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Hawking radiation of scalar and Dirac quanta from a Kerr black hole

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Abstract. Unruh's technique of replacing collapse by boundary conditions on the past horizon (the ξ–quantisation scheme) for the derivation of the well-known Hawking radiation is extended to the Kerr black hole for the scalar and especially for the spin half field. The expectation value of the energy momentum tensor is evaluated asymptotically in the ξ–vacuum state yielding explicitly the net Hawking flux of scalar and spin half quanta. The appropriate statistical distribution that emerges naturally for Dirac quanta validates the ξ–scheme for fermions and confirms the association of temperature with a Kerr black hole.

Keywords. Kerr black hole; Hawking effect; scalar fields; Dirac fields.

1. Introduction

The Hawking effect (Hawking 1975), that is, the quantum evaporation of a collapsing black hole is, by now, well known. Sometime ago Unruh (1976) showed that the same effect could be alternatively obtained by replacing the difficult time-dependent problem of collapse by a choice of appropriate boundary conditions on the past horizon of the black hole. In this scheme known as the ξ-quantisation scheme, Unruh rederived Hawking radiation for the scalar field in a Schwarzschild background.

The more general case of the Kerr black hole has also been considerably discussed in literature (for instance Unruh 1974; Ford 1975; De Witt 1975). It is well-known that a Kerr black hole will spontaneously emit particles in the so-called classical superradiant modes (Unruh-Starobinsky effect). Further Hawking's method extended to the Kerr black hole (e.g. De Witt 1975) yields particle creation in all modes and, as in the Schwarzschild case, leads to the concept of the temperature of a Kerr black hole.

The purpose of this paper is to extend Unruh's ξ-quantisation scheme to the scalar and especially to the Dirac field propagating in Kerr space time. The ξ-quantisation scheme is to be distinguished from the more conventional η-quantisation scheme. The schemes differ in their definition of positive frequency boundary conditions (see §2) and hence in their choice of vacuum states. The extension of these quantisation schemes to the massive spin half field is now made possible following the separation of Dirac equation in Kerr metric due to Chandrasekhar (1976). In an earlier work (Iyer and Kumar 1978a) employing the η-quantisation scheme, we exhibited the Unruh-Starobinsky effect for Dirac field in Kerr background. In this paper we obtain explicitly, in the ξ-scheme, the Hawking flux of scalar and Dirac quanta with appropriate statistics.

In §2 we briefly review the η-scheme for the scalar field in Kerr metric and then construct the ξ-modes and exhibit an alternative quantisation scheme. The entire treatment of this section follows closely the work of Unruh (1976) with straightforward modifications that ensue from the Kerr case. The massive spin half field theory in the ξ-scheme presented in §§3 and 4 is however completely new and, as mentioned above, is a sequel to the work of Chandrasekhar (1976) and our earlier work on the η-scheme. In §4 we evaluate asymptotically the vacuum expectation value of the energy-momentum tensor. The ξ-vacuum is then shown to yield the net Hawking flux of scalar and Dirac quanta which, in the low temperature limit, reduces to the particle creation in the classical superradiant modes.

Units are chosen so that $\hbar=c=G=k=1$

The metric signature is $(+, -, -, -)$.

2. ξ-quantisation in Kerr metric. The scalar field

2.1. *The η-quantisation*

The line element for Kerr metric representing the geometry of a rotating black hole of mass M and angular momentum Ma is given in the usual Boyer-Lindquist co-ordinates by

$$ds^2 = \left(1-\frac{2Mr}{|\rho|^2}\right) dt^2 - \frac{|\rho|^2}{\triangle} dr^2 - |\rho|^2 d\theta^2$$

$$-\left[(r^2+a^2)\sin^2\theta + \frac{2Mr}{|\rho|^2} a^2 \sin^4\theta\right] d\phi^2 + \frac{4Mr}{|\rho|^2} a \sin^2\theta \, d\phi \, dt;$$

$$\triangle = r^2 + a^2 - 2Mr, \quad \rho = r + i\, a \cos\theta. \tag{1}$$

The metric is singular for $\rho=0$ and $\triangle=0$. The singularity at $\triangle=0$, however, is a co-ordinate singularity and occurs at

$$r = r_\pm \equiv M \pm (M^2-a^2)^{1/2} \quad (a<M).$$

The greater solution $r=r_+$ defines the outer horizon of the Kerr black hole. The co-ordinate singularities can be removed by transformations analogous to the Eddington-Finkelstein transformations or by Kruskal-like transformations (Boyer and Lindquist 1967, Carter 1968). The singularity at $r=r_+$ can be removed by choosing for example,

$$u = t-r', \tag{2a}$$

$$v = t+r', \tag{2b}$$

$$\phi^+ = \phi - at/(r_+^2+a^2) \equiv \phi - \Omega t, \tag{2c}$$

where $(dr'/dr) = (r^2+a^2)/\triangle$; as $r \to \infty$, $r' \to \infty$; as $r \to r_+$, $r' \to -\infty$, and $\Omega = a/(r_+^2+a^2)$ is the frequency of dragging of inertial frames at the horizon. The expression for the line element in these co-ordinates is lengthy (Carter 1972) and need not be given here. As in the Schwarzschild case one introduces the Kruskal like co-ordinates

$$U = -4\,M \exp-(\kappa_+ u), \tag{3a}$$

$$V = \quad 4\,M \exp(\kappa_+ v)$$

where $$\kappa_+ = (r_+ - r_-)/2\,(r_+^2 + a^2), \tag{3b}$$

and employs them to construct the maximal extension of the Kerr metric. This extension is far more complicated than for the Schwarzschild (a=0) solution. Note however that from (3a) $U<0$; the extended manifold contains besides this region another exterior region with $U>0$ as in the Schwarzschild case. We make use of this extended U-plane for the construction of ξ-modes later.

The generally covariant scalar equation for a particle of mass μ is:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left((-g)^{\frac{1}{2}}\, g^{\alpha\beta}\frac{\partial\phi}{\partial x^\beta}\right) + \gamma\, R\phi + \mu^2\phi = 0, \tag{4}$$

where R is the curvature scalar and γ is any constant. [For μ=0 $\gamma=\frac{1}{6}$ the equation is conformally invariant]. In the exterior region which is of interest in our discussion here, R=0 and (4) reduces to the scalar equation with ' minimal coupling ' (γ=0). The asymptotic energy momentum tensor (§4) is also the same in the two cases. The curvature scalar term, therefore, need not be considered in the subsequent discussion.

