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On the alleged equivalence of certain field theories

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Abstract. We critically examine some recent claims that certain field theories with and without boson kinetic energy terms are equivalent. We point out that the crucial element in these claims is the finiteness or otherwise of the boson wavefunction renormalisation constant. We show that when this constant is finite, the equivalence proof offered in the literature fails in a direct way. When the constant is divergent, the claimed equivalence is only a consequence of improper use of divergent quantities.

Keywords. Field theory; boson wave function renormalisation constant; quantum mechanics.

1. Introduction

The classic paper of Nambu and Jona-Lasinio (1961) in which ideas from the theory of superconductivity were applied to relativistic Fermion field theories has led to many far-reaching consequences. Amongst other things, the paper introduced boson fields as collective coordinates to describe fermion-antifermion pairs. The original Nambu-Jona Lasinio model was in (3+1) dimensions, where the four-fermion interactions considered by them are non-renormalizable. The results therefore involved using a momentum cut-off Λ. To avoid this problem and for other reasons, many years later Gross and Neveu (1974) considered essentially the Nambu-Jona Lasinio model but in (1 + 1) dimensions, where it is renormalizable and has many interesting properties. As part of their work, Gross and Neveu also pointed out in a compact way how a fermionic system governed by the Lagrangian

$$\mathscr{L}_1 = \bar{\psi}\,(i\gamma.\delta - M)\,\psi + \frac{g^2}{2}\,(\bar{\psi}\,\psi)^2, \tag{1}$$

is equivalent to the Yukawa-like system

$$\mathscr{L}_2 = \bar{\psi}\,(i\gamma.\delta - M)\,\psi - g\,\bar{\psi}\,\psi\,\phi - \tfrac{1}{2}\phi^2. \tag{2}$$

Similar relations hold when the scalar bilinear form $\bar{\psi}\,\psi$ is replaced by other bilinear forms. Such equivalences are by now well-known, and can be proved easily through a variety of methods. The important point, for the purposes of our paper is that the boson field in (2) has no kinetic energy terms. Its field equation is just the constraint $\phi = - g\bar{\psi}\psi$, and it acts, loosely speaking, as a collective co-ordinate for the pair $\bar{\psi}\psi$. We have no quarrel with these results, or with any other features contained in these two excellent papers.

However, more recently, a new twist has been added by some authors to such equivalence relations. It is claimed, for instance, that not only are the systems $\mathscr{L}_1$ and $\mathscr{L}_2$ equivalent, but that both are equivalent to the genuine Yukawa system

$$\mathscr{L}_3 = \bar{\psi}\,(i\gamma.\delta - M)\,\psi - g\bar{\psi}\,\psi\phi - \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\,(\partial_\mu\,\phi)^2. \tag{3}$$

A compact and clever derivation to this effect, using functional integration methods is offered in the work of Eguchi (1976) (see also other references cited therein). Such ideas have been extended to other models in Eguchi's paper and more recently by Rajasekaran and Srinivasan (1977, 1978). One of the models treated in the latter work is the Amati-Testa (1974) model

$$\mathscr{L} = \bar{\psi}\,(i\gamma\cdot\delta + M)\,\psi,$$

under the constraint

$$j_\mu{}^i \equiv \bar{\psi}\,\gamma_\mu\,\frac{\lambda^i}{2}\,\psi, \tag{4}$$

where ψ are quark fields and λ^i, the SU(n) generators. It is claimed, applying the Eguchi proof, that this is equivalent to the familiar gauge theory

$$\mathscr{L} = \bar{\psi}\,(i\gamma\cdot\delta - M)\,\psi - \frac{1}{4}\,\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - g\bar{\psi}\,\gamma^\mu\,\frac{\lambda}{2}\,\psi\,\mathbf{A}^\mu. \tag{5}$$

Once again, if the term $-\frac{1}{4}\mathbf{G}_{\mu\nu}\,\mathbf{G}_{\mu\nu}$ were absent in (5), its equivalence to (4) may be trivially valid, but what is claimed is equivalence in the presence of that term.

