Statistical mechanics of bosons with attractive interaction

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Abstract. In order to obtain meaningful results for a system of bosons interacting via a potential containing a long-ranged attraction, the coupling of low momentum particles with high momentum particles has to be taken into account. The method of thermal Green's functions applied to this problem leads to involved graphical calculations even for temperatures close to the absolute zero. It is shown that the problem can be tackled much more simply and profitably by obtaining an effective hamiltonian for low momentum particles in a manner similar to that adopted in the recent renormalisation group approach to critical phenomena. The occurrence of anomalous pairings is avoided by performing suitable Bogolubov transformations. The procedure gives the quasiparticle spectrum besides yielding an explicit expression for the free energy in the low temperature limit. The results agree with those derived by Green's function methods using partial summations.

Keywords. Bosons; attractive interaction; renormalisation group; Green's functions.

1. Introduction

Attempts at understanding the behaviour of liquid ^{4}He at a microscopic level have generated over the years a large amount of work on the properties of a system of interacting bosons (see e.g. Gobble and Kobe 1974). For a theoretical model to be a good representative of liquid ^{4}He, the interparticle potential should incorporate a weak, long-range attraction in addition to a repulsion at small distances. In most of the work on interacting bosons, however, the attractive part of the potential and its consequences for a many-body bound state and a first-order phase transition have received little attention. Most of the treatments are essentially refined versions of the Bogolubov approximation (Bogolubov 1947) and break down if the zero-momentum scattering amplitude associated with the two-body potential is allowed to become negative. To get meaningful results in the latter case, the part of the hamiltonian neglected in these approximations has to be taken into account as was first pointed out by Huang (1959). He used perturbation theory of the ground state to take into account certain terms in the hamiltonian and showed that, provided suitable restrictions were imposed on the attractive part of the interaction, a well defined long wavelength excitation spectrum as well as a many-body bound state at zero pressure could be obtained. Sawada and Vasudevan (1961) have tried to overcome the difficulty associated with the negative scattering length by performing a unitary transformation which has the effect of modifying the single-particle state which is supposed to carry the condensate. Results similar to those of Huang have been obtained by Singh (1974) by the equations-of-motion method using generalised Hartree-Fock factorisation.

It is a matter of considerable importance to extend the investigation of the problem to non-zero temperatures. One can then discuss the properties of the system at various temperatures and may hope to learn something about the nature of phase transtitions possible in the model. A few years ago Singh and Kumar (1970) formulated the problem in terms of thermal Green's functions and worked out the low temperature properties of the system. Their procedure, however, turned out to be inconvenient on account of two reasons: (i) the free energy was calculated indirectly by performing suitable integrations on the single-particle Green's functions (ii) for a consistent calculation, cumbersome graphical calculations had to be performed since the appearance of anomalous pairings considerably increased the number of diagrams to be evaluated.

It is desirable to have a method of calculation which avoids the above complexities. The aim of this paper is to show that a direct and simplified calculation of the free energy is possible by manipulating the perturbation expansion of the thermodynamic potential in a manner similar to that adopted in the recent renormalisation group approach to critical phenomenon (Wilson and Kogut 1974; Singh 1975, 1978). The essential point about the Bose system with attractive interaction is that the low momentum part of the hamiltonian must be supplemented with contributions arising from the interaction of small momentum particles with large momentum particles. This is similar to the calculation of an effective hamiltonian in the renormalisation group approach by integrating out the small wavelength modes. Moreover, in our treatment, appearance of anomalous lines in the diagrams is avoided by first performing a Bogolubov transformation. This leads to considerable economy in the number of diagrams contributing in a given order and in their explicit evaluation.

A brief outline of the paper is as follows: In §2 we formulate the problem at non-zero temperatures and point out the difficulties associated with the attractive part of the interaction. In § 3 we use the expansion of the grand partition function to derive an effective hamiltonian for small momenta. The leading low temperature terms in the free energy and other quantities are obtained in § 4. Section 5 contains some remarks on the difficulties which must be overcome in order to discuss the behaviour of the system near the Bose-Einstein transition temperature.

2. Formulation of the problem with attractive interactions

The hamiltonian for an essembly of bosons, enclosed in a volume V and interacting through a two-body potential $u(r)$ is (in units such that $\hbar = m = 1$)

$$H = \sum_k \frac{k^2}{2} a^\dagger_k a_k$$

$$+ \frac{1}{4V} \sum_{k_1 \ldots k_4} [u(k_1 - k_4) + u(k_1 - k_3)] a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} \delta_{Kr}(k_1 + k_2 - k_3 - k_4) \quad (1)$$

where a_k, $a_k^\dagger$ denote the annihilation and creation operators for the single-particle state of momentum $\mathbf{k}$; $u(\mathbf{k})$ is the Fourier transform of $u(\mathbf{r})$ and δ_{Kr} denotes the

Kronecker symbol. The thermodynamic potential of the system is given by

$$\Omega = -\frac{1}{\beta} \ln \left[\text{Tr} \exp \left(-\beta H' \right) \right] \tag{2}$$

$$H' = H - \mu \sum_k a^\dagger_k a_k, \tag{3}$$

where μ denotes the chemical potential and β the inverse of the product of the Boltzmann constant and the absolute temperature.

For investigating states of the system characterized by the presence of a condensate in the single-particle zero-momentum state, one can, following Bogolubov (1947) replace $a_0/V^{1/2}$ and $a^\dagger_0/V^{1/2}$ by $n_0^{1/2}$ where n_0 denotes the density of particles in the zero momentum state. The hamiltonian H' can then be written as

$$H'(n_0) = H_2 + H_3 + H_4, \tag{4}$$

$$H_2 = \tfrac{1}{2} n_0^2 u(0) V - \mu n_0 V$$

$$+ \sum_k \left(\frac{k^2}{2} - \mu + n_0 u(0) + n_0 u(\mathbf{k}) \right) a^\dagger_k a_k$$

$$+ \frac{1}{2} \sum_k n_0 u(\mathbf{k}) [a^\dagger_k a^\dagger_{-k} + a_{-k} a_k], \tag{5}$$

$$H_3 = \frac{1}{2} \left(\frac{n_0}{V} \right)^{1/2} \sum_{k_1, k_2} [u(\mathbf{k}_1) + u(\mathbf{k}_2)]$$

$$\times (a^\dagger_{k_1} a^\dagger_{k_2} a_{k_1+k_2} + a^\dagger_{k_1+k_2} a_{k_2} a_{k_1}), \tag{6}$$

$$H_4 = \frac{1}{4V} \sum_{k_1 \ldots k_4} [u(\mathbf{k}_1 - \mathbf{k}_3) + u(\mathbf{k}_1 - \mathbf{k}_4)] \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$

$$\times \delta_{\text{Kr}} (\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4). \tag{7}$$

In (5) to (7) each k-summation excludes the point $\mathbf{k} = 0$. At a given temperature, n_0 is determined by requiring that the thermodynamic potential $\Omega(n_0)$ obtained by replacing H' in (2) by $H'(n_0)$ be minimum with respect to variations in n_0. The equation determining n_0 thus is

$$\partial \Omega / \partial n_0 = 0. \tag{8}$$

On carrying out the differentiation with respect to n_0, we find

$$\mu = n_0 u(0) + \frac{1}{V} \sum_k [u(\mathbf{k}) + u(0)] \langle a^\dagger_k a_k \rangle + \frac{1}{V} \sum_k u(\mathbf{k}) \langle a_k a_{-k} \rangle$$

$$+ \frac{n_0^{-1/2}}{V^{3/2}} \sum_{k_1, k_2} u(\mathbf{k}_1) \langle a^\dagger_{k_1} a^\dagger_{k_2} a_{k_1+k_2} \rangle, \tag{9}$$

where $\langle \ldots \rangle$ denotes thermodynamic average. As regards the two-body potential $u(\mathbf{r})$ we assume it to be of the type

$$u(\mathbf{r}) = v(\mathbf{r}) + w(\mathbf{r}) \tag{10}$$

where $v(\mathbf{r})$ is a short-ranged repulsive potential and $\omega(\mathbf{r})$ is a weak, long-ranged attractive potential. We assume the Fourier transforms of v and w to have the simple forms

$$v(\mathbf{k}) = \begin{bmatrix} v(0) & , & |\mathbf{k}| < k_r \\ 0 & , & |\mathbf{k}| > k_r, \end{bmatrix}$$

$$w(\mathbf{k}) = \begin{bmatrix} -w(0) & , & |\mathbf{k}| < k_0 \\ 0 & , & |\mathbf{k}| > k_0, \end{bmatrix} \tag{11}$$

where $v(0)$ and $w(0)$ are both positive, $[v(0) - w(0)] < 0$ and $k_0 \ll k_r$. The Fourier transform of the complete potential $u(\mathbf{r})$ is

$$u(\mathbf{k}) = [v(\mathbf{k}) + w(\mathbf{k})], \quad u(0) = [v(0) - w(0)] < 0. \tag{12}$$

