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A new approach to peratization technique

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Abstract. The usual method of peratization technique is to expand scattering length $A(\alpha)$ in Born series in powers of the coupling constant g. In this paper a new approach to the peratization technique has been discussed starting with the standard equation for the scattering length. As an application of the theory developed, the cases of inverse fourth power and a logarithmic singular potentials have been discussed.

Keywords. Peratization technique; inverse fourth power; logarithmic singular potentials; scattering length.

1. Introduction

Recently, the problem of scattering on the singular potentials has been widely discussed (Khuri and Pais 1964; Pais and Wu 1964; Calogero and Casssandro 1965, Aly *et al* 1964, 1965; Frank and Land 1970 a, b) mainly as a testing ground for the validity of the peratization technique introduced by Feinberg and Pais (1963) to deal with unrenormalizable field theories. The technique of peratization is designed to give meaning to a series in which each term is a divergent function of a parameter. It has been applied to the calculation of scattering length of various repulsive singular potentials.

For a singular potential $gV(r)$, the scattering length A considered as a function of the coupling constant g, has a singularity at $g=0$ due to the singular nature of the potential (Jabbur 1965). Hence, a power expansion of A in g is frustrated by infinite integrals. To overcome this difficulty, the technique of peratization has been devised.

For this, one parameter family of potentials $gV(r, \alpha)$ is introduced with the following two conditions

(a) $gV(r, \alpha)$ is nonsingular for $\alpha > 0$ and

(b) $gV(r, \alpha) \to gV(r)$ as $\alpha \to 0$.

A series for the scattering length in the coupling constant now exists for each value of α, such that

$$A(\alpha) = \sum_{n=1}^{\infty} A_n(\alpha)\, g^n, \tag{1}$$

where the coefficients $A_n(\alpha)$ are functions of α which diverge as $\alpha \to 0$. The peratization consists of summing up the series of most singular terms in $A_n(\alpha)$ in each power of g and to see finally whether the sum is finite, when the limit $\alpha \to 0$ is approached.

The usual procedure of obtaining the series (1) so far is through Born series. In this paper, we have obtained the series (1) by a different approach which is discussed in § 2. In § 3 and 4 some specific examples have been discussed. A few concluding remarks have been made in § 5.

2. Formulation of the method

Instead of obtaining series (1) by using Born series, we assume that each coefficient $A_n(a)$ in (1) is obtained from the function $A_n(r, a)$ in the limit $r \to \infty$, so that

$$\lim_{r \to \infty} A_n(r, a) = A_n(a), \tag{2}$$

and

$$\lim_{r \to \infty} A(r, a) = A(a). \tag{3}$$

From (2) and (3) we find that (1) can be obtained in the limit $r \to \infty$, if we get

$$A(r, a) = \sum_{n=1}^{\infty} A_n(r, a)\, g^n, \tag{4}$$

where $A(r, a)$ is the interpolating scattering length corresponding to the regularised potential $gV(r, a)$.

The standard equation for scattering length (Calogero 1967) for the potential $gV(r)$ is given as

$$A'(r) = -gV(r)[r+A(r)]^2 \tag{5}$$

Substituting (4) in (5) yields

$$\frac{d}{dr}\sum_{n=1}^{\infty} A_n(r, a) g^n = -gV(r, a)\left[r+\sum_{n=1}^{\infty} A_n(r, a)\cdot g^n\right]^2. \tag{6}$$

On equating equal powers of g on both sides, one gets simple first order differential equations for $A_n(r, a)$ which can easily be solved.

$$A'_1(r, a) = -V(r, a)\cdot r^2; \tag{7a}$$

$$A'_2(r, a) = -V(r, a)\{2r\, A_1(r, a)\}; \tag{7b}$$

$$A'_3(r, a) = -V(r, a)\{2r\, A_2(r, a) + A_1^2(r, a)\}; \tag{7c}$$

$$A'_4(r, a) = -V(r, a)\, 2\, \{r\, A_3(r, a) + A_1(r, a)\, A_2(r, a)\}. \tag{7d}$$

Generally the following two regularisation schemes have been used:

(i) θ regularisation

$$V_\theta(r, a) = V(r)\, \theta\, (r-a), \tag{8}$$

where $\theta\,(x)$ denotes the step function which is unity for non-negative values of x and zero otherwise. Obviously for θ regularisation, the limits of integration will be from $r = a$ to r in (7).

(ii) + regularisation

$$V_+\,(r,\,a) = V\,(r+a), \tag{9}$$

where $V\,(r)$ is considered as the limits as $a \to 0^+$ of the sequence. In this case the limits of integration will be from $r = 0$ to r in (7).

3. Application to the potential g/r^4

Now we apply the above theory to the simplest form of the singular potential i.e.

$$V(r) = g/r^4. \tag{10}$$

With θ regularisation of potential (10), (7a), (7b), (7c), ..., give,

$$A_1(r,\,a) = \frac{1}{r} - \frac{1}{a}\,; \tag{11a}$$

$$A_2(r,\,a) = 2\left\{\frac{1}{3r^3} - \frac{1}{2ar^2} + \frac{1}{6a^3}\right\}; \tag{11b}$$

$$A_3(r,\,a) = \left\{\frac{7}{15r^5} - \frac{1}{ar^4} + \frac{1}{3a^2r^3} + \frac{1}{3a^3r^2} - \frac{2}{15a^5}\right\}; \tag{11c}$$

$$A_4(r,\,a) = 2\left\{\frac{17}{105r^7} - \frac{4}{9ar^6} - \frac{4}{15a^2r^5} + \frac{1}{6a^3r^4} - \frac{1}{9a^4r^3}\right.$$

$$\left. - \frac{1}{15a^5r^2} + \frac{17}{630a^7}\right\}. \tag{11d}$$

From (2), (3), (4) and (11) we immediately get:

$$A(a) = -\frac{g}{a} + \frac{g^2}{3a^3} - \frac{2g^3}{15a^5} + \frac{17g^4}{315a^7} \tag{12}$$

The expression (12) is exactly the same as the one obtained by usual Born series method for the "θ regularisation" of potential g/r^4. The expression (12), is summed to obtain:

$$A(a) = -\,g^{1/2}\tanh\,(g^{1/2}/a), \tag{13}$$

which in the limit $a \to 0$ gives $-g^{1/2}$, the exact scattering length for the potential (10). Hence the peratization for the potential (10) is successful for this case.