Such results, if true, are clearly very interesting. Apart from their academic interest as startling results in the theory of quantum fields, some of the systems involved are important in their own right to particle physicists. Thus, the Yukawa theory in (3) is widely used, and is known to be renormalisable in (3 + 1) dimensions. The $(\bar{\psi}\,\psi)^2$ theory in (1) is also familiar, but is generally considered non-renormalisable. Their equivalence, if true, would be a major result. Similarly, the system in (5) is just quantum chromodynamics—the leading contender for a quark model of hadrons, and its alleged equivalence to the Amati-Testa model is also an important matter.

At the same time, several features of such 'equivalences' are disturbing. One would, at first glance, expect that a system such as (2) where $(\partial_\mu\,\phi)^2$ is not present would contain quite different physics from (3) which has a $(\partial_\mu\phi)^2$ term. In the latter case, the field ϕ has a non-trivial equation of motion and a canonical momentum. One can use the standard canonical quantisation procedure for ϕ. In the former case it obeys just a constraint equation and has no canonical momentum. Of course, the spirit of the Eguchi proof is that upon integrating over the ψ degrees of freedom, the field ϕ acquires $(\partial_\mu\phi)^2$ terms through radiative corrections. However, radiative correction terms are of higher order in powers of $\hbar$, and the notion of a canonical momentum which is explicitly proportional to $\hbar$ is disturbing in the normal quantisation framework. Furthermore, the proof offered by Eguchi and subsequent workers involves rescaling fields by divergent factors, and these factors are treated rather formally, if not casually in these papers. While the success of the renormalisation programme in QED has given us some confidence in dealing with infinite quantities, it is well known that they are fraught with peril.

These are merely misgivings and do not amount to solid criticism. But they motivate us, especially in view of the potential importance of these results, to examine critically the validity of Eguchi's proof of equivalence. That is the purpose of this article.

We begin in the next section with an example from non-relativistic quantum mechanics. It is designed to study the anatomy of the Eguchi method in its simplest context. On the one hand, the candidates chosen in § 2 are such that they can be clearly distinguished from one another by counting the degrees of freedom. At the same time, all the algebraic steps of the equivalence proof can be carried out here. No divergent radiative corrections occur to obscure the issues. Given that the starting systems are *a priori* distinct, the equivalence proof must of course fail for this example. That it does, but at the very last stage and in a fairly subtle way.

The lessons of this analysis help us greatly in § 3, in studying the more interesting field theoretic examples considered in the literature. We illustrate our arguments using the systems (1) to (3). We find that equivalence between such systems as (2) and (3) will not hold if the $(\partial_\mu \phi)^2$ term arising from radiative corrections has a finite coefficient. This shows clearly why the proof of Eguchi and successors *necessarily* relies on the presence of divergent quantities, which are then used to rescale fields. Yet, at the same time, if these quantities do diverge, we argue that rescaling fields by such divergences is neither permissible, nor implied by the usual renormalisation procedure. If the manipulations of the Eguchi proof were carried to their conclusion they would lead to absurd results in the presence of such divergences. We also show that if one tries to recast the last stages of the proof by handling divergent quantities carefully, the equivalence does not follow in any obvious way. From all this we conclude that the equivalence claimed between systems (2) and (3) is merely the result of improper use of divergent quantities. Our argument, with obvious changes in algebra, is equally applicable against similar equivalence claims for other models.

We should emphasize that from the outset we have no objection to the equivalence of pairs like (1) and (2). Our conflict is only with the more recent articles which equate systems like (2) with corresponding systems like (3) which have the added boson kinetic energy term. We also do not argue against the presence of fermion-antifermion bound states in systems like (1). But the possibility of such bosonic bound states does not necessarily lead to its equivalence to (3), contrary to what has been implied in this recent literature. In fact, we argue that these equivalence proofs can be examined in their own right without appealing to possible bound states, and are found wanting. We conclude that in view of these proofs not holding water, the possible equivalence of systems like (2) with (3) or (4) with (5) is at best an open question. In fact, conventional renormalisation analysis would indicate otherwise for the pair (2) versus (3).