The part H_2 of the hamiltonian (4) is that considered by Bogolubov in his original work. It is diagonalised by the canonical transformation.

$$a_k = u_k b_k - v_k b^\dagger_{-k}$$

$$a^\dagger_k = u_k b^\dagger_k - v_k b_{-k}, \tag{13}$$

where u_k, v_k are real numbers given by

$$u_k^2 = \tfrac{1}{2} (f_0(\mathbf{k})/E_0(\mathbf{k}) + 1), \tag{14}$$

$$v_k^2 = \tfrac{1}{2} (f_0(\mathbf{k})/E_0(\mathbf{k}) - 1), \tag{15}$$

$$u_k v_k = h_0(\mathbf{k})/2E_0(\mathbf{k}), \tag{16}$$

$$f_0(\mathbf{k}) = \frac{k^2}{2} - \mu + n_0 u(0) + n_0 u(\mathbf{k}), \tag{17}$$

$$E_0(\mathbf{k}) = [f_0^2(\mathbf{k}) - h_0^2(\mathbf{k})]^{1/2}, \tag{18}$$

$$h_0(\mathbf{k}) = n_0 u(\mathbf{k}). \tag{19}$$

This transformation brings H_2 into the form

$$H_2 = E_0 + \Sigma_k E_0(\mathbf{k}) b^\dagger_k b_k, \tag{20}$$

$$E_0 = \tfrac{1}{2} u(0) n^2_0 V - \mu n_0 V + \tfrac{1}{2} \Sigma_k [E_0(\mathbf{k}) - f_0(\mathbf{k})]. \tag{21}$$

At low temperatures, and to the first order in the interaction, μ according to (9) can be approximated by $n_0 u(0)$. The excitation energies $E_0(k)$ are consequently given by

$$E_0(k) = \left[\frac{k^2}{2}\left(\frac{k^2}{2} + 2n_0 u(k)\right)\right]^{1/2}.$$ (22)

For an interaction of the type (12) the excitation energy becomes imaginary for very small values of k. This means that the state of the system is unstable. Moreover, so long as $|v(0) - w(0)|$ is considered of the same order as $v(0)$, there is no way to overcome this instability occurring in the first order approximation. However, if $|v(0) - w(0)|$ is considered small in comparison with $v(0)$, it is possible that contributions to $f_0(k)$ and $h_0(k)$ arising from the H_3 and H_4 terms in the hamiltonian may remove the instability. We note that no difficulty arises with the spectrum (22) for $|k| > k_0$.

To take into account the terms H_3 and H_4 it is convenient to divide the momentum space into two parts; (i) the subspace $\mathcal{H}(q)$ comprising momenta q such that $0 < |q| < k_0$, and (ii) the subspace $\mathcal{H}(p)$ comprising momenta p such that $k_0 < |p| < k_r$. The spectrum (22) is well defined in the subspace $\mathcal{H}(p)$. The two-body potential (12) in the two subspaces is given by

$$u(q) = u(0), \quad u(p) = v(0).$$ (23)

On separating terms in the hamiltonian corresponding to small and large momenta, it can be rewritten in the form

$$H' = E + H_0(q) + H_0(p) + h,$$ (24)

$$E = \tfrac{1}{2} u(0)n^2{}_0 V - \mu n_0 V + \tfrac{1}{2} \sum_{|p| > k_0} [\epsilon(p) - f(p)],$$ (25

$$H_0(q) = \sum_{|p| < k_0} \frac{q^2}{2} a^\dagger{}_q a_q,$$ (26)

$$H_0(p) = \sum_{|p| > k_0} f(p) a^\dagger{}_p a_p + \tfrac{1}{2} \sum_{|p| < k_0} h_0(p) [a^\dagger{}_p a^\dagger{}_{-p} + a_{-p} a_p]$$

$$- \tfrac{1}{2} \sum_{|p| > k_0} [\epsilon(p) - f(p)],$$ (27)

$$h = H(q) + H(p) + H(p, q).$$ (28)

In the above equations

$$\epsilon(p) = [f^2(p) - h^2(p)]^{1/2},$$ (29)

$$f(\mathbf{p}) = \frac{p^2}{2} - \mu + n_0 u(0) + n_0 v(0), \tag{30}$$

$$h(\mathbf{p}) = n_0 v(0), \tag{31}$$

$$H(\mathbf{q}) = \sum_q (-\mu + n_0 u(0) + n_0 u(\mathbf{q})) a^\dagger_q a_q$$

$$+ \tfrac{1}{2} \sum_q n_0 \, u(0) \, [a^\dagger_q a^\dagger_{-q} + a_{-q} a_q]$$

$$+ \tfrac{1}{2} \left(\frac{n_0}{V}\right)^{1/2} \sum_{q_1, q_2} [u(\mathbf{q_1}) + u(\mathbf{q_2})] \, [a^\dagger_{q_1} a^\dagger_{q_2} a_{q_1 + q_2} + \text{cc.}]$$

$$+ \frac{1}{4V} \sum_{q_1, q_2, q_3} [u(\mathbf{q_1} - \mathbf{q_3}) + u(\mathbf{q_3} - \mathbf{q_2})] \, a^\dagger_{q_1} a^\dagger_{q_2} a_{q_3} a_{q_1 + q_2 - q_3}. \tag{32}$$

$H(\mathbf{p})$ is obtained from the last two terms in (32) by replacing all q's by p's and

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \left(\frac{n_0}{V}\right)^{1/2} \sum_{k_1, k_2} [u(\mathbf{k_1}) + u(\mathbf{k_2})] \, (a^\dagger_{k_1} a^\dagger_{k_2} a_{k_1 + k_2} + \text{cc.})$$

$$+ \frac{1}{4V} \sum_{k_1, k_2, k_3} [u(\mathbf{k_1} - \mathbf{k_3}) + u(\mathbf{k_3} - \mathbf{k_2})] \, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_1 + k_2 - k_3}. \tag{33}$$

In (33) all the k's in any particular term cannot belong either exclusively to the subspace $\mathcal{H}(q)$ or to the subspace $\mathcal{H}(p)$.

On differentiating Ω with respect to $f(\mathbf{p})$, $h(\mathbf{p})$, $f_0(\mathbf{q})$ and $h_0(\mathbf{q})$ where $f_0(\mathbf{q})$ and $h_0(\mathbf{q})$ are given by (17) and (19), one obtains the following exact relations

$$\frac{\partial \Omega}{\partial f(\mathbf{p})} = \langle a^\dagger_p a_p \rangle, \quad \frac{\partial \Omega}{\partial h(\mathbf{p})} = \langle a^\dagger_p a^\dagger_{-p} \rangle, \tag{34}$$

$$\frac{\partial \Omega}{\partial f_0(\mathbf{q})} = \langle a^\dagger_q a_q \rangle, \quad \frac{\partial \Omega}{\partial h_0(\mathbf{q})} = \langle a_q a_{-q} \rangle. \tag{35}$$

The average particle density n is given by

$$n = n_0 + n',$$

$$= n_0 + \sum_{|p| > k_0} \frac{\partial \Omega}{\partial f(\mathbf{p})} + \sum_{|q| < k_0} \frac{\partial \Omega}{\partial f_0(\mathbf{q})}. \tag{36}$$

Here n' denotes the density of particles in the non-zero momentum states.