The conserved current is

$$J_\mu = \frac{1}{2i}(\phi^*_{1,\mu}\phi_2 - \phi_1{}^*\,\phi_{2,\mu}), \tag{5}$$

which yields a time-independent inner product for any two solutions ϕ_1 and ϕ_2:

$$\langle\phi_2, \phi_1\rangle = \frac{1}{2i}\int (-g)^{\frac{1}{2}}\, g^{t\alpha}\,(\phi^*_{2,\alpha}\,\phi_1 - \phi^*_2\,\phi_{1,\alpha})\, d^3x. \tag{6}$$

In Kerr metric the solution of (4) is given by (Ford 1975)

$$\phi(\omega, m, \lambda, \epsilon; x) = \frac{\exp-(i\omega t)\exp(im\phi)\, S(\omega, m, \lambda; \theta)\, R(\omega, m, \lambda, \epsilon; r)}{(r^2 + a^2)^{1/2}}, \tag{7}$$

where λ is a separation constant and ϵ = I, II distinguishes the two types of solutions. The two types are characterised by the behaviour of the radial solutions near the horizon and at infinity.

$$R(\omega, m, \lambda, \mathrm{I}; r) \to \frac{1}{(2\pi|k|)^{1/2}}\begin{cases}\exp-(ikr') + A_{\mathrm{I}}(\omega, m, \lambda)\exp(ikr'), & r' \to \infty,\\ B_{\mathrm{I}}(\omega, m, \lambda)\exp(-i\tilde{\omega}r'), & r' \to -\infty,\end{cases}$$

$$R(\omega, m, \lambda, \text{II}; r) \to \frac{1}{(2\pi\,|\,\tilde{\omega}\,|)^{1/2}} \begin{cases} B_{\text{II}}(\omega, m, \lambda) \exp(ikr'), \; r' \to \infty, \\ \exp(i\tilde{\omega}r') + A_{\text{II}}(\omega, m, \lambda) \exp(-i\tilde{\omega}r'), \; r' \to -\infty, \end{cases}$$

where $\quad k = \omega\,(1-\mu^2/\omega^2)^{1/2}, \;\; \tilde{\omega} = \omega - m\,\Omega.$ (8)

The angular solutions satisfy the orthogonality condition:

$$\int d\Omega\; S^*(\omega, m, \lambda;\, \theta)\; S(\omega, m, \lambda';\, \theta) = \delta_{\lambda\lambda'}. \tag{9}$$

By construction of wave packets it is easily seen that the type I wave moves in from infinity in the distant past and in the remote future consists of a localised ingoing wave at the horizon and an outgoing wave at infinity. Type II solution is an unphysical wave that comes out of the past horizon of the black hole and in the distant future consists of two pieces; one ingoing at the horizon and the other outgoing at infinity.

The Wronskian relation following from the equation for R gives:

$$1 - |A_{\text{I}}(\omega, m, \lambda)|^2 = \frac{\tilde{\omega}}{k}\,|\,B_{\text{I}}(\omega, m, \lambda)\,|^2, \tag{10a}$$

$$k\, B_{\text{II}}(\omega, m, \lambda) = \tilde{\omega}\, B_{\text{I}}(\omega, m, \lambda). \tag{10b}$$

It is easy to verify that the above solutions satisfy the following orthogonality relations with respect to the inner product given by (6)

$$\langle \phi(\omega, m, \lambda, \epsilon), \phi(\omega', m', \lambda', \epsilon') \rangle = \kappa(\omega, m, \lambda, \epsilon)\, \delta(\omega-\omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}\, \delta_{\epsilon\epsilon'} \tag{11a}$$

$$\langle \phi^*(\omega, m, \lambda, \epsilon), \phi^*(\omega', m', \lambda', \epsilon') \rangle = -\,\kappa(\omega, m, \lambda, \epsilon)\, \delta(\omega-\omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}\, \delta_{\epsilon\epsilon'} \tag{11b}$$

$$\langle \phi(\omega, m, \lambda, \epsilon), \phi^*(\omega', m', \lambda', \epsilon') \rangle = 0;\; \kappa(\omega, m, \lambda, \epsilon)\, \kappa(\omega', m', \lambda', \epsilon') = 1 \tag{11c}$$

where the notation is

$$\begin{aligned} \kappa(\omega, m, \lambda, \epsilon) &= +1 \;\text{ for } \;\epsilon = \text{I}, \;\; \omega > \mu, \\ &\qquad\;\; \text{and } \;\epsilon = \text{II}, \;\; \tilde{\omega} > 0, |\,\omega\,| > \mu, \\ &= -1 \;\text{ for } \;\epsilon = \text{I}, \;\; \omega < -\mu, \\ &\qquad\;\; \text{and } \;\epsilon = \text{II}, \;\; \tilde{\omega} < 0, |\,\omega\,| > \mu. \end{aligned} \tag{12}$$

Notice the important point that for type II modes the sign of the norm is determined not by ω but by $\tilde{\omega}$. In other words for waves originating from the past horizon of the black hole the effective frequency is $\tilde{\omega}$ and not ω. This fact is responsible for many a characteristic difference of the Kerr black hole from the Schwarzschild case.

The above treatment is evidently the same for both the exterior regions of the extended Kerr manifold. We now introduce a label $(+, -)$ to indicate the sector where the solution is restricted. The (+) indicates the usual exterior region ($U<0$) and (—) indicates the other exterior region ($U>0$). The two regions have opposite time directions as in the Schwarzschild case and therefore $\kappa=+1$, for example, corresponds to a negative frequency mode in the (—) region and hence to a negative norm. We then define the generalised inner product for the extended manifold ($-\infty < U < \infty$) by

$$(\phi_1, \phi_2) \equiv \langle \phi_{1+}, \phi_{2+} \rangle - \langle \phi_{1-}, \phi_{2-} \rangle, \tag{13}$$

where $\phi_{1\pm}$ ($\phi_{2\pm}$) is the restriction of ϕ_1 (ϕ_2) to the ($\pm$) regions. As we shall see, the introduction of the generalised inner product (both for the scalar and spin half case) facilitates an elegant treatment of the ξ-quantisation scheme.

An arbitrary solution of the scalar equation in the extended manifold can be expanded as,

$$\begin{aligned}\Phi = \sum_{m,\lambda,\epsilon} \int_{\kappa=+1} d\omega\, [&a_+(\omega, m, \lambda, \epsilon)\, \phi_+(\omega, m, \lambda, \epsilon; x) \\ &+ b_+^\dagger(\omega, m, \lambda, \epsilon)\, \phi_+^*(\omega, m, \lambda, \epsilon; x)] \\ + \sum_{m,\lambda,\epsilon} \int_{\kappa=+1} d\omega\, [&a_-(\omega, m, \lambda, \epsilon)\, \phi_-^*(\omega, m, \lambda, \epsilon; x) \\ &+ b_-^\dagger(\omega, m, \lambda, \epsilon)\, \phi_-(\omega, m, \lambda, \epsilon; x)].\end{aligned} \tag{14}$$

The first (second) term obviously contributes in the +(—) regions only. Using the orthogonality of modes, (11), (13), we get:

$$\begin{aligned} a_+(\omega, m, \lambda, \epsilon) &= (\phi_+(\omega, m, \lambda, \epsilon), \Phi);\ \kappa=+1, \\ b_+^\dagger(\omega, m, \lambda, \epsilon) &= -(\phi_+^*(\omega, m, \lambda, \epsilon), \Phi);\ \kappa=+1, \\ a_-(\omega, m, \lambda, \epsilon) &= (\phi_-^*(\omega, m, \lambda, \epsilon), \Phi);\ \kappa=+1, \\ b_-^\dagger(\omega, m, \lambda, \epsilon) &= -(\phi_-(\omega, m, \lambda, \epsilon), \Phi);\ \kappa=+1. \end{aligned} \tag{15}$$