2. A non-relativistic quantum mechanics example

Consider the following three systems, governed by Lagrangians:

$$L_1 = \tfrac{1}{2}(\dot{x}^2 - w^2 x^2) + \tfrac{1}{2} g^2 x^4 - x^6, \tag{6}$$

$$L_2 = \tfrac{1}{2}(\dot{x}^2 - w^2 x^2) - g_1 x^2 y - \tfrac{1}{2} y^2 - x^6, \tag{7}$$

$$L_3 = \tfrac{1}{2}(\dot{x}^2 - w^2 x^2) - g_2 x^2 y - \frac{m^2}{2} y^2 + \tfrac{1}{2}\dot{y}^2 - x^6. \tag{8}$$

The choice of these systems is clearly motivated by analogy to the field theoretic example in (1) to (3). In the place of the fields $\psi(\mathrm{x}, t)$ and $\phi(\mathrm{x}, t)$, we have $x(t)$ and $y(t)$ respectively. The main difference is that the systems in (6) to (8) involve a finite number of dynamical variables (either one or two). Both x and y are functions of time t alone. They can also be thought of as boson fields in zero-space dimension. The extra term ($-x^6$) in (6) to (8) has no anologue in (1) to (3) but is necessary here to ensure that the Hamiltonian is bounded from below. [It is not needed in (1) because ψ is a Fermi field and the Hamiltonian for (1) is bounded from below as it stands, despite the $(\bar{\psi}\psi)^2$ term in $\mathscr{L}_1$ having a positive sign.] The presence of this additional term common to L_1, L_2 and L_3, while necessary, does not make any essential difference to our problem. Our task is (a) to establish that while systems L_1 and L_2 are indeed equivalent to each other, the system L_3 is distinct and different, (b) to attempt to 'prove' the equivalence of L_3 to L_1 and L_2 by using functional integral methods identical to that of Eguchi and point out where exactly the proof fails, as it must for this example, and (c) to draw inferences about the interesting field theoretic examples considered in the literature.

The statement (a) is intuitively evident, but since it is a crucial point it is worth double checking with canonical principles for quantising constrained systems. The Lagrangian L_2 leads to the following equations of motion:

$$\ddot{x} = - w^2 x - 2g_1 xy - 6x^5, \tag{9}$$

and

$$y = - g_1 x^2. \tag{10}$$

The second equation is clearly a constraint. Classically, the system has only one independent degree of freedom, say, $x(t)$. The variable $y(t)$ is just another name for $-g_1 x^2(t)$. Upon substituting $y=-g_1x^2$ into (9) one obtains the equation of motion for the system L_1. Thus L_1 and L_2 are classically equivalent. Similarly, the quantum theory of L_2, when properly constructed, is identical to that of L_1. For doing this, we use Dirac's theory for quantising constrained systems (Dirac 1950; Hanson and Regge 1974). Dirac's prescription calls for constructing the Dirac Bracket from the Poisson Bracket, and replacing the former rather than the latter by commutators to quantise the theory. To start with, the system L_2 has two co-ordinates x and y with their canonical momenta p_x and p_y. There are just two constraints which, using Dirac's notation, are

$$\phi_1 \equiv p_y = (\partial L_2/\partial \dot{y}) = 0, \tag{11}$$

and $$\phi_2 \equiv y + g_1 x^2 = 0, \tag{12}$$

whose Poisson Bracket is $\{\phi_1, \phi_2\} = -1$. Then, for any two classical observables, $A(x, p_x, y, p_y)$ and $B(x, p_x, y, p_y)$ the Dirac Bracket in this case reduces to

$$\{A, B\}_{\rm DB} = \{A, B\} - \{A, \phi_1\}\{\phi_2, B\} + \{A, \phi_2\}\{\phi_1, B\}. \tag{13}$$