3. Effective hamiltonian for small momenta

In order to obtain a meaningful hamiltonian for small momenta, we shall adopt a procedure similar to that used in the recent renormalisation group (RG) approach to critical phenomena. The basic ideas of the RG approach are quite general but their mathematical implementation has been attempted so far for *classical* fields only (see, e.g. Wilson and Kogut 1974). As we are interested in a problem involving a *quantum* field, only the basic ideas of the method are relevant for our purpose. We shall first recount these ideas briefly for the benefit of the reader who may not be familiar with them. Their application to the problem in hand will then be explained in detail.

The critical point of a many-body system is characterised by the fact that suitably defined correlation functions or susceptibilities exhibit singular behaviour at small wave-vectors or long wavelengths. The small wave-vector terms in the hamiltonian of a system are, therefore, considered to be of special importance in developing a theory of its critical behaviour. The basic idea of the RG approach is to somehow get rid of the large momentum terms in the hamiltonian so as to obtain an effective hamiltonian involving small momenta only. To achieve this purpose, a large upper cut-off momentum p_c is introduced and the momentum space is divided into two subspaces $\mathcal{H}(\mathbf{q})$ and $\mathcal{H}(\mathbf{p})$; $\mathcal{H}(\mathbf{q})$ is the 'small momentum' subspace comprising momenta $|\mathbf{q}| < p_c \zeta^{-1}$ while $\mathcal{H}(\mathbf{p})$ denotes the 'large momentum' subspace comprising momenta $|\mathbf{p}| > p_c \zeta^{-1}$ where ζ denotes an arbitrary scale factor greater than unity. The large momentum terms of the hamiltonian are eliminated by taking a partial trace of the density matrix exp $(-H_1)$ over the subspace $\mathcal{H}(\mathbf{p})$ where H_1 denotes the 'reduced' or temperature-dependent hamiltonian (βH). This process can, in general, be carried out only approximately. It will be explained below in the framework of perturbation theory for the case of the Bose system. The end result is that the partition function can be calculated from an effective hamiltonian H_2 which has the *same* form as H_1, but involves only small wave-vectors.

In studies relating to critical phenomena, the wave-vectors q are rescaled by the factor ζ so as to restore the original wave-vector range from zero to p_c. The parameters (e.g. chemical potential, interaction strength, etc.) of the new hamiltonian H_2 are obviously related to the parameters of the original hamiltonian H_1. These relations are called recursion relations.

The hamiltonian H_2 being of the same form as H_1, one can repeat the process of taking a partial trace and rescaling on H_2, thereby getting another new effective hamiltonian H_3. Repetitions yield a sequence of hamiltonians H_1, H_2, H_3.... If the sequence approaches a fixed point H^*, one concludes that the parameters in the original reduced hamiltonian correspond to a critical point of the system. By studying the recursion relations in the neighbourhood of the fixed point, one can draw conclusions about the critical behaviour of the system. For details we refer the reader to Ma's monograph (Ma 1976).

The idea of carrying out a partial trace on the density matrix is such a general one that one may use it to solve problems not necessarily concerned with the studies of critical behaviour. In what follows we use this idea to obtain an effective hamiltonian for a Bose system with attractive interactions which is the object of study of this paper.

We divide the momentum space into two subspaces $\mathcal{H}(\mathbf{q})$ and $\mathcal{H}(\mathbf{p})$ in the manner,

explained in §2. In (24) giving the hamiltonian H', we treat $(E+H_0(q)+H_0(p))$ as the unperturbed hamiltonian and h as a perturbation. The partition function may then be written in the form (Abrikosov *et al* 1963).

$$Z = \operatorname*{Tr}_{\mathcal{H}(q)} \operatorname*{Tr}_{\mathcal{H}(p)} \exp\left\{-\beta[H_0(p) + H_0(q) + E + h]\right\}$$

$$= Z_0 \operatorname*{Tr}_{\mathcal{H}(q)} \exp\left[-\beta H_0(q)\right] \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta \cdots \int_0^\beta d\tau_1 \dots d\tau_n \right.$$

$$\left. \cdot \left\langle T\, h(\tau_1) \dots h(\tau_n) \right\rangle\right]. \tag{37}$$

$$Z_0 = \operatorname*{Tr}_{\mathcal{H}(p)} \exp\left\{-\beta[E + H_0(p)]\right\} \tag{38}$$

Here $h(\tau)$ denotes the part h of the hamiltonian in the interaction representation, T denotes the time ordering operator and $\langle \dots \rangle$ denotes thermodynamic average calculated with the hamiltonian $H_0(p)$.

The contribution $F(1)$ of the first-order term in the sum in (37) is easily written down. One gets

$$F(1) = -\int_0^\beta d\tau\, [H(q, \tau) + g_1(q, \tau)], \tag{39}$$

where $$g_1(q, \tau) = \frac{1}{V}\Big[\sum_{p_1, p_2} [u(0) + u(p_1 - p_2)] \langle a^\dagger_{p_1}(\tau)\, a_{p_1}(\tau)\rangle \langle a^\dagger_{p_2}(\tau)\, a_{p_2}(\tau)\rangle$$

$$+ \tfrac{1}{2} [u(p_1 - p_2) + u(p_1 + p_2)] \langle a^\dagger_{p_1}(\tau)\, a^\dagger_{-p_1}(\tau)\rangle \langle a_{p_2}(\tau)\, a_{-p_2}(\tau)\rangle\Big]$$

$$+ \frac{1}{V}\Big[\sum_{q, p} [u(0) + u(p - q)]\, a^\dagger_q(\tau)\, a_q(\tau) \langle a^\dagger_p(\tau)\, a_p(\tau)\rangle$$

$$+ \tfrac{1}{4} [u(q - p) + u(q + p)] [a^\dagger_q(\tau)\, a^\dagger_{-q}(\tau) \langle a_p(\tau)\, a_{-p}(\tau)\rangle$$

$$+ a_{-q}(\tau)\, a_q(\tau) \langle a^\dagger_p(\tau)\, a^\dagger_{-p}(\tau)\rangle]\Big], \tag{40}$$

and $H(q, \tau)$ denotes the part $H(q)$ of the interaction (28) in the interaction representation. The second-order term in (37) can be written as

$$F(2) = \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2\, [H(q, \tau_1) + g_1(q, \tau_1)]\, [H(q, \tau_2) + g_1(q, \tau_2)]$$

$$+ \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \langle [H(p, \tau_1) + H(p, q, \tau_1)]\, [H(p, \tau_2)$$

$$+ H(p, q, \tau_2)]\rangle_c, \tag{41}$$

where $H(\mathbf{p}, \tau)$ and $H(\mathbf{p}, \mathbf{q}, \tau)$ denote $H(\mathbf{p})$ and $H(\mathbf{p}, \mathbf{q})$ in the interaction representation and the suffix C stands for connected graphs.

The basic structure of the graphs representing the various first-order terms in (40) is shown in figure 1 while that of the connected graphs representing the second term in (41) is shown in figure 2. The solid lines represent single-particle Green functions (normal or anomalous) with momenta $|\mathbf{p}| > k_0$. The dotted lines represent either a_q or $a^\dagger_q$. A wavy line represents a factor $\sqrt{n_0}$. The graphs (c) and (d) in figure 2 give contributions containing three a_q's and four a_q's respectively, i.e. similar to the last two terms in $H(\mathbf{q})$. We shall ignore these graphs in comparison with graphs such as (a) in figure 2 which involve two condensate operators and are quadratic in a_q's. This is justified because, for a weak interaction, the a_q's are small in comparison with the condensate operators.

