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Classical solutions of a model of quark confinement

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Abstract. We find the classical solutions of a model of quark confinement defined by the vanishing of colour currents. Both plane-wave type of solutions extending all over space as well as string-type of solutions confined to restricted regions of space are found.

Keywords. Confinement of quarks; colour currents; classical solutions; strings.

1. Introduction

An elegant model of quark confinement can be constructed by the defining equations:

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0;$$

$$\bar{\psi}\gamma^{\mu}\lambda_i\psi = 0;\ i = 1\ldots 8.$$

Here, ψ is the quark field which is a triplet under colour SU(3) group and λ_i are the generators of this group in the triplet representation. The model has exact quark confinement since the second equation demands the vanishing of the colour current densities at all space-time points.

This model was introduced by Amati and Testa (1974) and some of its remarkable properties were studied recently (Rajasekaran and Srinivasan 1978). We used the functional method of Kikkawa and Eguchi to show its equivalence to quantum chromodynamics and also analysed the structure of the Green's functions of the model.

In this brief paper, we restrict ourselves to a classical viewpoint. We regard the above defining equations as describing a classical field system. Discovery of interesting solutions to classical field systems is an important recent development in field theory and this provides a motivation for the present study. What type of solutions does the model of quark confinement allow? We find both plane-wave type of solutions extending all over space as well as string type of solutions confined to restricted regions of space.

2. Plane-wave solutions

We shall restrict ourselves to the SU(2) subgroup of the colour SU(3). The quark field will be denoted by ψ_i^α, where i goes over 1 and 2 and stands for the SU(2) index while α goes over 1, ... 4 and stands for the Dirac index. We have*

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0, \tag{1}$$

$$\bar{\psi}\gamma_\mu \boldsymbol{\tau}\psi = 0, \tag{2}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$, τ_1, τ_2, τ_3 being the Pauli matrices.

We first solve the constraint eq. (2). It is convenient to rewrite this equation in the following form:

$$\sum\nolimits_{i,j} \psi_i^\dagger\, \boldsymbol{\tau}_{ij}\, \psi_j = 0 \tag{3}$$

$$\sum\nolimits_{i,j} \psi_i^\dagger\, \boldsymbol{\tau}_{ij}\, (\gamma_0\, \boldsymbol{\gamma})\, \psi_j = 0 \tag{4}$$

where the isospin indices have been written explicitly. Eq. (3) implies that the two Dirac spinors ψ_1 and ψ_2 satisfy

$$\begin{aligned} &\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 = 0 \\ &\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1 = 0 \\ &\psi_1^\dagger \psi_2 - \psi_2^\dagger \psi_1 = 0. \end{aligned} \tag{5}$$

In other words, ψ_1 and ψ_2 have equal normalisation and are mutually orthogonal:

$$\psi_1^\dagger \psi_1 = \psi_2^\dagger \psi_2; \tag{6a}$$

$$\psi_1^\dagger \psi_2 = \psi_2^\dagger \psi_1 = 0. \tag{6b}$$

To solve (4) we may take ψ_1 and ψ_2 to be of the form:

$$\psi_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \\ 0 \end{pmatrix}; \; \psi_2 = \begin{pmatrix} \eta_1 \\ \eta_2 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

Since the matrix $\gamma_0\, \boldsymbol{\gamma}$ connects the upper two components with the lower two, it is clear that (7) satisfies (4). By taking $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ to be the two orthonormal two-component spinors we satisfy (6).

A particular choice is

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} f(x); \; \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(x); \tag{8}$$

*We use the Dirac matrices and other conventions of Bjorken and Drell (1964).

so that we may write the complete solution as

$$\psi_i{}^a = \delta_{ia} f(x) \tag{9}$$

Note the mixing between the internal symmetry index i and the kinematic index α which is also characteristic of the other classical solutions of field theory being intensively studied at the present time (Wu and Yang 1969, t'Hooft 1974, Belavin *et al* 1975).

However, the solutions given in (7), (8) or (9) are space-independent*, as can be verified by inserting in the Dirac equation (1), and the time-dependence is given by $\exp(-imt)$.

Solutions with non-trivial space-dependence can be obtained now by applying Lorentz boost to the above solutions. Since the constraint eq. (2) is Lorentz-covariant, the Lorentz-boosted solutions continue to satisfy this equation, while at the same time they will develop the space-time dependence $\exp(-ip{\cdot}x)$, thus satisfying (1) also. These Lorentz-boosted solutions are nothing but the usual Dirac plane-waves, except that the two orthogonal solutions corresponding to spin up and spin down should be assigned for the isospin indices $i = 1$ and 2 respectively. So, we have

$$\psi_1 = \begin{pmatrix} \xi \\ \dfrac{\sigma{\cdot}\mathbf{p}}{p_0 + m}\xi \end{pmatrix} \exp(-ip{\cdot}x); \; \psi_2 = \begin{pmatrix} \eta \\ \dfrac{\sigma{\cdot}\mathbf{p}}{p_0 + m}\eta \end{pmatrix} \exp(-ip{\cdot}x);$$

$$p_0 = (\mathbf{p}^2 + m^2)^{1/2} \tag{10}$$

where ξ and η are the two-component spinors which are mutually orthogonal and have equal normalisation:

$$\xi^\dagger\xi = \eta^\dagger\eta; \; \xi^\dagger\eta = \eta^\dagger\xi = 0. \tag{11}$$