Note that

$$\{A, \phi_1\} = (\partial A/\partial y); \; \{A, \phi_2\} = -2g_1 x\,(\partial A/\partial p_x) - \partial A/\partial p_y,$$

and similarly for B. Hence

$$\{A, B\}_{\rm DB} = \{A, B\} - \left(\frac{\partial A}{\partial y}\frac{\partial B}{\partial p_y} - \frac{\partial B}{\partial y}\frac{\partial A}{\partial p_y}\right)$$

$$+\left[(-2g_1 x)\frac{\partial A}{\partial y}\frac{\partial B}{\partial p_x} - \frac{\partial A}{\partial p_x}(-2g_1 x)\frac{\partial B}{\partial y}\right]$$

$$= \left(\frac{\partial A}{\partial x} - 2g_1 x\frac{\partial A}{\partial y}\right)\frac{\partial B}{\partial p_x} - \left(\frac{\partial B}{\partial x} - 2g_1 x\frac{\partial B}{\partial y}\right)\frac{\partial A}{\partial p_x}. \tag{14}$$

Notice that $dy/dx = -2g_1 x$ if y were set equal to $-g_1 x^2$. Thus we see that the Dirac Bracket in (14) is just the Poisson Bracket that would result if we were to discard y as an independent degree of freedom, and set $p_y = 0$ and $y = -g_1 x^2$ in all observables. That is, we insert the constraints (11) and (12) into every observable. The Hamiltonian for the system L_2 becomes

$$\begin{aligned} H_2 &= \dot{x}p_x + \dot{y}p_y - L_2(x, \dot{x}, y), \\ &= \dot{x}p_x - L_2(x, \dot{x}, -g_1 x^2), \\ &= \tfrac{1}{2}p_x^2 + \tfrac{1}{2}w^2 x^2 - g_1^2(x^4/2) + x^6. \end{aligned} \tag{15}$$

These manoeuvres are done already at the classical level to eliminate the constraints and bring the system to the canonical Hamiltonian form. We quantise the system only after this by setting $[x, p_x] = i\hbar$. The complete set of commuting observables is just one operator x. Wave functions $\psi(x)$ depend only on x even for the system L_2. The operator y can be defined, but it is always equal to $-g_1 x^2$. The quantum system corresponding to L_2 is thus identical to that in L_1 and its Hamiltonian as given in (15) is the same as what would arise for L_1. By contrast, the system L_3 has two genuine equations of motion:

$$\begin{aligned} \ddot{x} &= -w^2 x - 2g_2 xy - 6x^5, \text{ and} \\ \ddot{y} &= -m^2 y - g_2 x^2. \end{aligned} \tag{16}$$

Thus, there are two separate degrees of freedom. The quantum theory of L_3 will have two independent commuting operators x and y and wave functions will be functions $\psi(x, y)$ of two variables. Thus, L_3 and L_2 differ in a very basic sense in terms of number of degrees of freedom. For no finite choice of parameters g_1, g_2, m^2, etc. or rescaling of variables x and y, can the quantum (or classical) system L_3 be made equivalent to L_2 or L_1.

We have perhaps belaboured the point in applying Dirac's powerful machinery to such trivial systems. But we wanted to outline the logic, using time-honoured canonical quantisation procedures, which establishes that L_3 is not equivalent to L_1 and L_2. It will be useful to remember this logic in studying the more complex field theory examples.

As long as we are being careful, it is also worth disposing of a mathematical red-herring. There are theorems which map a plane, in some sense, onto a real line. Similarly, in the quantum-theoretic context, it is possible to map the Hilbert space of functions $\psi(x, y)$ of two variables into functions of just one variable η. These theorems may cause anxiety about the sanctity of the number of degrees of freedom as a distinguishing feature of different systems. However, we are assured by mathematicians that such mappings are necessarily non-differentiable. Thus, a classical system with smooth time evolution (say, the trajectory of a particle on a plane) described by $x(t)$, $y(t)$ will necessarily be a non-differentiable function $\eta(t)$ in terms of the equivalent single parameter. A similar pathology will result for Heisenberg operators when the Hilbert space of $\psi(x, y)$ is mapped to functions of just one variable. Therefore, for purposes of meaningful physical or dynamical description, these theorems do not violate our intuitive notion that a particle in n-dimensional space, classical or quantum, needs n co-ordinate variables and no less.