The part of the interaction hamiltonian which gives rise to the graphs in figure 1 and the graphs a, b, e and f in figure 2 is

$$H_{\text{eff}} = H_3(\mathbf{p}) + H_4(\mathbf{p}) + H_3(\mathbf{p}, \mathbf{q}) + H_4(\mathbf{p}, \mathbf{q}), \tag{42}$$

$$H_3(\mathbf{p}) = \left(\frac{n_0}{V}\right)^{1/2} \sum_{\substack{p_1, p_2 \\ |\mathbf{p}_1 + \mathbf{p}_2| > k_0}} v(0)\, [a^\dagger_{p_1}\, a^\dagger_{p_2}\, a_{p_1 + p_2} + \text{cc.}), \tag{43}$$

$$H_4(\mathbf{p}) = \frac{1}{4V} \sum_{\substack{p_1, p_2, p_3 \\ |\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3| > k_0}} [u(\mathbf{p}_1 - \mathbf{p}_3) + u(\mathbf{p}_2 - \mathbf{p}_3)]\, a^\dagger_{p_1}\, a^\dagger_{p_2}\, a_{p_3}\, a_{p_1 + p_2 - p_3}, \tag{44}$$

(a)   (b)

Figure 1. Basic structure of first-order graphs representing various terms in equation (40).

(a)   (b)

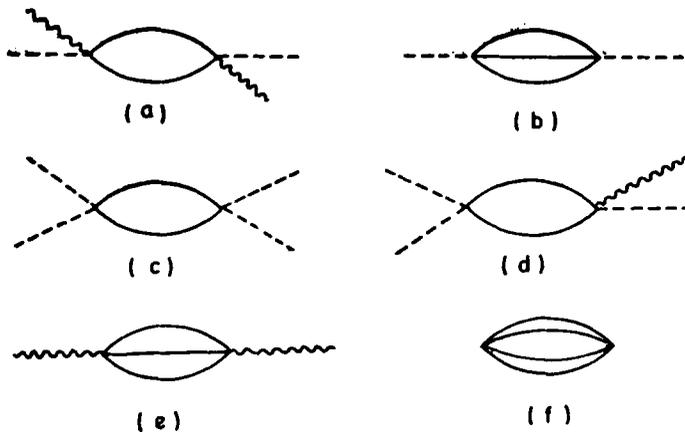(c)   (d)

(e)   (f)

Figure 2. Basic structure of second-order graphs representing the second term in equation (41).

$$H_3(\mathbf{p}, \mathbf{q}) = \left(\frac{n_0}{V}\right)^{1/2} \sum_{\substack{q,p \\ |\mathbf{p}+\mathbf{q}| > k_0}} \left\{ [(u(0) + v(0)) \, a^\dagger_p \, a^\dagger_q \, a_{p+q} \right.$$

$$\left. + v(0) \, a^\dagger_p \, a^\dagger_{-p-q} \, a_{-q}] + \text{cc.} \right\} \tag{45}$$

$$H_4(\mathbf{p}, \mathbf{q}) = \frac{1}{V} \sum_{p,q} \left\{ [(u(0) + u(\mathbf{p}-\mathbf{q})] \, a^\dagger_q \, a^\dagger_p \, a_p \, a_q \right.$$

$$+ \frac{1}{4} [u(\mathbf{p}-\mathbf{q}) + u(\mathbf{p}+\mathbf{q})] \, [a^\dagger_q \, a^\dagger_{-q} \, a_p \, a_{-p} + a^\dagger_p \, a^\dagger_{-p} \, a_q \, a_{-q}] \right\}$$

$$+ \frac{1}{2V} \sum_{\substack{p_1 \neq p_2 \\ |\mathbf{q}+\mathbf{p}_1-\mathbf{p}_2| > k_0}} [u(\mathbf{q}-\mathbf{p}_2) + u(\mathbf{p}_2-\mathbf{p}_1)] \, a^\dagger_q \, a^\dagger_{p_1} \, a_{p_2} \, a_{q+p_1-p_2}$$

$$+ \frac{1}{2V} \sum_{\substack{p_1 \neq p_2 \\ |\mathbf{p}_1+\mathbf{p}_2-\mathbf{q}| > k_0}} [u(\mathbf{p}_1-\mathbf{q}) + u(\mathbf{p}_2-\mathbf{q})] \, a^\dagger_{p_1} \, a^\dagger_{p_2} \, a_q \, a_{p_1+p_2-q}. \tag{46}$$

The operators a_p, $a^\dagger_p$ in (42) can be written in terms of the quasi-particle operators b_p, $b^\dagger_p$ defined through (13) to (16). The first-order and the second order graphs can then be redrawn as shown in figures 3 and 4. The internal lines now denote the pairings of quasiparticle operators and the circles represent the modified vertices. We have taken note of the fact that only normal quasiparticle pairings exist, the unperturbed anomalous pairings being zero.

The contribution of each graph can be calculated with the help of the following general rules.

(i) Arrange vertices associated with times $\tau_1 > \tau_2 > \tau_3 \ldots$ along a line so that the time decreases as we proceed from left to right.

(ii) With each incoming arrow of momentum $|\mathbf{p}| > k_0$ at a vertex labelled by time τ associate a factor $\exp[\epsilon(\mathbf{p})\tau]$ and with each outgoing arrow of momentum $\mathbf{p}$ associate a factor $\exp[-\epsilon(\mathbf{p})\tau]$ where $\epsilon(\mathbf{p})$ denotes the quasiparticle energy (29).

(iii) With each internal line of momentum $|\mathbf{p}| > k_0$ associate a factor $[1+n(p)]$ or $n(p)$ according as the line is directed from left to right or from right to left; $n(p)$ denotes the quasi-particle distribution function

$$n(p) = [\exp(\beta \, \epsilon(\mathbf{p})) - 1]^{-1}. \tag{47}$$

Figure 3. First-order graphs having quasiparticle pairings. The circles represent the modified vertices.
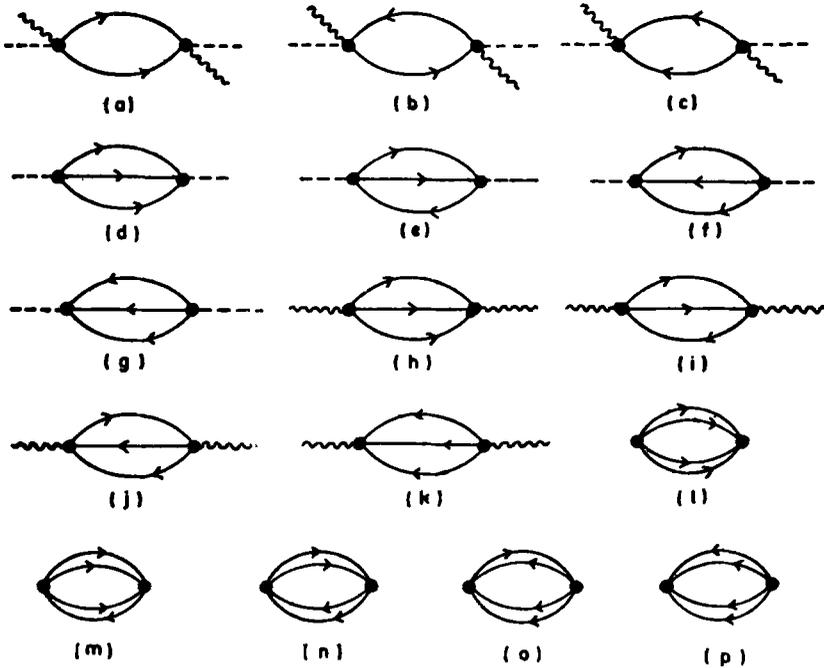
Figure 4. Second order-graphs having quasiparticle pairings. The circles represent the modified vertices.

Figure 5. Typical second-order graphs.

(iv) With each wavy line associate a factor $(n_0/V)^{1/2}$.

(v) With each external (dotted) line of momentum q at a vertex associate a factor $a^\dagger_q \exp [\epsilon_0 (q) \tau]$ or $a_q \exp [- \epsilon_0 (q) \tau]$ where $\epsilon_0 (q) = q^2/2$.

(vi) With each diagram associate a numerical factor corresponding to the number of ways in which the external lines can be chosen.

(vii) Integrate over the times of all vertices and sum over independent momenta of external and internal lines.

