_0}}\eta - \frac{a_0\sqrt{a_0}}{\sqrt{2}\sqrt{\triangle_0}}\eta. \tag{21}$$

The effective intersoliton potential $U(Q, R)$ which is given by $2E(Q, R)$—$2E(Q, \infty)$ is therefore

$$U \simeq -\frac{2a_0Q}{\omega_0^2}\epsilon + \frac{Q}{2\omega_0^2}\epsilon - \frac{\omega_0}{2\sqrt{2}\sqrt{a_0}}\epsilon$$

$$-\frac{\sqrt{a_0}}{2\sqrt{2}\sqrt{\triangle_0}}\eta - \frac{\sqrt{2}a_0\sqrt{a_0}}{\sqrt{\triangle_0}}\eta. \tag{22}$$

This potential vanishes for large separations of the solitons. Substituting for ϵ from (18) and eliminating ω_0 between (22) and (12), U can, in principle, be expressed as a function of Q and R.

The dependence on R can be easily obtained from (22), because only ϵ and η depend on R, and that too, through the factor exp $[-\sqrt{2a}\,R]$. The potential therefore has an exponential form in R with an inverse range $\sqrt{2a}$ $[=(m^2-\omega^2)^{1/2}]$. The range is seen to be an increasing function of the frequency and approaches infinity as the frequency approaches its maximum value m. There will be further discussion on the range in the next section.

However, the dependence of the potential on the charge of the solitons is not so transparent, and will be dealt with in § 4.

4. Conclusion

For an arbitrary frequency, the explicit dependence of the potential on the charge is difficult to find because of the implicit relation between the charge Q and the frequency ω.

We first consider the two limiting cases when the frequency is close to its maximum or its minimum value. In these cases, it is found that it is possible to express U as an analytic function of Q.

Take the case when $\omega \sim \omega_{\max}$ $(= m)$, then a_0 is small and is a convenient parameter for expanding various quantities in.
To leading order

$$\epsilon \sim -\frac{a_0}{\omega_0}\eta;\ \omega_0 \sim m\left(1-\frac{a_0}{m^2}\right);\ \triangle_0 \sim 1-4a_0.$$

Therefore, $$Q \sim \frac{\omega_0}{\sqrt{2}}[2\sqrt{a_0} + 2a_0\sqrt{a_0}], \tag{23}$$

and $$U \sim -\frac{a_0\sqrt{a_0}}{2\sqrt{2}\,m^2}(1+3m^2)\eta. \tag{24}$$

We express a_0 in terms of Q by inverting (23) (Remember Q is small when a_0 is, and $Q \to 0$ as $a_0 \to 0$). On squaring the expression for Q, we have

$$Q^2 \simeq 2m^2a_0 - 4a_0^2 + 4m^2a_0^2.$$

This is a quadratic in a_0. Its solutions are

$$a_0 = \frac{-2m^2 \pm [4m^4+16Q^2(m^2-1)]^{1/2}}{8(m^2-1)}.$$

Since $Q \to 0$ as $a_0 \to 0$, only the positive root should be considered.

$$\text{Therefore, } a_0 = \frac{-2m^2+2m^2\left(1+\dfrac{4Q^2(m^2-1)}{m^4}\right)^{1/2}}{8(m^2-1)}.$$

In the limit that we are considering now, this is

$$a_0 \sim (Q^2/2m^2) + O(Q^4).$$

Therefore, $a_0\sqrt{a_0} \sim [(Q^2/2m^2) + O(Q^4)]\ [(Q^2/2m^2) + O(Q^4)]^{1/2}$.

Substituting this in (24), we have

$$U = -Q^2/(2\sqrt{2}m^4)\ [(Q^2/2m^2)+O(Q^4)]^{1/2}\ (1+3m^2) \exp\left[-\sqrt{2a}\ R\right]. \quad (25)$$

This is an attractive potential and is an even function of Q.

We now study the weak coupling limit of U. The dependence of U on the coupling constants b and c is through m and Q only. In terms of the original quantities, (25) becomes

$$U = -\ Q^2b^4/(2\sqrt{2}c^2m^4)\ [(Q^2b^2/2cm^2)+O(Q^4)]^{1/2} \times (1+(3b^2/m^2c)) \exp\left(-\sqrt{2a}\ R\right) \quad (26)$$

where $$Q = \frac{\omega_0}{2\sqrt{2}\sqrt{c}} \ln\left(\frac{b+\sqrt{4a_0c}}{b-\sqrt{4a_0c}}\right).$$

Making b small but keeping it finite, we see that as c is allowed to approach zero, Q takes a constant value. Then the factor c in the denominator of (26) causes U to diverge.