Having thus satisfied ourselves that L_3 in having two degrees of freedom, cannot be equivalent to L_1 and L_2, we can be confident that attempts to prove such equivalence by alternate path integral methods must necessarily fail for this example. To see exactly where this failure occurs let us apply Eguchi's method as adapted to this example.

Let us begin with L_3 and evaluate $\exp(iW[J, \eta])$ which generates the n-point correlation functions of the quantum system, i.e.

$$\frac{(-i)^{n+m}}{\exp(iW_3)} \frac{\partial^{n+m}(\exp iW_3[J, \eta])}{\partial J(t_1)\ldots\partial J(t_n)\,\partial\eta(t_1')\ldots\partial\eta(t_m')}\bigg|_{J=\eta=0}$$

$$= \langle 0|T[x(t_1)\ldots x(t_n)\,y(t_1')\ldots y(t'_m)]|0\rangle. \tag{17}$$

This functional $W_3[J, \eta]$ is given, as is well known, by the path integral (Abers and Lee 1973),

$$\exp[iW_3(J, \eta)]$$
$$= \int D[x(t)]\,D[y(t)]\,\exp\left\{i\int dt\,[L_3 + J(t)\,x(t) + \eta(t)\,y(t)]\right\}. \tag{18}$$

Here, and in subsequent steps, we set $\hbar = 1$ and ignore overall constants multiplying $\exp(iW)$ since they have no consequence. We insert (8) for L_3 and use the compact vector-space notation, $\langle f|g\rangle \equiv \int f(t)\,g(t)\,dt$, for any two functions f and g, and any operator A, and $\langle f|A|g\rangle \equiv \int f(t)\,A\,g(t)\,dt$.

Then

$$\exp\,[iW_3(J,\eta)] = \int D[x]\; D[y]\; \exp\,[i\,(\langle x\,|\,M\,|\,x\rangle - g_2\langle x^2\,|\,y\rangle$$

$$+\langle y\,|\,O\,|\,y\rangle - \langle x^3\,|\,x^3\rangle + \langle J\,|\,x\rangle\;\langle \eta\,|\,y\rangle)], \tag{19}$$

where $M \equiv \frac{-1}{2}\left(\frac{\partial^2}{\partial t^2} + w^2\right)$, $O \equiv -\frac{1}{2}\left(\frac{\partial^2}{\partial t^2} + m^2\right)$ (20)

and $\int dt\;(\dot{y})^2 = -\int y\,\frac{\partial^2 y}{\partial t^2}\,dt,$

on integration by parts.
Note that,

$$\exp\,[i(\langle J\,|\,x\rangle - \langle\, x^3\,|\,x^3\,\rangle)] = \exp\left[i\int dt\; J\,(t)\; x(t)\right]$$

$$\sum_n\left(-\,i\int x^6\; dt'\right)^n 1/n!$$

$$= \sum_n\,(1/n!)\left[\,i\int dt'\;(\partial^6/\partial\,J^6(t'))\,\right]^n \exp\left(i\int\; dt\,J\,(t)\;x\,(t)\,\right)$$

$$= \exp\left\{i\int\; dt'\;\;[\partial^6/\partial\,J^6\,(t')]\right\}\exp\,(i\,\langle\, J\,|\,x\,\rangle). \tag{21}$$

The last equation is rather compactly written. The first exponent is understood to be expanded, and the functional derivatives $\partial/\partial\,J(t')$ are to act on $\exp\,(i\,\langle\, J|\;x\,\rangle)$. Thus,

$$\exp\,(iW_3) = \int D[y]\;\;\exp\;\;\left\{i\Big[\,\langle\, y|\;O\,|\,y\,\rangle + \langle\,\eta\,|\,y\,\rangle\, +\right.$$

$$\left. +\int\, dt'\;(\partial^6/\partial\,J^6)\,\Big]\right\}$$

$$\cdot\int\, D\;[x]\;\;\exp\;\{\;i\;[\;\langle\; x\,|\,M - g_2 y\,|\,x\,\rangle + \langle\; J\,|\,x\;\rangle]\;\}. \tag{22}$$