Using the above rules it is easy to see that at low temperatures a very small number of all the graphs needs to be retained. Consider, for example, the two second-order graphs (a) and (b) shown in figure 5. They arise from the H_3 (p, q) part of the hamiltonian (42) when written in terms of the quasi-particle operators, viz.

$$H_3 (\mathbf{p}, \mathbf{q}) = (n_0/V)^{1/2} \sum_{\substack{p, q \\ |\mathbf{p}+\mathbf{q}| > k_0}} \{ [V_1 a_q b_{-p} b_{p+q} + V_2 a^\dagger_q b^\dagger_p b_{p+q}$$

$$+ V_3 a^\dagger_q b^\dagger_p b^\dagger_{-p-q}] + \text{c.c.} \}, \qquad (48)$$

$$V_1 = - [u(0) + v(0)] u_{p+q} v_p + v(0) u_{-p-q} u_p, \tag{49}$$

$$V_2 = [u(0) + v(0)] [u_p u_{p+q} + v_p v_{p+q}] - v(0) [u_{-q-p} v_p + v_{-p-q} u_p], \tag{50}$$

$$V_3 = - [u(0) + v(0)] u_p v_{p+q} + v(0) v_p v_{-p-q}. \tag{51}$$

The contributions of these graphs are

$$\text{(a)} = \frac{2n_0}{V} \sum_{p, q} \frac{V_1^2}{\epsilon(p+q) + \epsilon(p) - \epsilon_0(q)} (1+n(p))(1+n(p+q))$$

$$\times \left(1 + \frac{\exp\{-\beta[\epsilon(p)+\epsilon(p+q)-\epsilon_0(q)] -1\}}{\beta[\epsilon(p)+\epsilon(p+q)-\epsilon_0(q)]} \right) \left[\int_0^\beta d\tau a^\dagger_q(\tau) a_q(\tau) \right], \tag{52}$$

$$\text{(b)} = - \frac{2n_0}{V} \sum_{p, q} \frac{V_2^2}{\epsilon_0(q) + \epsilon(p) - \epsilon(p+q)} n(p)(1+n(p+q))$$

$$\times \left(1 - \frac{\exp\{\beta[\epsilon(p)+\epsilon_0(q)-\epsilon(p+q)] -1\}}{\beta[\epsilon(p)+\epsilon_0(q)-\epsilon(p+q)]} \right) \left[\int_0^\beta d\tau a^\dagger_q(\tau) a_q(\tau) \right]. \tag{53}$$

Assuming the temperature to be so low that $\beta \epsilon(p) \gg 1$, the distribution function $n(p) \to 0$. For such temperatures we can ignore the contributions of graphs like (b) in figure 5 which contain one or more $n(p)$ factors. Since an $n(p)$ factor arises from an internal line directed from right to left, we ignore all graphs containing such lines. The only graphs to be considered at low temperatures are consequently those having all internal lines directed from left to right. The factor $(1+n(p))$ associated with each of these lines will be approximated by unity.

It is not difficult to see that if $(nv^3(0))^{1/2}$ is regarded as a dimensionless weak-interaction parameter, the contribution of a typical three-line graph such as the one shown in figure 6 is smaller by one order of magnitude as compared with the contribution of a 2-line graph. In what follows we shall assume $(nv^3(0))^{1/2}$ to be small. All 3-line graphs will consequently be ignored.

Graphs without (dotted) external lines contribute to the constant term E in (24). Considering, for example, the graph (b) in figure 3 we find that its contribution is smaller than the last term in (25) by the factor $(nv^3(0))^{1/2}$. The same holds for the contributions of graphs (h) and (l) in figure 4. We ignore all these graphs. Thus finally we are left with the graphs shown in figure 7.

The contributions of the first-order graphs (a), (b) and (c) in figure 7 are

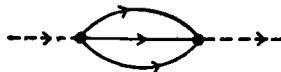$$\text{(a)} = \frac{u(0)+v(0)}{V} \sum_{p, q} v_p^2 \int_0^\beta d\tau a^\dagger_q(\tau) a_q(\tau), \tag{54}$$

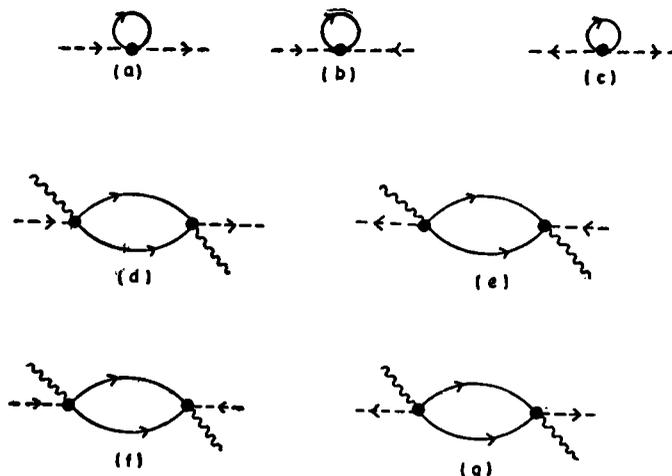Figure 6. A typical three-line graph.

Figure 7. First-order and second-order graphs to be evaluated finally.

$$(b) = -\frac{v(0)}{2V} \sum_{p,q} u_p v_p \int_0^\beta d\tau a_{-q}^\dagger(\tau) a_{-q}^\dagger(\tau), \tag{55}$$

$$(c) = -\frac{v(0)}{2V} \sum_{p,q} u_p v_p \int_0^\beta d\tau a_{-q}(\tau) a_q(\tau). \tag{56}$$

In writing the contributions (54) to (56) we have assumed $|\,q\,| \ll k_0$. This is a valid approximation since at very low temperatures only small momenta are important. The coefficients of the operator terms in (54) to (56) are of the order of magnitude $nv(0)(nv^3(0))^{1/2}$. As $(nv^3(0))^{1/2}$ is assumed to be a small quantity, we can consider $v(0)(nv^3(0))^{1/2}$ to be of the same order of magnitude as $u(0)$. The terms (54) to (56) then become comparable to the first two terms of $H(q)$ in (32). As we shall see, contributions of the same order arise from the second-order graphs (d), (e), (f) and (g) in figure 7 also. One can, therefore, hope that if all terms of the same order are collected together, the difficulty with the low momentum spectrum mentioned in § 2 may disappear.

The contributions of the second-order graphs in figure 7 are

$$(d) = \sum_{q,p} \frac{V_1^2}{\epsilon(p)} \int_0^\beta d\tau\, a_q^\dagger(\tau)\, a_q(\tau), \tag{57}$$

$$(e) = \sum_{p,q} \frac{V_3^2}{\epsilon(p)} \int_0^\beta d\tau\, a_q^\dagger(\tau)\, a_q(\tau) + \sum_{p,q} \frac{V_3^2}{\epsilon(p)}, \tag{58}$$

$$(f) = -\sum_{p,q} \frac{V_1 V_3}{\epsilon(p)} \int_0^\beta d\tau\, a_q^\dagger(\tau)\, a_{-q}^\dagger(\tau), \tag{59}$$

$$(g) = -\sum_{p,q} \frac{V_1 V_3}{\epsilon(p)} \int_0^\beta d\tau\, a_{-q}(\tau)\, a_q(\tau), \tag{60}$$

where we have again assumed $|\mathbf{q}| \ll k_0$. The second term in (58) will be ignored since it makes a contribution to E/V of order V^{-1}.