Combining the two parts in (10) into an eight-component Dirac spinor-isospinor, the general solution of the quark-confinement model defined by (1) and (2) is therefore

$$\psi = \begin{pmatrix} \xi \\ \dfrac{\sigma{\cdot}\mathbf{p}}{p_0 + m}\xi \\ \eta \\ \dfrac{\sigma{\cdot}\mathbf{p}}{p_0 + m}\eta \end{pmatrix} \exp(-ip{\cdot}x); \; p_0 = (\mathbf{p}^2 + m^2)^{1/2} \tag{12}$$

where $\mathbf{p}$ is arbitrary. Here the upper four elements correspond to isospin index $i = 1$ and the lower four elements to $i = 2$. In the same way, we get another set of

*This is under the condition that the solution is finite everywhere. If we relax this condition, we can get space-dependent solutions. The solutions given in the next section belong to this latter category.

solutions based on the negative energy spinors;

$$\psi' = \begin{pmatrix} -\dfrac{\sigma\cdot\mathbf{p}}{|p_0|+m}\,\xi \\ \xi \\ -\dfrac{\sigma\cdot\mathbf{p}}{|p_0|+m}\,\eta \\ \eta \end{pmatrix} \exp(-ip\cdot x); \quad p_0 = -(\mathbf{p}^2+m^2)^{1/2}. \tag{13}$$

We thus see that the original eight-fold multiplicity of the usual plane-wave solutions of (1) has been reduced to a two-fold multiplicity because of the constraint of vanishing colour density (2). It is remarkable that (12) and (13) are colour isospinors, nevertheless they have zero colour density. These solutions (12) and (13) with arbitrary $\mathbf{p}$ form a complete set of solutions of the model of quark-confinement and they may be expected to play an important role in the quantized version of the model.

At first sight, it might be thought that since these solutions are non-vanishing everywhere, the model cannot possibly describe confinement of quarks. This point can be clarified only in a quantum-field-theoretical treatment. In fact, we have already shown that the Green's functions of the quantised version of the model do describe non-propagating confined quarks (Rajasekaran and Srinivasan 1978). In the present paper, we restrict ourselves to just one more question. How do solutions to (1) and (2) look, if one confines them to a restricted region of space, as for example within a hadron? We look for this type of solution in the next section.

3. String solutions

We choose cylindrical polar coordinates (ρ, θ, z) with z axis along the string. We again start with solution (7) and inserting it into (1), we get

$$\begin{aligned} &(i\partial_0 - m)\,\xi_i = 0;\ i = 1, 2; \\ &-\frac{\partial \xi_1}{\partial z} = \exp(-i\theta)\left(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\theta}\right)\xi_2; \\ &\exp(i\theta)\left(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\theta}\right)\xi_1 = \frac{\partial\xi_2}{\partial z}; \end{aligned} \tag{14}$$

and exactly same equations for (η_1, η_2) also. The solutions which vanish for $\rho\to\infty$ are

$$\begin{aligned} \xi_1 &= A \exp(in\theta - ikz - imt)\, K_n(k\rho) \\ \xi_2 &= -\, i A \exp(i(n+1)\theta - ikz - imt)\, K_{n+1}(k\rho) \\ \eta_1 &= A \exp(-\, i(n+1)\theta + ikz - imt)\, K_{n+1}(k\rho) \\ \eta_2 &= i A \exp(-\, in\theta + ikz - imt)\, K_n(k\rho). \end{aligned} \tag{15}$$

Here, K_n are the hyperbolic Bessel functions (Gradshteyn and Ryzhik 1965), k is an arbitrary real parameter, n is zero or an integer and A is a constant. The orthogonality condition

$$\xi_1^* \eta_1 + \xi_2^* \eta_2 = 0, \tag{16}$$

is automatically satisfied, and the normalisation condition

$$\xi_1^* \xi_1 + \xi_2^* \xi_2 = \eta_1^* \eta_1 + \eta_2^* \eta_2 \tag{17}$$

is also satisfied because of our choice of the same constant A for both ξ_i and η_i.

The solution (15) describes an infinite string along z axis and vanishes exponentially for large distances in the direction perpendicular to the string, however, it is singular along the axis of the string.

The above set of solutions (i.e. (7) and (17)) describe static strings with vanishing colour density everywhere. By applying Lorentz boosts, it is possible to generate moving strings.

4. Discussion

The main purpose of the present paper is to point out that by associating the spin-up and the spin-down Dirac spinors with the two possible isospin states, one can generate solutions with vanishing isospin density everywhere. It is possible to do this both for plane-wave type of solutions as well as for string-type of solutions. The precise significance of these solutions in the context of quantum field theory is yet to be investigated. In particular, it would be interesting to find possible connections of the colourless Dirac-field string found here with the gauge field strings of Nielsen and Olesen (1973) or of Nambu (1978).

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