Therefore, the force is strong in the weak c-coupling limit. It was observed to be so for the Higgs, sine-Gordon and the double sine-Gordon systems too (Rajaraman 1977; Joseph and Shenoi 1978).

Now we consider the case when $\omega \sim \omega_{\min}$, i.e. $4a_0 \sim 1$. Then $\triangle_0(\equiv 1-4a_0)$ is the convenient small parameter. To leading order, we have

$$\epsilon \sim -\frac{\sqrt{\triangle_0}}{4\omega_0}\eta;\ \omega_0^2 \sim \tfrac{1}{2}(2m^2-1)+\frac{\triangle_0}{2};\ a_0 \sim \tfrac{1}{4}(1-\triangle_0)$$

$$\text{Therefore, } Q \sim -\frac{\omega_0}{2\sqrt{2}} \ln \triangle_0 \quad (27)$$

$$Q \to \infty \text{ as } \omega \to \omega_{\min}$$

$$U \sim -\frac{1}{4\sqrt{2}\sqrt{\triangle_0}} \cdot \eta$$

Other terms in U are of order $(\sqrt{\triangle_0})\ln\triangle_0$ or less, and are therefore negligible as compared to $1/\sqrt{\triangle_0}$ because $\ln\triangle_0 \ll 1/\triangle_0$ for small $\triangle_0$.
Inverting (27),

$$\frac{1}{\triangle_0} \sim \exp\{2\sqrt{2}[(Q^2/\omega_0{}^2)+O(1)]^{1/2}\}$$

$$\text{therefore } U \sim -(1/4\sqrt{2})\exp\{\sqrt{2}[(Q^2/\omega_0{}^2)+O(1)]^{1/2}\}\,\exp(-\sqrt{2a}\,R). \tag{28}$$

This is also attractive and an even function of Q. Again, on rewriting (28) in terms of the original quantities, we get

$$U \sim -(1/4\sqrt{2})\exp\{\sqrt{2}[(Q^2b^2/c\omega_0{}^2)+O(1)]^{1/2}\}\exp-(\sqrt{2a}\,R).$$

This is also seen to be strong in the weak c-coupling limit.

For arbitrary frequencies, as mentioned, an explicit, analytic dependence of U on Q is difficult to obtain. So, we have calculated U and Q for a set of values of ω in the allowed range, and have thus obtained numerically the charge dependence of U for a fixed R. The result is shown in figure 3. The potential is seen to be attractive at large distances for all values of the charge.

It was shown in § 3 that the potential has an exponential form in R with inverse range $(m^2-\omega^2)^{1/2}$. For the real Higgs, sine-Gordon and the double sine-Gordon systems, the intersoliton forces were interpreted as being due to the exchange of free mesons (Rajaraman 1977; Joseph 1978), because the inverse range was found to be equal to the mass of the free meson. However, in view of the results obtained in this paper, we cannot generalise this interpretation to all intersoliton forces. In the present situation, it may perhaps be possible that the exchange of bound mesons is responsible for the force between the charged solitons. These bound mesons would have masses lower than m, the mass of the free meson, and would lead to ranges of the type obtained in our case.

In evaluating the forces, we introduce a slope discontinuity at $x=0$. This can be physically interpreted as a point force applied at the near ends of the two solitons in order to hold them static. In similar calculations on uncharged solitons, in the lite-

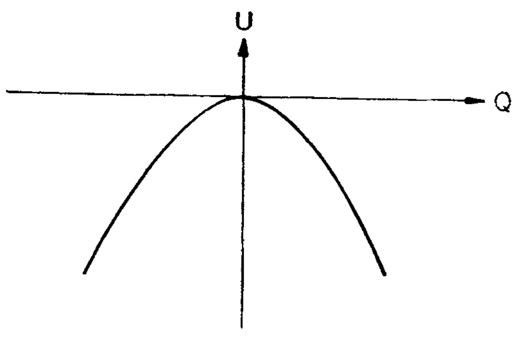

Figure 3. Inter-soliton potential U versus the total charge Q for a fixed separation R.

rature, such a discontinuity has been introduced at the centre of each soliton. The choice of where to place these discontinuities is, at the present state of the art, quite arbitrary. There is no sanctity to holding the solitons at their centres as done in the papers by Perring and Skyrme (1962), Rajaraman (1977), Forster (1977) over our procedure of holding them by their tails. Indeed, one can place the discontinuities at any point on each soliton. The physics, of course, differs from choice to choice, since the extent of polarization of one soliton by the other varies. These points although not mentioned in the early work of Perring and Skyrme (1962), have been emphasised by Rajaraman (1977). This arbitrariness in the location of the discontinuity is an inherent weakness of such methods to find intersoliton forces. But there is no better method available now. Furthermore, in the case of the $\lambda\phi^4$ kinks, a calculation of the potential by applying the discontinuity at $x=0$ as we have done here, gives the same functional form, range, sign and coupling constant dependence of $V(R)$ as holding them at their respective centres.* This is shown in the appendix. From this example, it appears that the above mentioned essential features are not effected by this arbitrariness.