The Lagrangian density for the scalar field is

$$\mathscr{L} = \frac{\sqrt{-g}}{2}(g^{\alpha\beta}\, \Phi^*_{,\alpha}\, \Phi_{,\beta} - \mu^2\, \Phi^*\, \Phi). \tag{16}$$

The conjugate momenta are defined by

$$\begin{aligned} \Pi_\pm &= \pm \frac{\partial \mathscr{L}}{\partial \Phi_{,t}} = \pm \frac{\sqrt{-g}}{2} g^{\alpha t}\, \Phi^*_{,\alpha}, \\ \Pi^*_\pm &= \pm \frac{\partial \mathscr{L}}{\partial \Phi^*_{,t}} = \pm \frac{\sqrt{-g}}{2} g^{t\beta}\, \Phi_{,\beta}. \end{aligned} \tag{17}$$

The second quantisation is effected by imposing equal time commutation relations ($t = t'$):

$$[\Phi_\pm(x), \Phi_\pm(x')] = [\Pi_\pm(x), \Pi_\pm(x')] = [\Phi_\pm(x), \Pi^\dagger_\pm(x')] = 0$$

$$[\Phi_\pm(x), \Pi_\pm(x')] = i\delta(x-x'). \tag{18}$$

and similarly for hermitian conjugates. Using the above commutators we obtain

$$[a_\pm(\omega, m, \lambda, \epsilon), a^\dagger_\pm(\omega', m', \lambda', \epsilon')] = \delta(\omega-\omega')\,\delta_{mm'}\,\delta_{\lambda\lambda'}\,\delta_{\epsilon\epsilon'};\ \kappa=+1,$$

$$[b_\pm(\omega, m, \lambda, \epsilon), b^\dagger_\pm(\omega', m', \lambda', \epsilon')] = \delta(\omega-\omega')\,\delta_{mm'}\,\delta_{\lambda\lambda'}\,\delta_{\epsilon\epsilon'};\ \kappa=+1. \tag{19}$$

All other commutators vanish. The η-vacuum is defined by

$$a_\pm(\omega, m, \lambda, \epsilon)\,|0\rangle_\eta = b_\pm(\omega, m, \lambda, \epsilon)\,|0\rangle_\eta = 0;\ \kappa = +1. \tag{20}$$

Equation (14), (19), (20) describe the η-quantisation scheme. In this case the positive frequency mode is defined via the Killing vector $\eta=\partial/\partial t$ which is time-like outside the 'stationary limit' $[r=M+(M^2-a^2\cos^2\theta)^{1/2}]$. In simple words it means $\exp(-i\omega t)$ $(\omega > 0)$ is a positive frequency mode for waves originating in the asymptotically flat region. For type II solutions the effective frequency is $\tilde{\omega}$ and not ω as mentioned earlier. Thus for $\kappa = +1$, ϕ_+ (ϕ_+^*) has positive (negative) norm. In contrast, in the other exterior region for $\kappa = +1$, ϕ_-^* (ϕ_-) has positive (negative) norm. This is the basis of the field expansion given by (14).

Quantisation schemes may differ in their choice of normal modes (since this choice is governed by the definition of positive frequency) and therefore in their specification of the vacuum of the field. It turns out that the above η-vacuum refers to a primordial black hole in that it does not give rise to the spontaneous Hawking emission and is therefore not appropriate for a realistic collapse problem. For the Kerr black hole however the η-vacuum does give the Unruh-Starobinsky effect arising due to the rotation of the black hole (Unruh 1974; Ford 1975; Iyer and Kumar 1978a).

As shown by Unruh the collapse problem can be handled alternatively in the framework of the so-called ξ-quantisation scheme. This scheme differs from the η-scheme essentially in the definition of positive frequency for modes originating from the past horizon of the black hole (type II solutions). The positive frequency in this case is defined via the vector $\xi = \partial/\partial U$. This means that for type I mode $\exp(-i\omega t)$ $(\omega > 0)$ is still a positive frequency mode but for type II mode the positive frequency solutions behave like $\exp(-i\omega' U)$ $(\omega'>0)$. We next construct these ξ-modes and exhibit the alternative quantisation scheme. The new vacuum (ξ-vacuum) will then be shown in §4 to yield the Hawking radiation.

2.2. *The ξ-quantisation*

Positive and negative frequency modes can be defined in terms of their analytic properties. The positive (negative) frequency ξ-mode (for type II) can be characterised as one that is analytic and bounded in the lower (upper) half U-plane (Unruh 1976;

Fulling 1977). [For type I the ξ-mode is the same as the η-mode]. Such modes can be constructed from type II solutions of both the exterior regions. From (7) and (8) near the past horizon the type II η-modes go as

$$\phi_{\pm} \longrightarrow \frac{\exp(-i\omega t)\exp(im\phi)\,S(\omega, m, \lambda)\exp(i\tilde{\omega}r')}{[2\pi\,|\,\tilde{\omega}\,|\,(r_+^2+a^2)]^{1/2}}. \tag{21}$$

In terms of Kruskal co-ordinates well-behaved at $r=r_+$, (2), (3), we have

$$\phi_{\pm} \longrightarrow \frac{\exp(im\phi^+)\,S(\omega, m, \lambda)\exp[(i\tilde{\omega}/\kappa_+)\ln(|U|/4M)]}{[2\pi\,|\,\tilde{\omega}\,|\,(r_+^2+a^2)]^{1/2}}. \tag{22}$$

To construct a positive frequency ξ-mode consider

$$\hat{\phi}(\omega, m, \lambda, \mathrm{II}) \equiv \frac{\exp(\pi\tilde{\omega}/2\kappa_+)}{\left(2\sinh\dfrac{\pi\,|\,\tilde{\omega}\,|}{\kappa_+}\right)^{\frac{1}{2}}}\,\phi_+(\omega, m, \lambda, \mathrm{II});\; U<0,$$

$$\equiv \frac{\exp(-\pi\tilde{\omega}/2\kappa_+)}{\left(2\sinh\dfrac{\pi\,|\,\tilde{\omega}\,|}{\kappa_+}\right)^{\frac{1}{2}}}\,\phi_-(\omega, m, \lambda, \mathrm{II});\; U>0. \tag{23}$$

On the past horizon this is the restriction near real U in the lower half U-plane of the analytic and bounded function that goes as

$$\sim \frac{\exp(-\pi\tilde{\omega}/2\kappa_+)}{\left(2\sinh\dfrac{\pi\,|\,\tilde{\omega}\,|}{\kappa_+}\right)^{\frac{1}{2}}}\exp[(i\tilde{\omega}/\kappa_+)\ln U], \tag{24}$$

with a cut along the negative real U-axis. Thus $\hat{\phi}(\omega, m, \lambda, \mathrm{II})$ is a positive frequency ξ-mode *for all values of* $\tilde{\omega}$. In the same manner $\hat{\phi}^*(\omega, m, \lambda, \mathrm{II})$ is the restriction near real U in the upper-half U-plane of the function

$$\sim \frac{\exp(-\pi\tilde{\omega}/2\kappa_+)}{\left(2\sinh\dfrac{\pi\,|\,\tilde{\omega}\,|}{\kappa_+}\right)^{\frac{1}{2}}}\exp[(-i\tilde{\omega}/\kappa_+)\ln U], \tag{25}$$

and is therefore a negative frequency ξ-mode *for all* $\tilde{\omega}$.