On substituting for $u_p{}^2$ and $u_p\, v_p$ from (14) and (16) in (54) to (56), we obtain for $g_1 (\mathbf{q}, \tau)$ defined by (40) the result

$$g_1 (\mathbf{q}, \tau) = \frac{v(0)}{V} \sum_{p,q} \left\{ \frac{1}{2} \left(\frac{f(\mathbf{p})}{\epsilon(\mathbf{p})} - 1 \right) a^\dagger_q (\tau)\, a_q (\tau) \right.$$

$$\left. - \frac{1}{4} \frac{h(\mathbf{p})}{\epsilon(\mathbf{p})} [a^\dagger_q (\tau)\, a_{-q} (\tau) + a_{-q} (\tau)\, a_q (\tau)] \right\}. \tag{61}$$

On using the expressions (49) and (51) for V_1 and V_3 the total contribution of second-order graphs can be written as

$$\int_0^\beta d\tau\, g_2 (\mathbf{q}, \tau)\, g_2 (\mathbf{q}, \tau) \tag{62}$$

$$= n_0 v^2 (0) \sum_{p,q} \frac{f(\mathbf{p})\, [f(\mathbf{p}) - h(\mathbf{p})]}{V \epsilon^3 (\mathbf{p})}\, a^\dagger_q (\tau)\, a_q (\tau)$$

$$- n_0 v^2 (0) \sum_{p,q} \frac{h(\mathbf{p})\, [f(\mathbf{p}) - h(\mathbf{p})]}{2\, V \epsilon^3 (\mathbf{p})}\, [a^\dagger_q (\tau)\, a^\dagger_{-q} (\tau) + a_{-q} (\tau)\, a_q (\tau)]. \tag{63}$$

Substituting in (37) the contributions of the first and second order terms we find

$$Z = Z_0 \, \mathrm{Tr} \exp [-\beta H_0 (\mathbf{q})] \left\{ 1 - \int_0^\beta d\tau\, [H(\mathbf{q}, \tau) + g_1 (\mathbf{q}, \tau) - g_2 (\mathbf{q}, \tau)] \right.$$

$$+ \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2\, [H(\mathbf{q}, \tau_1) + g_1 (\mathbf{q}, \tau_1)] \, [H(\mathbf{q}, \tau_2) + g_1 (\mathbf{q}, \tau_2)]$$

$$\left. + \text{higher order terms} \right\}. \tag{64}$$

The connected graphs arising from the third and higher order terms in (37) give contributions to g_1 and g_2 in (64) which are of higher order in the small parameter $(n v^3 (0))^{1/2}$. They will be ignored. It is, however, important to discuss the role of disconnected graphs arising in various orders. If we had carried out a complete trace of the density matrix rather than a partial one, the well-known connected-graph theorem (see e.g. Abrikosov *et al* 1963) tells us that the disconnected graphs simply lead to exponentiation of the contribution of the connected graphs. However, when only a partial trace is carried out, the contributions of the connected graphs contain time-ordered products of small momentum operators $a_q (\tau)$, $a^\dagger_q (\tau)$, and the usual method of proving the connected-graph theorem does not work. In this case, one has to verify that the disconnected graphs do lead to exponentiation of the contribution of connected graphs. An illustration is provided by the third term in the parenthesis in (64). It represents the contribution of disconnected graphs in second order but has the form of the time-ordered product of the contributions of connected graphs in first order (*cf* equation (39)). To establish full exponentiation of the connected-graph contributions $g_1 (\mathbf{q})$ and $g_2 (\mathbf{q})$ we must first show that disconnected graphs in

third and fourth orders have the effect of adding a term g_2 to each of the factors $(H+g_1)$ in the third term in (64). This is somewhat tedious but easy to check. It is more difficult to show that disconnected graphs arising in higher orders produce terms in (64) which may be identified with the third and higher order terms in the expansion of exp $[-\beta (H_0+h_1)]$ where h_1 denotes $(H+g_1+g_2)$. We do not know of any general method of proving this result. For the much simpler case of a classical spin field, Wilson and Kogut (1974) have remarked that it is rather cumbersome to establish the corresponding result. We shall assume exponentiation of the contribution of connected graphs to hold to all orders. An indirect justification for this assumption will emerge in § 4 when we show that the results obtained on its basis are in complete agreement with those obtained by Green's function methods.

Assuming exponentiation, (64) can be written as

$$Z = Z_0 \exp \left[-\beta H_e(\mathbf{q})\right] \tag{65}$$

where the effective hamiltonian $H_e(\mathbf{q})$ is given by

$$H_e(\mathbf{q}) = H_0(\mathbf{q}) + H(\mathbf{q}) + g_1(\mathbf{q}) + g_2(\mathbf{q})$$

$$= \sum_q f_e(\mathbf{q}) a^\dagger_q a_q + \tfrac{1}{2} \sum_q h_e(\mathbf{q}) (a_q a_{-q} + a^\dagger_{-q} a^\dagger_q); \tag{66}$$

$$f_e(\mathbf{q}) = \frac{q^2}{2} - \mu + n_0 u(0) + n_0 u(\mathbf{q}) + C_1(\mathbf{q}); \tag{67}$$

$$h_e(\mathbf{q}) = n_0 u(\mathbf{q}) + C_2(\mathbf{q}); \tag{68}$$

$$C_1(\mathbf{q}) = \frac{v(0)}{V} \sum_p \left\{ \frac{f(\mathbf{p})}{2 \epsilon(\mathbf{p})} - \frac{1}{2} - \frac{n_0 v(0) f(\mathbf{p}) [f(\mathbf{p}) - h(\mathbf{p})]}{\epsilon^3(\mathbf{p})} \right\}; \tag{69}$$

$$C_2(\mathbf{q}) = \frac{v(0)}{V} \sum_p \left\{ - \frac{h(\mathbf{p})}{2 \epsilon(\mathbf{p})} + \frac{n_0 v(0) h(\mathbf{p}) [f(\mathbf{p}) - h(\mathbf{p})]}{\epsilon^3(\mathbf{p})} \right\}. \tag{70}$$

The effective hamiltonian (66) is of the same form as the Bogolubov hamiltonian (5). It is diagonalised by the transformation (13). The result is

$$H_e(\mathbf{q}) = \tfrac{1}{2} \sum_q [E(\mathbf{q}) - f_e(\mathbf{q})] + \sum_q E(\mathbf{q}) b^\dagger_q b_q \tag{71}$$

$$E(\mathbf{q}) = [f_e^2(\mathbf{q}) - h_e^2(\mathbf{q})]^{1/2}. \tag{72}$$

The thermodynamic potential Ω is now easily calculated. We find

$$\Omega = \frac{u(0)}{2V} N_0^2 - \mu N_0 + \frac{1}{2} \sum_p [\epsilon(\mathbf{p}) - f(\mathbf{p})]$$

$$+ \frac{1}{\beta} \sum_q \ln (1 - \exp [-\beta E(\mathbf{q})]) + \frac{1}{2} \sum_q [E(\mathbf{q}) - f_e(\mathbf{q})]$$

$$+ \frac{1}{\beta} \sum_p \ln (1 - \exp [-\beta \epsilon(\mathbf{p})]) \tag{73}$$

P.—5

4. Excitations and thermodynamics at low temperatures

The expression (72) for the small momentum excitations involves the unknown quantities n_0 and μ which are to be determined according to (36) and (8). For the thermodynamic potential in these equations we can now use the approximate expression (73). At low temperatures, the last term in (73) can be ignored. The last but one term can also be neglected in comparison with the other temperature-independent terms because the q vectors occupy a small volume in momentum space. The approximate expression to be used for Ω thus is

$$\Omega = \tfrac{1}{2}u(0)N_0^2/V - \mu N_0 + \tfrac{1}{2}\sum_p [\epsilon(\mathbf{p}) - f(\mathbf{p})]$$

$$+ \frac{1}{\beta}\sum_q \ln(1 - \exp(-\beta E(\mathbf{q})). \tag{74}$$

Using (74) in (8) and (36), and remembering that $E(\mathbf{q})$ is a functional of $f(\mathbf{p})$, $h(\mathbf{p})$, $f_0(\mathbf{q})$ and $h_0(\mathbf{q})$, we get

$$n' = \frac{1}{2V}\sum_p \left(1 - \frac{f(\mathbf{p})}{\epsilon(\mathbf{p})}\right) + A_1 + A_1 \frac{v(0)}{2V}\sum_p \left[\left(\frac{1}{\epsilon(\mathbf{p})} - \frac{f^2(\mathbf{p})}{\epsilon^2(\mathbf{p})}\right)\right.$$