We have chosen to put the discontinuity at $x=0$ because it makes the calculations tractable and analytic. Corresponding calculations with discontinuities at the soliton centres would be much more cumbersome, and numerical methods may have to be employed. It might also not be worth taking that much trouble over what is after all a model system in two dimensions. In any case, as argued above, that calculation would not be any less arbitrary than what we have done. Note that $\rho(x)$ and ω are different in our solution from those in the free solitons. Our procedure has therefore permitted the polarisation of each soliton along its entire length due to the presence of the other. This is what one would physically expect. In that sense, our procedure of placing the discontinuities at $x=0$ may even be superior to what has been done in earlier literature.

Acknowledgements

The author wishes to thank Prof. R Rajaraman for suggestions and stimulating discussions and the NCERT for financial aid.

Appendix

We calculate here the potential between $\lambda\phi^4$ kinks by applying the slope discontinuity at $x=0$.

The two-dimensional ϕ^4 field theory is described by

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}m^2\phi^2-\tfrac{1}{4}\lambda\phi^4-\tfrac{1}{4}m^4/\lambda.$$

In terms of rescaled fields and coordinates $\phi\to\sqrt{\lambda}/m\,\phi$, $x_\mu\to mx_\mu$

$$\mathscr{L} = m^4/\lambda\,[\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}\,\phi^2-\tfrac{1}{4}\phi^4-\tfrac{1}{4}].$$

*R Rajaraman: Private communication

We construct the kink-antikink configuration (with the two held a distance R apart) by introducing a discontinuity in slope at $x=0$. The kink solution centred at $x = -R/2$ is given by

$$\phi(x) = \tanh\,[(x+R/2)/\sqrt{2}].$$

This is taken to be the profile for $x < 0$. For $x > 0, \phi$ is fixed by symmetry

i.e. $\phi(x) = \phi(-x)$.

Therefore, $\phi(x) = \tanh\left(\frac{-x+R/2}{\sqrt{2}}\right)$ for $x > 0$.

$$= -\tanh\left(\frac{x-R/2}{\sqrt{2}}\right).$$

This is the antikink centred at $x=R/2$. The slope discontinuity at $x=0$ is

$$(d\phi/dx)_{x=0+} - (d\phi/dx)_{x=0-} = -\sqrt{2}\,\mathrm{sech}^2\,[R/(2\sqrt{2})].$$

The energy of the kink-antikink configuration, with the two held a distance R apart, is given by

$$E(R) = m^3/\lambda \int_{-\infty}^{+\infty} [\tfrac{1}{2}\,(d\phi/dx)^2 - \tfrac{1}{2}\,\phi^2 + \tfrac{1}{4}\phi^4 + \tfrac{1}{4}]\,dx.$$

This is easily evaluated to yield

$$E(R) = \sqrt{2}\,m^3/\lambda\,[-\tfrac{1}{3}\tanh^3(R/2\sqrt{2}) + \tanh(R/2\sqrt{2}) + \tfrac{2}{3}]$$

As $R \to \infty$, $\tanh(R/2\sqrt{2}) \to 1$ and

Therefore, $E(\infty) = ((4\sqrt{2}\,m^3)/3\lambda)$.

We now determine the large separation behaviour of $E(R)$. For large x,

$$\tanh x \sim 1 - 2/\exp(2x)\,[1 - \exp(-2x) + \exp(-4x) + \ldots]$$

Therefore for large R,

$$\tanh(R/2\sqrt{2}) \sim 1 - 2\,[\exp(-R/\sqrt{2}) - \exp(-\sqrt{2}\,R) + \ldots]$$

and $$\tanh^3(R/2\sqrt{2}) \sim 1 - 6\exp(-R/\sqrt{2}) + 18\exp(-\sqrt{2}\,R) + \ldots$$

Therefore, $E(R) \sim \sqrt{2}\,m^3/\lambda\,[4/3 - 4\exp(-\sqrt{2}\,R) + \ldots]$ for large R.

The kink-antikink potential, $V(R)$, which is defined as $E(R)-E(\infty)$, is

$$V(R) \sim -(4\sqrt{2}\, m^3/\lambda) \exp(-\sqrt{2}\, mR) \text{ for large } R,$$

where we have reverted to the original variables.

This potential, as mentioned, has the same functional form, range, sign and coupling constant dependence as the potential obtained by placing the slope discontinuities at $x=\pm R/2$.

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