The above characterisation of positive frequency mode regards U as the appropriate 'time co-ordinate' for modes on the past horizon. The physical justification for this choice has been discussed by Unruh on the basis of particle-detector models. This analysis indicates that a freely falling detector near the horizon will respond to the presence of positive frequency modes related to the 'time co-ordinate' U (and not t), that is to ξ-modes.

It is easy to verify that the above ξ-modes satisfy the following orthonormality relations with respect to the generalised inner product (13).

$$(\hat{\phi}(\omega, m, \lambda, \mathrm{II}), \hat{\phi}(\omega', m', \lambda', \mathrm{II}) = \delta(\omega-\omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'};\ \kappa=\pm 1, \tag{26a}$$

$$(\hat{\phi}^*(\omega, m, \lambda, \mathrm{II}), \hat{\phi}^*(\omega', m', \lambda', \mathrm{II}) = -\delta(\omega-\omega')\delta_{mm'}\, \delta_{\lambda\lambda'};\ \kappa=\pm 1, \tag{26b}$$

$$(\hat{\phi}(\omega, m, \lambda, \mathrm{II}), \hat{\phi}^*(\omega', m', \lambda', \mathrm{II})) = 0;\quad \kappa=\pm 1. \tag{26c}$$

Thus the norm of a positive (negative) frequency ξ-mode is positive (negative). An arbitrary field solution over the extended manifold can now be expanded alternatively as,

$$\begin{aligned}\Phi = \sum_{m,\lambda} \int_{\kappa=+1} d\omega\ & [a_+(\omega, m, \lambda, \mathrm{I})\, \phi_+(\omega, m, \lambda, \mathrm{I}; x) \\ & + b_+^\dagger(\omega, m, \lambda, \mathrm{I})\, \phi_+^*(\omega, m, \lambda, \mathrm{I}; x) + a_-(\omega, m, \lambda, \mathrm{I})\, \phi_-^*(\omega, m, \lambda, \mathrm{I}; x) \\ & + b_-^\dagger(\omega, m, \lambda, \mathrm{I})\, \phi_-(\omega, m, \lambda, \mathrm{I}; x)] \\ & + \sum_{m,\lambda} \int_{\kappa=\pm 1} d\omega\ [\hat{a}(\omega, m, \lambda, \mathrm{II})\, \hat{\phi}(\omega, m, \lambda, \mathrm{II}; x) \\ & + \hat{b}^\dagger(\omega, m, \lambda, \mathrm{II})\, \hat{\phi}^*(\omega, m, \lambda, \mathrm{II}; x)].\end{aligned} \tag{27}$$

Note that for type I, the ξ-expansion is the same as the η-expansion. Using the orthonormality of the ξ-modes (26), we obtain

$$\hat{a}(\omega, m, \lambda, \mathrm{II}) = (\hat{\phi}(\omega, m, \lambda, \mathrm{II}), \Phi);\ \kappa=\pm 1. \tag{28a}$$

$$\hat{b}^\dagger(\omega, m, \lambda, \mathrm{II}) = -(\hat{\phi}^*(\omega, m, \lambda, \mathrm{II}), \Phi);\ \kappa=\pm 1. \tag{28b}$$

Using (13), (23) and (15) we get:

$$\hat{a}(\omega, m, \lambda, \mathrm{II}) = p(\tilde{\omega})a_+(\omega, m, \lambda, \mathrm{II}) - p(-\tilde{\omega})b_-^\dagger(\omega, m, \lambda, \mathrm{II});\ \kappa=+1, \tag{29a}$$

$$\hat{b}(\omega, m, \lambda, \mathrm{II}) = p(^{\omega})b_+(\omega, m, \lambda, \mathrm{II}) - p(-\tilde{\omega})a_-^\dagger(\omega, m, \lambda, \mathrm{II});\ \kappa=+1, \tag{29b}$$

$$\begin{aligned}\hat{a}(\omega, m, \lambda, \mathrm{II}) &= p(-\tilde{\omega})a_-(-\omega, -m, \lambda, \mathrm{II}) \\ &\quad - p(\tilde{\omega})\, b_+^\dagger(-\omega, -m, \lambda, \mathrm{II});\ \kappa=-1,\end{aligned} \tag{29c}$$

$$\begin{aligned}\hat{b}(\omega, m, \lambda, \mathrm{II}) &= p(-\tilde{\omega})b_-(-\omega, -m, \lambda, \mathrm{II}) \\ &\quad - p(\tilde{\omega})a_+^\dagger(-\omega, -m, \lambda, \mathrm{II});\ \kappa=-1,\end{aligned} \tag{29d}$$

where $$p(\tilde{\omega}) = \frac{\exp(\pi\tilde{\omega}/2\kappa_+)}{(2\sinh \pi|\tilde{\omega}|\kappa_+)^{1/2}}.$$

From these one readily obtains

$$[\hat{a}(\omega, m, \lambda, \mathrm{II}), \hat{a}^{\dagger}(\omega', m', \lambda', \mathrm{II})] = \delta(\omega-\omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'};\ \kappa=\pm 1.$$

$$[\hat{b}(\omega, m, \lambda, \mathrm{II}), \hat{b}^{\dagger}(\omega', m', \lambda', \mathrm{II})] = \delta(\omega-\omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'};\ \kappa=\pm 1. \tag{30}$$

All other commutators are zero. The ξ-vacuum is defined by

$$a_{\pm}(\omega, m, \lambda, \mathrm{I})\,|\,0\,\rangle_{\xi} = b_{\pm}(\omega, m, \lambda, \mathrm{I})\,|\,0\,\rangle_{\xi} = 0;\ \kappa=+1,$$

$$\hat{a}(\omega, m, \lambda, \mathrm{II})\,|\,0\,\rangle_{\xi} = \hat{b}(\omega, m, \lambda, \mathrm{II})\,|\,0\,\rangle_{\xi} = 0;\ \kappa=\pm 1. \tag{31}$$

3. ξ-quantisation of Dirac field in Kerr metric

The η-quantisation of Dirac field in Kerr metric was developed in an earlier work (Iyer and Kumar 1978a). Here we only quote the relevant results, adapting them to the extended manifold. The Dirac equation in a curved geometry is

$$\gamma^{\mu}\nabla_{\mu}\psi + i\mu\psi=0, \tag{32}$$

where $$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \tag{33}$$

The conserved current

$$J^{\mu} = \bar{\psi}_1\, \gamma^{\mu}\, \psi_2, \tag{34}$$

yields a definition of time-independent inner product

$$\langle\psi_1, \psi_2\rangle \equiv \int (\sqrt{-g})\, \bar{\psi}_1\, \gamma^t\, \psi_2\, d^3 x. \tag{35}$$

In Chandrasekhar's representation the Dirac wave function in Kerr metric separates into radial and angular parts:

$$\psi\,(\omega, m, \lambda, \epsilon; x) = \exp\,(-i\omega t)\exp\,(im\,\phi)\left(\frac{(S^-(\theta)\,R^-(r)}{(\sqrt{2})\,\rho^*}, \frac{S^+(\theta)\,R^+(r)}{(\sqrt{\triangle})}, -\frac{S^-(\theta)\,R^+(r)}{(\sqrt{\triangle})}, -\frac{S^+(\theta)\,R^-(r)}{(\sqrt{2})\,\rho}\right)^T. \tag{36}$$