The functional integral over $[x]$ has quadratic form, and can be exactly integrated to yield

$$\int\, D[x]\;\;\exp\,\{\,i\;[\;\langle\; x\;|\;M - g_2 y\;|\;x\rangle + \langle\; J\;|\;x\rangle]\;\}$$

$$\simeq [\det\,(M\; - g_2 y)]^{-1/2}\;\exp\;\;[(-i/4)\;\langle\; J\,|\,(M\; - g_2 y)^{-1}\;|\;J\,\rangle], \tag{23}$$

again suppressing overall constants. This last step has used the fact that

$$M - g_2 y \equiv -\tfrac{1}{2}\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) - g_2 y,$$

is a real symmetric operator. Further

$$[\det (M - g_2 y)]^{-1/2}$$
$$= [\det M]^{-1/2} \exp\left[-\tfrac{1}{2} \operatorname{Tr} \ln \left(1 - \frac{1}{M} g_2\, y\right)\right]$$
$$\equiv [\det M]^{1/2} \exp\, [\, i\, V\, (g_2\, y)], \tag{24}$$

where the functional

$$V\, [g_2\, y] \equiv -i \sum_{n=1}^{\infty} \frac{1}{2n} \operatorname{Tr} \left[\frac{1}{M} g_2\, y\right]^n \tag{25}$$

Note that the expression,

$$\operatorname{Tr} \left[\frac{1}{M} g_2\, y\right]^n$$

stands in more explicit notation for

$$(g_2)^n \int dt_1 \ldots \int dt_n\, G(t_1\, t_2)\, y\, (t_2)\, G\, (t_2\, t_3)\, y\, (t_3) \ldots G\, (t_n\, t_1)\, y\, (t_1)$$

where $G\, (t_1, t_2) \equiv 1/M$

$$= \int (d\nu/\pi)\, \frac{\exp\, (i\, \nu\, (t_1 - t_2))}{\nu^2 - \omega^2 + i\epsilon}. \tag{26}$$

The term $V[g_2 y]$ represents the radiative contributions to the effective action of the y-variable upon integrating over x. Following the Eguchi procedure, we separate from $V[g_2 y]$ those pieces which are similar in form to

$$\langle\, y \mid O \mid y\, \rangle \equiv \tfrac{1}{2} \int dt\, y \left(- \frac{\partial^2}{\partial t^2} - m^2\right) y.$$

Such pieces arise from the

$$\operatorname{Tr} \left(\frac{1}{M} g_2\, y\right)^2$$

term in the expansion (25). We have

$$- \frac{i}{4} \operatorname{Tr} \left[\frac{1}{M} g_2\, y\right]^2$$
$$= - i g_2^{\,2} \int dt_1 dt_2 \frac{d\nu}{2\pi}\, \frac{d\nu''}{2\pi}\, y\, (t_1)\, y\, (t_2) \times$$
$$\frac{\exp\, [i\, (\nu' - \nu'')\, (t_1 - t_2)]}{(\, (\nu')^2 - \omega^2 + i\epsilon)\, ((\nu'')^2 - \omega^2 + i\epsilon\,)}$$
$$\equiv g_2^{\,2} \int dt_1 dt_2 \frac{d\nu}{2\pi}\, y\, (t_1) \exp\, [i\nu\, (t_1 - t_2)]\;\; K(\nu) y(t_2), \tag{27}$$

where

$$K(\nu) \equiv (1/2\pi) \int \frac{-id\nu'}{((\nu')^2 - \omega^2 + i\epsilon)((\nu' - \nu)^2 - \omega^2 + i\epsilon)} \tag{28}$$