$$\left. + n_0 v(0)\left(\frac{f(\mathbf{p})}{\epsilon^3(\mathbf{p})} - \frac{2(f(\mathbf{p}) - h(\mathbf{p}))}{\epsilon^3(\mathbf{p})} + \frac{3f(\mathbf{p})\,[f(\mathbf{p}) - h(\mathbf{p})]^2}{\epsilon^5(\mathbf{p})}\right)\right]$$

$$- A_4 \frac{v(0)}{2V}\sum_p \left[\frac{f(\mathbf{p})\,h(\mathbf{p})}{\epsilon^3(\mathbf{p})} - n_0 v(0)\left(\frac{f(\mathbf{p})}{\epsilon^3(\mathbf{p})} - \frac{2[f(\mathbf{p}) - h(\mathbf{p})]}{\epsilon^3(\mathbf{p})}\right.\right.$$

$$\left.\left. - \frac{3f(\mathbf{p})\,[f(\mathbf{p}) - h(\mathbf{p})]^2}{\epsilon^5(\mathbf{p})}\right)\right] \tag{75}$$

$$\mu = nu(0) + \frac{v(0)}{2V}\sum_p \left[\left(\left(\frac{p^2}{2} - \mu + n_0 u(0)\right)\Big/ \epsilon(\mathbf{p})\right) - 1\right]$$

$$- A_1 \frac{v^2(0)}{2V}\sum_p (f(\mathbf{p}) - h(\mathbf{p}))^2/\epsilon^3(\mathbf{p})$$

$$+ (A_1 - A_4)\left[u(0) + \frac{v^2(0)}{V}\sum_p \frac{[f(\mathbf{p}) - h(\mathbf{p})]^3}{2\epsilon^5(\mathbf{p})}[2h(\mathbf{p}) - f(\mathbf{p})]\right], \tag{76}$$

$$A_1 = \int \frac{f_e(\mathbf{q})}{E(\mathbf{q})}\{\exp[\beta E(\mathbf{q})] - 1\}^{-1}\frac{d^3q}{(2\pi)^3}, \tag{77}$$

$$A_u = \int \frac{h_e(\mathbf{q})}{E(\mathbf{q})}\{\exp[\beta E(\mathbf{q})] - 1\}^{-1}\frac{d^3q}{(2\pi)^3}. \tag{78}$$

The coefficients multiplying A_1 and A_4 in (75) are easily seen to be proportional to the small parameter $(nv^3(0))^{1/2}$ which we shall denote by s. In the first term in (75) we write $f(\mathbf{p})$, $h(\mathbf{p})$ and $\epsilon(\mathbf{p})$ as

$$f(\mathbf{p}) = f_0(\mathbf{p}) + \delta f(\mathbf{p}), \tag{79}$$

$$h(\mathbf{p}) = h_0(\mathbf{p}) + \delta h(\mathbf{p}), \tag{80}$$

$$\epsilon(\mathbf{p}) = \epsilon_0(\mathbf{p}) + \delta \epsilon(\mathbf{p}), \tag{81}$$

$$f_0(\mathbf{p}) = p^2/2 + nv(0), \tag{82}$$

$$h_0(\mathbf{p}) = nv(0), \tag{83}$$

$$\epsilon_0(\mathbf{p}) = [f_0^2(\mathbf{p}) - h_0^2(\mathbf{p})]^{1/2} \tag{84}$$

$$\delta f(\mathbf{p}) = nu(0) - \mu - n'[u(0) + v(0)], \tag{85}$$

$$\delta h(\mathbf{p}) = - n'v(0), \tag{86}$$

$$\delta \epsilon(\mathbf{p}) = [f_0(\mathbf{p})\delta f(\mathbf{p}) - h_0(\mathbf{p})\,\delta h(\mathbf{p})]/\epsilon_0(\mathbf{p}) \tag{87}$$

and treat $\delta f(\mathbf{p})$, $\delta h(\mathbf{p})$ and $\delta \epsilon(\mathbf{p})$ as small in comparison with $f_0(\mathbf{p})$, $h_0(\mathbf{p})$ and $\epsilon_0(\mathbf{p})$. We then find

$$n' = n'_\infty + A_1 + O(sA_1) + (sA_4), \tag{88}$$

$$n'_\infty = \frac{2n}{3\pi^2} [nv^3(0)]^{1/2} [\nu^3 - (\nu^2 + 1)^{3/2}(\nu^2 + 1)^{1/2}], \tag{89}$$

where ν given by

$$\nu^2 = k_0^2/4nv(0), \tag{90}$$

can be assumed to be a quantity of order unity.

In the expression (76) for μ, the second term on using (81) can be written as

$$\frac{v(0)}{2V} \sum_p \left(\frac{p^2}{2\epsilon_0(\mathbf{p})} - 1 \right) + \frac{n'v^2(0)}{2V} \sum_p \frac{p^4}{4\epsilon_0^3(\mathbf{p})} + O(s\mu). \tag{91}$$

Substituting for n' the first two terms of (88) and replacing $f(\mathbf{p})$ and $h(\mathbf{p})$ in the rest of the terms of (76) by $f_0(\mathbf{p})$ and $h_0(\mathbf{p})$ respectively, we obtain

$$\mu = \mu_\infty + (A_1 - A_4) \left[u(0) - \frac{v^2(0)}{2\pi^2}(k_r - k_0) + \frac{v(0)\,(nv^3(0))^{1/2}}{\pi^2} f(\nu) \right], \tag{92}$$

$$\mu_\infty = n \left[u(0) - \frac{v^2(0)\,(k_r - k_0)}{2\pi^2} + \frac{v(0)\,(nv^3(0))^{1/2}}{\pi^2} F(\nu) \right], \tag{93}$$

$$f(\nu) = (\nu^2 + 1)^{1/2} + \frac{5}{2}(\nu^2 + 1)^{-1/2} - \frac{1}{2}(\nu^2 + 1)^{-3/2} - \nu, \qquad (94)$$

$$F(\nu) = \frac{2}{3}\nu^3 + 2(\nu^2 + 1)^{1/2} - \frac{2}{3}(\nu^2 + 1)^{3/2} - \nu. \qquad (95)$$

In order to calculate the leading low temperature terms of various thermodynamic quantities we can ignore the temperature dependence of the $E(\mathbf{q})$'s in (74). The $E(\mathbf{q})$'s are then given by (72) with

$$f_e(\mathbf{q}) = \frac{q^2}{2} - \mu_\infty + nu(0) + nu(\mathbf{q}) + C_1(\mathbf{q}), \qquad (96)$$

$$h_e(\mathbf{q}) = nu(0) + C_2(\mathbf{q}). \qquad (97)$$

On replacing $f(\mathbf{p})$, $h(\mathbf{p})$ and $\epsilon(\mathbf{p})$ by $f_0(\mathbf{p})$, $h_0(\mathbf{p})$ and $\epsilon_0(\mathbf{p})$ respectively, the expressions (69) and (70) for $C_1(\mathbf{q})$ and $C_2(\mathbf{q})$ give

$$C_1(\mathbf{q}) = -\frac{nv^2(0)(k_r - k_0)}{\pi^2} + \frac{nv(0)(nv^3(0))^{1/2}}{\pi^2} f_1(\nu), \qquad (98)$$

$$C_2(\mathbf{q}) = -\frac{nv^2(0)(k_r - k_0)}{2\pi^2} + \frac{nv(0)(nv^3(0))^{1/2}}{\pi^2} f_2(\nu), \qquad (99)$$

$$f_1(\nu) = 3(\nu^2+1)^{1/2} + (\nu^2+1)^{-1/2} + (2/3)\nu^3 - 2/3(\nu^2+1)^{3/2} - 2\nu, \qquad (100)$$

$$f_2(\nu) = (\nu^2+1)^{1/2} + (\nu^2+1)^{-1/2} - \nu. \qquad (101)$$

Comparison of (93) with (98) and (99) shows that

$$\mu_\infty = nu(0) + C_1(0) - C_2(0). \qquad (102)$$

The Hugenholtz-Pines theorem (Hugenholtz and Pines 1959) is thus satisfied in the approximation considered. Substituting for μ_∞ and $C_2(\mathbf{q})$ from (102) and (99) in (96) and (97), we get

$$f_e(\mathbf{q}) = \frac{q^2}{2} + n\left(u(0) - \frac{v^2(0)(k_r - k_0)}{2\pi^2}\right) + \frac{nv(0)(nv^3(0))^{1/2}}{\pi^2} f_2(\nu), \qquad (103)$$

$$h_e(\mathbf{q}) = n\left(u(0) - \frac{v^2(0)(k_r - k_0)}{2\pi^2}\right) + \frac{nv(0)(nv^3(0))^{1/2}}{\pi^2} f_2(\nu). \qquad (104)$$