The angular functions $S^{\pm}(\theta)$ satisfy

$$\int d\Omega\ [S^{+*}(\omega, m, \lambda'; \theta)\, S^{+}(\omega, m, \lambda,; \theta) + S^{-*}(\omega, m, \lambda'; \theta)\, S^{-}(\omega, m, \lambda; \theta)] = \delta_{\lambda\lambda'}. \tag{37}$$

The behaviour of $R^{\pm}(r)$ asymptotically and near the horizon can be obtained from the coupled radial equations for $R^{\pm}$. We choose the two linearly independent solutions characterized as below:

$$R_{\mathrm{I}}^{+}(\omega, m, \lambda) \xrightarrow[r \to \infty]{} N_{\mathrm{I}}\left[\exp(-i\alpha) - \frac{\omega}{\mu}(1-(1-\mu^2/\omega^2)^{1/2})\, B_{\mathrm{I}} \exp(i\alpha)\right],$$

$$R_{\mathrm{I}}^{+}(\omega, m, \lambda) \xrightarrow[r \to r_+]{} N_{\mathrm{I}}\, A_{\mathrm{I}} \exp(-i\tilde{\omega} r'), \tag{38a}$$

$$R_{\mathrm{II}}^{+}(\omega, m, \lambda) \xrightarrow[r \to \infty]{} N_{\mathrm{II}}\left[-\frac{\omega}{\mu}(1-(1-\mu^2/\omega^2)^{1/2})\, B_{\mathrm{II}} \exp(i\alpha)\right],$$

$$R_{\mathrm{II}}^{+}(\omega, m, \lambda) \xrightarrow[r \to r_+]{} N_{\mathrm{II}}\, [\beta^*\, \Delta^{1/2} \exp(i\tilde{\omega} r') + A_{\mathrm{II}} \exp(-i\tilde{\omega} r'], \tag{38b}$$

where $\alpha = \omega\, (1-\mu^2/\omega^2)^{1/2}\left(r' - \frac{\mu^2 M}{\omega^2-\mu^2} \ln r\right),$

$$\beta = \frac{\lambda + i\mu r_+}{(r_+ - M) - 2i\tilde{\omega}\,(r_+^2 + a^2)},$$

$$N_{\mathrm{I}} = \left[2\pi \frac{\omega^2}{\mu^2}\left(1-\frac{\mu^2}{\omega^2}\right)^{1/2}\left(1-\left(1-\frac{\mu^2}{\omega^2}\right)^{1/2}\right)\right]^{-1/2},$$

$$N_{\mathrm{II}} = \pi^{-1/2}.$$

R^- can be obtained from the coupled radial equations. In terms of wave packets the two modes have a similar interpretation as for the scalar case. The coefficients in the above solutions satisfy the following ' Wronskian relations ':

$$1 - |B_{\mathrm{I}}|^2 = \pi\, N_{\mathrm{I}}^2\, |A_{\mathrm{I}}|^2, \tag{39a}$$

$$\pi\, N_{\mathrm{I}}^2\, A_{\mathrm{I}} = B_{\mathrm{II}}. \tag{39b}$$

The above solutions form an orthonormal set with respect to the inner product given by (35):

$$\langle \psi(\omega, m, \lambda, \epsilon), \psi(\omega', m', \lambda', \epsilon') \rangle = \delta(\omega - \omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}\, \delta_{\epsilon\epsilon'}; \quad \kappa = \pm 1 \tag{40}$$

It is important to note that the Dirac wave has positive-definite norm for all ω unlike the scalar wave where a positive (negative) frequency mode has positive (negative) norm. This basic distinction leads to characteristic differences between the scalar and spin half fields, for instance, the ' absence of superradiance ' theorem for the spin half field (Unruh 1973; Lee 1977; Martellini and Treves 1977; Iyer and

Kumar 1978b). We retain this feature of positive-definiteness of norm in the extended manifold also by defining the generalised inner product as:

$$(\psi_1, \psi_2) \equiv \langle \psi_{1+}, \psi_{2+} \rangle + \langle \psi_{1-}, \psi_{2-} \rangle. \tag{41}$$

This should be contrasted with the scalar case where the generalised norm is indefinite (equation (13)).

In the extended manifold an arbitrary Dirac field may be expanded in terms of these η-modes as:

$$\Psi = \sum_{m,\lambda,\epsilon} \int_{\kappa=+1} d\omega \, [a_+(\omega, m, \lambda, \epsilon)\, \psi_+(\omega, m, \lambda, \epsilon; x) + b_+^\dagger(\omega, m, \lambda, \epsilon)$$

$$\psi_+(-\omega, -m, \lambda, \epsilon; x) + a_-(\omega, m, \lambda, \epsilon)\, \psi_-(-\omega, -m, \lambda, \epsilon; x)$$

$$+ b_-^\dagger(\omega, m, \lambda, \epsilon)\, \psi_-(\omega, m, \lambda, \epsilon; x)] \tag{42}$$

From (42), (41), (40)

$$a_\pm(\omega, m, \lambda, \epsilon) = (\psi_\pm(\pm\omega, \pm m, \lambda, \epsilon), \Psi); \;\; \kappa = +1, \tag{43a}$$

$$b_\pm^\dagger(\omega, m, \lambda, \epsilon) = (\psi_\pm(\mp\omega, \mp m, \lambda, \epsilon), \Psi); \;\; \kappa = +1. \tag{43b}$$

The Lagrangian density of the spin-half field is

$$\mathscr{L} = (-g)^{1/2}\, (i\bar{\Psi}\, \gamma^\mu \nabla_\mu \Psi - \mu\bar{\Psi}\Psi). \tag{44}$$

The conjugate momenta are defined by

$$\Pi_\pm = \frac{\partial \mathscr{L}}{\partial \Psi_{,t}} = (i\sqrt{-g})\, \bar{\Psi}\, \gamma^t, \tag{45a}$$

$$\Pi_\pm^* = 0. \tag{45b}$$

Field quantisation is effected by imposition of equal-time anti-commutation relations: $(t = t')$.

$$\{\Psi_\pm^a(x), \Psi_\pm^b(x')\} = \{\Pi_\pm^a(x), \Pi_\pm^b(x')\} = 0,$$

$$\{\Pi_\pm^a(x), \Psi_\pm^b(x')\} = i\,\delta(x - x')\delta^{ab}. \tag{46}$$

The $a_\pm$, $a_\pm^\dagger$, $b_\pm$, $b_\pm^\dagger$ are time-independent and (43) and (46) yield

$$\{a_\pm(\omega, m, \lambda, \epsilon), a_\pm^\dagger(\omega', m', \lambda', \epsilon')\} = \delta(\omega - \omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}\, \delta_{\epsilon\epsilon'}; \; \kappa = +1,$$

$$\{b_\pm(\omega, m, \lambda, \epsilon), b_\pm^\dagger(\omega', m', \lambda', \epsilon')\} = \delta(\omega - \omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}\, \delta_{\epsilon\epsilon'}; \; \kappa = +1. \tag{47}$$