Note that $K(\nu)$ is finite and well defined. So are its Taylo rcoefficients $K_n \equiv (\partial^n K(O)/\partial \nu^n)$ in

$$K(\nu) = K_0 + 1/2\ \nu^2 K_2 + (1/4!)\ \nu^4 K_4 + \ldots\ldots \tag{29}$$

(Odd terms do not occur as can be checked).
Inserting (29) into (27) we can write,

$$-\frac{ig_2^2}{4}\mathrm{Tr}\left[\frac{1}{M}\ y\right]^2$$

$$= g_2^2 \int dt_1\, dt_2 \frac{d\nu}{2\pi} y(t_1) \left(\sum_n \frac{K_n \nu^n}{n!}\right) \exp\,[i\,\nu\,(t_1 - t_2)]\ y(t_2)$$

$$= g_2^2 \int dt\ y(t) \left(K_0 - \tfrac{1}{2} K_2 \frac{\partial^2}{\partial t^2}\right) y(t) + [n \geqslant 4 \text{ terms}]. \tag{30}$$

Bearing all this in mind, let us go back to (22) with (23) and (24) inserted. We have,

$$\exp(iW_3) = (\det M)^{-1/2} \int \mathrm{D}[y] \exp\,[i]\,(\langle y \mid O \mid y \rangle + \langle \eta \mid y \rangle$$

$$+ V(g_2 y)]$$

$$\exp\left[i \int dt\ \frac{\partial^6}{\delta J^6(t)}\right] \exp\left[-\frac{i}{4}\langle J \mid (M - g_2 y)^{-1} \mid J \rangle\right]. \tag{31}$$

Let us separate the two terms explicitly shown in (30) and lump the other terms of (31) together to write

$$\exp(iW_3, [J, n]) = (\det M)^{-1/2} \int D[y] \exp \Big\{ i\,(\langle y \mid O \mid y \rangle$$

$$+ \langle \eta \mid y \rangle + \int dt\ y(t) \left(g_2^2 K_0 - \tfrac{1}{2} g_2^2 K_2 \frac{\partial^2}{\partial t^2}\right) y(t)$$

$$+ \tilde{V}[g_2\, y, J]) \Big\}, \tag{32}$$

where $\tilde{V}[g_2 y, J]$ is a functional of both $y(t)$ and $J(t)$ and contains all remaining terms in (31) not explicitly shown in (32). The important point to bear in mind is that $\tilde{V}[g_2 y, J]$ depends on $y(t)$ only through the combination $g_2 y(t)$ where g_2 is the coupling constant.

An exactly similar evaluation may be made for

$$\exp(iW_2[J,\eta]) \equiv \int D[y]\,D[x]\,\exp\left[i\int dt(L_2 + J(t)x(t) + \eta(t)y(t)\right]. \tag{33}$$

The only difference between L_2 and L_3 is that the term $\frac{1}{2}\dot{y}^2 - \frac{1}{2}m^2y^2$ in L_3 is replaced by just $-\frac{1}{2}y^2$ in L_2. Clearly, following the steps shown above, we will get

$$\exp(i\,W_2\,[J,\eta]) = (\det M)^{-1/2}\int D[y]\,\exp\Big\{\Big(i\,(-\tfrac{1}{2}\,\langle y|y\rangle) + \langle\eta|y\rangle + \int dt\; y(t)\;[g_1^2K_0 - \tfrac{1}{2}\,g_1^2K_2\,(\partial^2/\partial t^2)]\;y(t) + \tilde{V}[g_1\,y,J]\Big)\Big\}. \tag{34}$$

Comparing (32) with (34), we see that $W_3[J,\eta]$ differs from $W_2[J,\eta]$ only in that (32) contains $\langle y\,|\,O\,|\,y\rangle$ in the place of $-\frac{1}{2}\langle y\,|\,y\rangle$ in (34). Now, if the quantum systems corresponding to L_3 and L_2 were equivalent, the n-point functions of the type shown) in (17) should be equal in the two theories, except perhaps for an overall scaling on the variables x and y. In other words $\exp(iW_3\,[J(t),\,\eta(t)])$ should at least equal α $\exp(iW_2\,[\beta J(t),\,\gamma\eta(t)])$ where α, β and γ are some constants. In Eguchi's work and in its successors along the same theme, such an equivalence is claimed through appropriate rescaling of fields. Let us attempt the same trick here.