It is convenient to write the above results in terms of the scattering amplitudes associated with the total potential $u(\mathbf{r})$ and the repulsive part $v(\mathbf{r})$. The zero momentum scattering amplitude a associated with $v(\mathbf{r})$ is given by the Born series

$$4\pi a = v(0) - \frac{k_r v^2(0)}{2\pi^2} + \dots \qquad (105)$$

while the scattering amplitude a_T associated with the total potential is

$$4\pi a_T = u(0) - \frac{v^2(0)(k_r - k_0)}{2\pi^2} - \frac{k_0{}^2 u^2(0)}{2\pi^2}. \tag{106}$$

Remembering that $|u(0)| \ll v(0)$, we can drop the last term in (106). As a_T is negative, we shall denote it by $(-b)$. In terms of a and b, (103) and (104) become

$$f_e(\mathbf{q}) = \frac{q^2}{2} - 4\pi bn + 32na \,(\pi na^3)^{1/2} f_2(\nu), \tag{107}$$

$$h_e(\mathbf{q}) = -4\pi bn + 32na \,(\pi na^3)^{1/2} f_2(\nu), \tag{108}$$

$$\nu = k_0 a / (16\pi na^3)^{1/2}. \tag{109}$$

The excitations $E(\mathbf{q})$ are given by

$$E(\mathbf{q}) = \left[\frac{q^2}{2} \left(\frac{q^2}{2} + 2h_e(\mathbf{q}) \right) \right]^{1/2}. \tag{110}$$

They are well defined in the limit $q \to 0$ provided

$$\left[-\frac{b}{a} + \frac{2k_0 a}{\pi} \frac{f_2(\nu)}{\nu} \right] > 0. \tag{111}$$

The function $f_2(\nu)$ (eq. (101)) is a monotonically decreasing function of ν having the limiting forms

$$f_2(\nu) \xrightarrow[\nu \to 0]{} 2, \quad f_2(\nu) \xrightarrow[\nu \to \infty]{} \left(\frac{3}{2\nu} \right). \tag{112}$$

The inequality (111) implies a restriction on the density for given values of the parameters (b/a) and $k_0 a$. For example, if $(b/a) = k_0 a = 0\cdot 1$, the inequality (111) is satisfied for densities such that $\nu < 0\cdot 8$, i.e. $(16\pi na^3)^{1/2} > 0\cdot 12$. We conclude that a system of bosons with attractive interaction can be in stable equilibrium only if the density is greater than a certain minimum value determined by (111).

We next calculate the free energy F given by

$$F = \Omega + \mu N. \tag{113}$$

Substituting for Ω from (74), we get

$$F/V = \tfrac{1}{2}(n^2 + n'^2)u(0) + [\mu - nu(0)]n' + \frac{1}{2V} \sum_p [\epsilon(\mathbf{p}) - f(\mathbf{p})]$$

$$+ \frac{1}{\beta V} \sum_q \ln(1 - \exp[-\beta E(\mathbf{q})]). \tag{114}$$

To extract the leading low temperature terms it is convenient to write

$$\mu = \mu_\infty + \mu_T, \tag{115}$$

$$n' = n'_\infty + n'_T, \tag{116}$$

μ_T and n'_T denoting the temperature dependent parts of μ and n' in (92) and (88). The first and second terms in (114) are then seen to make temperature-dependent contributions of order $(n'_\infty u(0)n'_T)$ and $(\mu_\infty n'_T)$ respectively. Since μ_∞ is an order larger than $(n'_\infty u(0))$, the leading low temperature contribution comes from the second term on replacing μ by μ_∞. It is easily seen to be of the same order as the last term in (114). A contribution of the same order arises from the third term in (114) also on writing

$$\frac{1}{2V}\sum_p [\epsilon(\mathbf{p})-f(\mathbf{p})] = \frac{1}{2V}\sum_p [\epsilon_0(\mathbf{p})-f_0(\mathbf{p})] - \frac{n'v(0)}{2V}\sum_p \left(\frac{p^2}{2\epsilon_0(\mathbf{p})}-1\right)$$
$$+ n'_\infty [-\mu + nu(0) - n'u(0)]. \tag{117}$$

Collecting terms of the same order, we obtain

$$\frac{F}{V} = \frac{E_0}{V} + \frac{1}{\beta}\int \frac{d^3q}{(2\pi)^2} \ln\{1 - \exp[-\beta E(\mathbf{q})]\}, \tag{118}$$

$$\frac{E_0}{V} = \frac{1}{2}n^2 u(0) + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}[\epsilon_0(\mathbf{p}) - f_0(\mathbf{p})], \tag{119}$$

E_0 evidently represents the ground state energy. On carrying out the integration, we find for E_0 the explicit result

$$\frac{E_0}{V} = 2\pi a n^2 \left[-\frac{b}{a} + \frac{32}{15\pi}k_0 a \frac{F_1(\nu)}{\nu}\right], \tag{120}$$

$$F_1(\nu) = \frac{15}{2}\left(\frac{\nu^5}{5} - \frac{(\nu^2+1)^{5/2}}{5} + \frac{(\nu^2+1)^{3/2}}{3} + \frac{\nu^3}{6} - \frac{\nu}{8}\right). \tag{121}$$

Like $f_2(\nu)$, $F_1(\nu)$ is also a monotonically decreasing function having the limiting forms

$$F_1(\nu) \xrightarrow[\nu\to 0]{} 1,$$

$$F_1(\nu) \xrightarrow[\nu\to\infty]{} \frac{15}{16}\left(\frac{1}{\nu} - \frac{5}{48\nu^2} + \cdots\right). \tag{122}$$

Equation (120) is identical with the result obtained by Singh and Kumar (1967) by the Green function method but differs from the result of Huang (1959) as regards the function $F_1(\nu)$ due to a mistake in his subtraction procedure. The ground state energy E_0 has been plotted as a function of the volume per particle by Singh and Kumar (1967) for a few typical values of the ratio $(b/a)/(k_0 a)$. The plots show the existence of an N-body bound state at zero pressure.

Equation (118) implies that at low temperatures the thermodynamic behaviour of the system is identical with that of a gas of quasi-particles having the dispersion relation $E(\mathbf{q})$. This accords with the point of view adopted by Landau (1941) in his semi-phenomenological theory of liquid He II.

The behaviour of thermodynamic functions following from (118) has been discussed in detail by Singh and Kumar (1970). We refer the interested reader to their work.

5. Conclusion

The chief motivation behind the work reported in this paper has been a desire to use techniques developed in connection with the renormalisation group theory of critical phenomena to solve problems not necessarily concerned with the behaviour of many-body systems near critical points. The particular example of a system of bosons with attractive interaction was chosen on account of its complexity. The calculations carried out in this paper demonstrate that, at low temperatures, the thermodynamics of the system can be derived far more simply by integrating out the short-wavelength modes so as to obtain an effective hamiltonian than by using Green's function methods with partial resummations (Singh and Kumar 1970).

It may be pointed out that the attractive part of the interaction leads to difficulties not only in the calculation of the long-wavelength part of the excitation spectrum at low temperatures but also in calculating the behaviour of the system near a critical point where once again the long-wavelength modes dominate. The studies of critical behaviour reported in recent years (Wilson and Kogut 1974; Singh 1975, 1978) assume that the interaction parameter associated with the quartic (4-operator) term in the hamiltonian is positive; otherwise one does not find an interesting fixed point. It needs to be shown that, as in the case of low temperatures, the bare, attractive interaction at small $\mathbf{q}$ gets renormalised to an effective positive interaction near the critical temperature also.

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