All other anti-commutators vanish. The η-vacuum is defined by

$$a_{\pm}(\omega, m, \lambda, \epsilon) | 0\rangle_{\eta} = b_{\pm}(\omega, m, \lambda, \epsilon) | 0\rangle_{\eta} = 0; \quad \kappa = +1. \tag{48}$$

The next task is to construct the positive (negative) frequency ξ-modes appropriate to the spin half field quantisation. Consider

$$\hat{\psi}_{(1)}(\omega, m, \lambda, \mathrm{II}) \equiv \frac{\exp(\pi \tilde{\omega}/2\kappa_+)}{(2 \cosh \pi \tilde{\omega}/\kappa_+)^{1/2}} \psi_+(\omega, m, \lambda, \mathrm{II}); \quad U < 0,$$

$$\equiv \frac{\exp(-\pi \tilde{\omega}/2\kappa_+)}{(2 \cosh \pi \tilde{\omega}/\kappa_+)^{1/2}} \psi_-(\omega, m, \lambda, \mathrm{II}); \quad U > 0. \tag{49}$$

The reduction of $\hat{\psi}_{(1)}(\omega, m, \lambda, \mathrm{II})$ on the past horizon can be read off directly from (36) and (38). It can then be proved that $\hat{\psi}_{(1)}(\omega, m, \lambda, \mathrm{II})$ on the past horizon is the restriction near real U in the lower half U-plane the analytic and bounded function

$$N_{\mathrm{II}} \exp(im\phi^+) \exp(-\pi \tilde{\omega}/2\kappa_+) \exp[(i\tilde{\omega}/\kappa_+)(\ln U/4m)] \times$$

$$(S^-/2^{1/2} \rho_+^*, S^+ \beta^*, -S^- \beta^*, -S^+/2^{1/2} \rho_+)^T \tag{50}$$

where $\quad \rho_+ \equiv r_+ + ia \cos\theta$

with a cut along the negative real U-axis $\hat{\psi}_{(1)}(\omega, m, \lambda, \mathrm{II})$ is therefore a positive frequency ξ-mode for the Dirac field *for all* $\tilde{\omega}$.

In a similar manner, to construct a negative frequency ξ-mode, consider

$$\hat{\psi}_{(2)}(-\omega, -m, \lambda, \mathrm{II}) \equiv \frac{\exp(\pi \tilde{\omega}/2\kappa_+)}{(2 \cosh \pi \tilde{\omega}/\kappa_+)^{1/2}} \psi_+(-\omega, -m, \lambda, \mathrm{II}); \; U<0$$

$$\equiv \frac{-\exp(\pi \tilde{\omega}/2\kappa_+)}{(2 \cosh \pi \tilde{\omega}/\kappa_+)^{1/2}} \psi_-(-\omega, -m, \lambda, \mathrm{II}); \; U > 0. \tag{51}$$

On the past horizon $\hat{\psi}_{(2)}(-\omega, -m, \lambda, \mathrm{II})$ can be seen to be the restriction near real U in the upper half U-plane of the analytic and bounded function that goes as

$$\sim -(U/|U|) \exp(-\pi \tilde{\omega}/2\kappa_+) \exp[(-i\tilde{\omega}/\kappa_+) \ln U], \tag{52}$$

with a cut along the negative real U-axis ($\sim$ stands for constant factors similar to those in (50)). $\hat{\psi}_{(2)}(-\omega, -m, \lambda, \mathrm{II})$ is therefore a negative frequency ξ-mode *for all* $\tilde{\omega}$. Note that the normalisation factors in (49), (51) have been so chosen that we obtain an orthonormal set of ξ-modes with respect to the extended inner product (41).

$$(\hat{\psi}_{(a)}(\omega, m, \lambda, \mathrm{II}), \hat{\psi}_{(b)}(\omega', m', \lambda', \mathrm{II})) = \delta(\omega - \omega') \delta_{mm'} \delta_{\lambda\lambda'} \delta_{ab};$$

$$a, b = 1, 2; \; \kappa = \pm 1. \tag{53}$$

It is important to realise that the difference in the normalisation factors here from the scalar case arises due to the positive-definiteness of norm of the Dirac field.

The ξ-modes constructed above provide an alternative expansion for an arbitrary Dirac field over the extended manifold:

$$\Psi = \sum_{m,\lambda} \int_{\kappa=+1} d\omega \, [a_+(\omega, m, \lambda, \mathrm{I}) \psi_+(\omega, m, \lambda, \mathrm{I}; x) + b_+^\dagger (\omega, m, \lambda, \mathrm{I}) \psi_+(-\omega, -m, \lambda, \mathrm{I}; x) + a_-(\omega, m, \lambda, \mathrm{I}) \psi_-(-\omega, -m, \lambda, \mathrm{I}; x) + b_-^\dagger (\omega, m, \lambda, \mathrm{I}) \psi_-(\omega, m, \lambda, \mathrm{I}; x)] + \sum_{m,\lambda} \int_{\kappa=\pm 1} d\omega \, [\hat{a}(\omega, m, \lambda, \mathrm{II}) \hat{\psi}_{(1)} (\omega, m, \lambda, \mathrm{II}; x) + \hat{b}^\dagger (\omega, m, \lambda, \mathrm{II}) \hat{\psi}_{(2)} (-\omega, -m, \lambda, \mathrm{II}; x)]. \tag{54}$$

From the orthonormality of ξ-modes (53)

$$\hat{a}(\omega, m, \lambda, \mathrm{II}) = (\hat{\psi}_{(1)} (\omega, m, \lambda, \mathrm{II}), \Psi); \; \kappa = \pm 1,$$

$$\hat{b}^\dagger (\omega, m, \lambda, \mathrm{II}) = (\hat{\psi}_{(2)} (-\omega, -m, \lambda, \mathrm{II}), \Psi); \; \kappa = \pm 1. \tag{55}$$

Next using (41), (49), (51) and (43), we obtain the Bogoliubov transformations:

$$\hat{a}(\omega, m, \lambda, \mathrm{II}) = q(\tilde{\omega})\, a_+(\omega, m, \lambda, \mathrm{II}) + q(-\tilde{\omega})\, b_+^\dagger (\omega, m, \lambda, \mathrm{II}); \; \kappa = +1, \tag{56a}$$

$$\hat{b}(\omega, m, \lambda, \mathrm{II}) = q(\tilde{\omega})\, b_+(\omega, m, \lambda, \mathrm{II}) - q(-\tilde{\omega})\, a_-^\dagger (\omega, m, \lambda, \mathrm{II}); \; \kappa = +1, \tag{56b}$$

$$\hat{a}(\omega, m, \lambda, \mathrm{II}) = q(\tilde{\omega})\, b_+^\dagger (-\omega, -m, \lambda, \mathrm{II}) + q(-\tilde{\omega})\, a_- (-\omega, -m, \lambda, \mathrm{II}); \; \kappa = -1, \tag{56c}$$

$$\hat{b}(\omega, m, \lambda, \mathrm{II}) = q(\tilde{\omega})\, a_+^\dagger (-\omega, -m, \lambda, \mathrm{II}) - q(-\tilde{\omega})\, b_- (-\omega, -m, \lambda, \mathrm{II}); \; \kappa = -1 \tag{56d}$$

where
$$q(\tilde{\omega}) = \frac{\exp(\pi\tilde{\omega}/2\kappa_+)}{(2\cosh(\pi\tilde{\omega}/\kappa_+))^{1/2}},$$