It is interesting to note here that for the + regularization ($V(r)$ replaced by $V(r+a)$), the application of the technique discussed in §2 leads to the final result

$$A(a) = -\frac{g}{3a} + \frac{g^2}{45a^3} - \frac{2g^3}{945a^5} + \dots,$$

or $$A(a) = -g^{1/2} \coth(g^{1/2}/a) + a, \tag{14}$$

and $$\lim_{a\to 0} A(a) = -g^{1/2} = A \tag{15}$$

Equation (15) shows that even for + regularization of potential (10), the technique developed in §2 is successful.

4. Application of the theory to a logarithmic singular potential

In order to see whether the theory developed in §2 works for other potentials, we apply it to a physically more realistic potential which is logarithmically singular in nature and is given by:

$$V(r) = g\,\frac{\ln^2 r}{r^4} \tag{16}$$

The above potential has been discussed by Aly *et al* (1964) in connection with the peratization technique. It can be easily shown that the scattering amplitude for the potential exists at zero energy. The peratized scattering length for θ regularization of potential (16) can be written directly through the application of the theorem given by Spector (1966) as:

$$A(a) = -g^{1/2}(\ln a)\tanh\left(g^{1/2}\frac{\ln a}{a}\right). \tag{17}$$

Different amplitudes $A_n(a)$ are obtained with the help of (2) and (7) as:

$$A_1(a) = -\frac{g}{a}\left\{\ln^2 a + 2\ln a + 2\right\}; \tag{18a}$$

$$A_2(a) = \frac{g}{a^3}\left\{\frac{\ln^4 a}{3} + \frac{7\ln^3 a}{9} + \frac{17\ln^2 a}{18} + \frac{17\ln a}{27} + \frac{17}{18}\right\}; \tag{18b}$$

$$A_2(a) = -\frac{g}{a^5}\left\{\frac{2\ln^6 a}{15} + \dots\right\}. \tag{18c}$$

We find that the above expressions are identical to the expressions obtained by Aly *et al* (1964) by the usual Born series method which further shows the correctness of the technique developed by us.

The peratization technique mainly consists of first retaining in the expansion of regularised scattering length $A(\alpha)$, terms of the highest singularity in α in each order in g, and then of summing these terms after which α is allowed to approach zero. By first order peratization, we mean that only the most singular term in each order in g is retained.

Isolating the leading singularities in α, we get:

$$A(\alpha) = -g^{1/2}(\ln\alpha)\left\{g^{1/2}\frac{\ln\alpha}{\alpha} - \frac{g^{3/2}}{3}\frac{\ln^3\alpha}{\alpha^3} + \frac{2g^{5/2}}{15}\frac{\ln^5\alpha}{\alpha^5} - \ldots\right\}$$

or
$$A(\alpha) = -g^{1/2}(\ln\alpha)\tanh\left(g^{1/2}\frac{\ln\alpha}{\alpha}\right). \tag{19}$$

In the limit $\alpha \to 0$, this gives

$$A \sim -g^{1/2}(\ln\alpha), \tag{20}$$

which is not defined in the limit $\alpha \to 0$, showing thereby that the first order peratization does not succeed in this case.

5. Concluding remarks

The agreement of our results (12) and (18) with the results obtained by the usual Born series method shows the correctness of our method. One distinct advantage of our technique is that the calculations for the scattering length are comparatively easier than other methods. It may be noted that the potentials considered in this paper as examples contain only one term, hence the summation (1) of the power series in g is taken over from $n=1$ to ∞. However, for the potentials which consist of two or more terms with one term independent of the coupling constant g (e.g. $V(r) = 1r^4(g \cdot e^{2/r} + g')$) the summation (1) (as well as 2) should be taken over from $n=0$ to ∞. In such cases we find that (7a) is a first order non-linear differential equation. However, other equations are simple linear differential equations which can be solved by integrating factor method.

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References

Aly H H, Riazuddin and Zimmerman A H 1964 *Phys. Rev.* **B136** 1174

Aly H H, Riazuddin and Zimmerman A H 1965 *Nuovo Cimento* **35** 324

Calogero F 1967 *Variable phase approach to potential scattering* (New York: Academic Press) p. 69

Calogero F and Cassandro M 1965 *Nuovo Cimento* **37** 760
Feinberg G and Pais A 1963 *Phys. Rev.* **131** 2724
Frank W M and Land D J 1970a *J. Math. Phys.* **11** 2041
Frank W M and Land D J 1970b *J. Math. Phys.* **11** 2058
Jabbur R J 1965 *Phys. Rev.* **B138** 1525
Khuri N N and Pais A 1964 *Rev. Mod. Phys.* **36** 590
Pais A and Wu T T 1964 *J. Math. Phys.* **5** 799
Spector R M 1966 *J. Math. Phys.* **7** 2103