To complete the ξ-quantisation scheme of the Dirac field we check the anti-commutation relations of the new operators $\hat{a}$ and $\hat{b}$. Using the anti-commutation relations of $a_\pm$, $b_\pm$, etc. We readily verify that

$$\{\hat{a}(\omega, m, \lambda, \mathrm{II}), \hat{a}^\dagger(\omega', m', \lambda', \mathrm{II})\} = \delta(\omega - \omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}; \; \kappa = \pm 1,$$

$$\{\hat{b}(\omega, m, \lambda, \mathrm{II}), \hat{b}^\dagger(\omega', m', \lambda', \mathrm{II})\} = \delta(\omega - \omega')\, \delta_{mm'}\, \delta_{\lambda\lambda'}; \; \kappa = \pm 1. \tag{57}$$

All other anti-commutators are zero. The spin half ξ-vacuum is defined by

$$a_{\pm}(\omega, m, \lambda, \mathrm{I})\,|\,0\,\rangle_{\xi} = b_{\pm}(\omega, m, \lambda, \mathrm{I})\,|\,0\,\rangle_{\xi}\ ;\ \kappa = +1,$$

$$\hat{a}\,(\omega, m, \lambda, \mathrm{II})\,|\,0\,\rangle_{\xi} = \hat{b}\,(\omega, m, \lambda, \mathrm{II})\,|\,0\,\rangle_{\xi}\ ;\ \kappa = \pm 1. \tag{58}$$

4. Vacuum expectation value of energy-momentum tensor

4.1. *The scalar field*

The stress energy tensor density of a scalar field in curved space time given by

$$T^{\mu\nu} = (-g)^{\frac{1}{2}}/2)\ [(g^{\mu\alpha}\,g^{\nu\beta} + g^{\mu\beta}\,g^{\nu\alpha} - g^{\alpha\beta}\,g^{\mu\nu})\,\partial_\alpha\,\Phi^\dagger\,\partial_\beta\,\Phi + \mu^2 g^{\mu\nu}\,\Phi^\dagger\,\Phi]. \tag{59}$$

To exhibit the difference between the two schemes we give the expectation value of $T^{\mu\nu}$ in both the η and ξ-vacuum states in the usual exterior region (+). The η-vacuum case is well known (Unruh 1974; Ford 1975) and we omit the details. From (59), (14), (19) and (20) one gets:

$$\begin{aligned}{}_\eta\langle\,0\,|\,T^{\mu\nu}\,|\,0\,\rangle_\eta = (-g)^{1/2}/2\Big[(g^{\mu\alpha}\,g^{\nu\beta} + g^{\mu\beta}\,g^{\nu\alpha} - g^{\alpha\beta}\,g^{\mu\nu})\\ \sum_{m,\lambda,\epsilon}\ \int_{\kappa=+1} d\omega\,\phi_{+,\alpha}\,(\omega, m, \lambda, \epsilon)\,\phi^*_{+,\beta}\,(\omega, m, \lambda, \epsilon)\\ + \mu^2 g^{\mu\nu} \sum_{m,\lambda,\epsilon}\ \int_{\kappa=+1} d\omega\,\phi_+\,(\omega, m, \lambda, \epsilon)\,\phi^*_+\,(\omega, m, \lambda, \epsilon)\Big].\end{aligned} \tag{60}$$

Using the asymptotic forms (8) of the solutions (7) we obtain

$$\begin{aligned}\mathop{\mathrm{Lt}}_{r\to\infty}\ {}_\eta\langle\,0\,|\,T^{\mu\nu}\,|\,0\,\rangle_\eta = -\frac{\sin\theta}{2\pi}\sum_{m,\lambda}\Big[\int_{\kappa=+1} d\omega\,|\,S\,|^2\,\omega\,(1-|\,A_I\,|^2)\\ - \int_{\kappa=+1} d\omega\,|\,S\,|^2\,\frac{k\omega}{|\,\tilde{\omega}\,|}\,|\,B_{\mathrm{II}}\,|^2\Big].\end{aligned} \tag{61}$$

The outgoing energy flux across a surface at infinity is

$$\frac{dE}{dt} = \int_{r\to\infty} d\theta\,d\phi\,\langle\,0\,|\,T^{rt}\,|\,0\,\rangle \tag{62}$$

From (61), using the Wronskian relations (10), we get

$$\frac{dE}{dt} = -\frac{1}{2\pi} \sum_{m,\lambda} \int_{\substack{\omega\tilde{\omega}<0 \\ |\omega|>\mu}} d\omega \, |\omega| \, (1-|A_{\rm I}|^2), \tag{63}$$

$$= -\frac{1}{2\pi} \Big[\sum_{m\Omega>\mu} \int_{\mu}^{m\Omega} d\omega \, |\omega| \sum_{\lambda} (1-|A_{\rm I}|^2)$$

$$+ \sum_{m\Omega<-\mu} \int_{m\Omega}^{-\mu} d\omega \, |\omega| \sum_{\lambda} (1-|A_{\rm I}|^2) \Big] \tag{64}$$

which using (10a) gives

$$dE/dt > 0. \tag{65}$$

This gives the scalar particle emission in the classical super-radiant modes. Next for the ξ-vacuum we obtain

$$_{\xi}\langle 0 | T^{\mu\nu} | 0 \rangle_{\xi} =$$

$$(-g)^{1/2}/2 \Big[(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\alpha\beta} g^{\mu\nu})$$

$$\Big(\sum_{m,\lambda} \int_{\kappa=+1} d\omega \, \phi_{+,\alpha}(\omega, m, \lambda, {\rm I}) \, \phi^{*}_{+,\beta}(\omega, m, \lambda, {\rm I})$$

$$+ \sum_{m,\lambda} \int_{\kappa=\pm 1} d\omega \, p^2(\tilde{\omega}) \, \phi_{+,\alpha}(\omega, m, \lambda, {\rm II}) \, \phi^{*}_{+,\beta}(\omega, m, \lambda, {\rm II}) \Big)$$

$$+ \mu^2 g^{\mu\nu} \Big(\sum_{m,\lambda} \int_{\kappa=+1} d\omega \, \phi_{+}(\omega, m, \lambda, {\rm I}) \, \phi^{*}_{+}(\omega, m, \lambda, {\rm I})$$

$$+ \sum_{m,\lambda} \int_{\kappa=\pm 1} d\omega \, p^2(\tilde{\omega}) \, \phi_{+}(\omega, m, \lambda, {\rm II}) \, \phi^{*}_{+}(\omega, m, \lambda, {\rm II}) \Big) \Big]. \tag{66}$$

Using (66), (7) and (8), (62) then yields

$$\frac{dE}{dt} = \frac{1}{2\pi} \sum_{m,\lambda} \Big[-\int_{\kappa=+1} d\omega \; \omega \, (1-|A_{\rm I}|^2)$$

$$+ \int_{\kappa=\pm 1} d\omega \, p^2(\tilde{\omega}) \frac{k\omega}{|\tilde{\omega}|} |B_{\rm II}|^2 \Big]. \tag{67}$$

Using Wronskian relations (10), a straightforward manipulation puts (67) in the form,

$$\frac{dE}{dt}=\frac{1}{2\pi}\Big[\sum_{m,\lambda}\int_{\mu}^{\infty}d\omega\,\frac{\omega}{\exp(2\pi\tilde{\omega}/\kappa_+)-1}\,[(1-|A_{\rm I}(\omega,m,\lambda)|^2)$$
$$+(1-|A_{\rm I}(-\omega,-m,\lambda)|^2)]. \tag{68}$$

Using (10a) it is easily established that

$$dE/dt>0. \tag{69}$$

Equation (68) gives the Hawking radiation of scalar quanta from a Kerr black hole, with its Planckian spectrum associated with the temperature

$$T=\frac{\kappa_+}{2\pi}=\frac{r_+-r_-}{4\pi(r_+{}^2+a^2)}, \tag{70}$$

in agreement with previous calculations based on Hawking's method (De Witt and references therein). In the limit when T is small (i.e. when the mass of the black hole is large) (68) reduces exactly to (64), that is, the Hawking effect reduces to the Unruh-Starobinsky effect.

4.2. *The spin-half field*

The energy momentum tensor density for the spin half field is given by

$$T_{\mu\nu}=\frac{i}{4}(-g)^{1/2}\,[\overline{\Psi}\,\gamma_\mu\,\nabla_\nu\,\Psi+\overline{\Psi}\,\gamma_\nu\,\nabla_\mu\,\Psi$$
$$-(\nabla_\mu\overline{\Psi})\,\gamma_\nu\,\Psi-(\nabla_\nu\overline{\Psi})\,\gamma_\mu\,\Psi]. \tag{71}$$

The calculation of the expectation value of $T_{\mu\nu}$ in the η-vacuum for the usual exterior region is given in Iyer and Kumar (1978a). The results are:

$${}_\eta\langle 0\,|\,T_{rt}\,|\,0\rangle_\eta\xrightarrow[r\to\infty]{}-\frac{\sin\theta}{2^{1/2}\pi}\sum_m\int_{\substack{\omega\tilde{\omega}<0\\|\omega|>\mu}}d\omega\,|\omega|\sum_\lambda(|S^+|^2+|S^-|^2)\times$$
$$(1-|B_{\rm I}|^2). \tag{72}$$

Therefore $$\frac{dE}{dt}=\frac{1}{2^{1/2}\pi}\Big[\sum_m\int_{\substack{\omega\tilde{\omega}<0\\|\omega|>\mu}}d\omega\,|\omega|\sum_\lambda(1-|B_{\rm I}(\omega,m,\lambda)|^2)\Big], \tag{73}$$

$$=\frac{1}{2^{1/2}\pi}\Big[\sum_{m\Omega>\mu}\int_{\mu}^{m\Omega}d\omega\,|\omega|\sum_\lambda(1-|B_I(\omega,m,\lambda)|^2)$$
$$+\sum_{m\Omega<-\mu}\int_{m\Omega}^{-\mu}d\omega\,|\omega|\sum_\lambda(1-|B_I(\omega,m,\lambda)|^2)\Big]. \tag{74}$$

Using the Wronskian relation, (39a), we have

$$dE/dt > 0, \tag{75}$$

which is the Unruh-Starobinsky effect for the Dirac quanta.

Next employing the ξ-expansion (54), of the Dirac field and going over to Chandrasekhar's representation the expectation value of $T_{\mu\nu}$ in the ξ-vacuum can be obtained in a manner similar to the η-case. The final result is:

$$_{\xi}\langle 0 \,|\, T_{rt} \,|\, 0 \rangle_{\xi} \xrightarrow[r\to\infty]{}$$

$$\frac{1}{2^{1/2}\pi}\Big[\sum_{m\ \kappa=+1}\int d\omega\ \omega \sum_{\lambda}(|S^+|^2+|S^-|^2)(|B_I|^2-1)+\sum_{m\ \kappa=\pm1}\int d\omega\ \omega\, q^2(\tilde{\omega})$$

$$\times\sum_{\lambda}(|S^+|^2+|S^-|^2)\Big\{1-\frac{\omega^2}{\mu^2}\Big[1-\Big(1-\frac{\mu^2}{\omega^2}\Big)^{1/2}\Big]^2\Big\}\,|B_{\rm II}|^2\Big], \tag{76}$$

where $\quad S^{\pm} \equiv S^{\pm}(-\omega, -m, \lambda),\ B_{\rm I,\,II} \equiv B_{\rm I,\,II}(-\omega, -m, \lambda).$

Using Wronskian relation (39b), and the equation above, (62) gives

$$\frac{dE}{dt}=\frac{1}{2^{1/2}\pi}\Big[\sum_{m\ \kappa=+1}\int d\omega\,\omega\sum_{\lambda}(1-|B_{\rm I}|^2)-\sum_{m\ \kappa=\pm1}\int d\omega\,\omega q^2(\tilde{\omega})\sum_{\lambda}(1-|B_{\rm I}|^2)\Big] \tag{77}$$

which after a straightforward manipulation can be brought in the form

$$\frac{dE}{dt}=\frac{1}{2^{1/2}\pi}\sum_{m,\lambda}\int_{\mu}^{\infty}d\omega\ \frac{\omega}{\exp(2\pi\tilde{\omega}/\kappa_+)+1}\ [(1-|B_{\rm I}(\omega, m, \lambda)|^2)$$

$$+(1-|B_{\rm I}(-\omega, -m, \lambda)|^2)]. \tag{78}$$

Again using (39a) it is seen that

$$dE/dt > 0. \tag{79}$$

Equation (78) is the main new result of this work, based on the ξ-quantisation scheme for Dirac field developed in §3. It explicitly gives the Hawking radiation of Dirac quanta from a rotating black hole. The appropriate Fermi-Dirac distribution has emerged as naturally for the Dirac case as the corresponding Bose-Einstein distribution emerges for the scalar case. The temperature associated with the emission is the same as for the scalar case, as would be naturally desired if the black hole is to be characterised by a temperature. For a massive Kerr black hole, that is, in the $T\to 0$ limit, (78) reduces exactly to (74) showing as before that in this limit Hawking effect reduces to the Unruh-Starobinsky effect. Thus for a cold Kerr black hole the thermal radiation dies down and one is left with the spontaneous particle creation in the classical super-radiant modes arising purely from the rotation of the black hole.

5. Conclusion

The extension of Unruh's ξ-quantisation scheme has been shown to yield Hawking emission of scalar and spin-half quanta from a Kerr black hole with proper boson, fermion differences. The η-vacuum yields particle creation in the classical super-radiant modes whereas the ξ-vacuum yields the total Hawking radiation. This strengthens the belief (Unruh 1976; Fulling 1977) that the η-vacuum is relevant to the problem of a primordial black hole whereas the ξ-vacuum is appropriate to the problem of a realistic gravitational collapse. The appropriate statistical distribution that has emerged naturally for Dirac quanta establishes the validity of the ξ-scheme for fermions and confirms the association of temperature with a Kerr black hole.

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