The exponent in (32) has the form

$$\int dt\,\{(g_2^2\,K_2 + 1)\,\tfrac{1}{2}\,\dot{y}^2 + [g_2^2\,K_0 - (m^2/2)]y^2\} + \langle\eta\,|\,y\rangle + \tilde{V}[g_2y,J], \tag{35}$$

while the exponent in (34) has the form

$$\int dt\,\{(g_1^2,\,K_2)\,\tfrac{1}{2}\,\dot{y}^2 + (g_1^2K_0 - \tfrac{1}{2})\,y^2\} + \langle\eta\,|\,y\rangle + \tilde{V}[g_1\,y,J]. \tag{36}$$

In (35) and (32) let us rescale variables to y_R, g_{2R} and η_R given by

$$[y_R(t)/y(t)] = (g_2/g_{2R}) = [\eta(t)/\eta_R(t)] = (g_2^2\,K_2 + 1)^{1/2}. \tag{37}$$

Similarly, in (34) and (36) let us rescale to y_R, g_{1R} and η_R given by

$$[y_R(t)/y(t)] = (g_1/g_{1R}) = [\eta(t)/\eta_R(t)] = (g_1^2K_2)^{1/2}. \tag{38}$$

Let us further suppose we can choose m^2, the parameter in L_3 such that

$$\frac{g_2^2\,K_0 - (m^2/2)}{g_2^2\,K_2 + 1} = (g_1^2\,K_0 - \tfrac{1}{2})/(g_1^2\,K_2) \equiv (-\,\mu^2/2). \tag{39}$$

Then clearly, both $\exp(iW_3)$ in (32) and $\exp(iW_2)$ in (34) could be cast into the common form

$$\exp(iW_3\,[J,\eta_R]) \propto \int D\,[y_R]\,\exp\Big\{i\int dt\,[\tfrac{1}{2}\,\dot{y}_R^2 - (\mu^2/2)\,y_R^2] + i\,\langle\eta_R\,|\,y_R\rangle + i\,\tilde{V}\,[g_{2R}y_R,J]\Big\} \tag{40}$$

and

$$\exp\,(iW_2\,[J,\eta_R]) \propto \int D[y_R]\,\exp\Big\{i\int dt\,[\tfrac{1}{2}\dot{y}_R^{\,2} - (\mu^2/2)\,y_R^{\,2}] + i\langle\eta_R\,|\,y_R\rangle + i\tilde{V}[g_{1R}y_R, J]\Big\}. \tag{41}$$

Of course, the factors which scale y into y_R in (37) and (38) are different in the two cases, but this will only lead to an inconsequential difference in the constant of proportionality in front of the functional integral (40) and (41). Similarly the fact that η is related to η_R by a different factor will only lead to a time-independent constant of proportionality between n-point functions of the two systems. It would appear then that the systems L_2 and L_3, which we have taken pains to establish as basically distinct, yield equivalent forms for the generating functional exp $[iW(J, \eta)]$. This apparent paradox is resolved by noting that although the functional form of the path integrals in (40) and (41) are the same, the allowed range of the parameters is non-overlapping. Note from (37) and (38) that

$$g_{2R} = g_2/(g_2^{\,2}K_2+1)^{1/2}, \text{ while } g_{1R} = g_1/(g_1^{\,2}K_2)^{1/2} = (1/\sqrt{K_2}).$$

Recall further that for our system, the constant K_2 as obtained from (28) and (29) is a finite constant. Thus, for no finite choice of the original couplings g_1 and g_2 can the renormalised coupling g_{1R} and g_{2R} be equal. Despite right hand sides of (40) and (41) looking identical, for no finite rescaling of variables y (or the currents η) will the functional $W_3[J, \eta]$ be equivalent to $W_2[J, \eta]$. The two systems L_2 and L_3 are indeed distinct, as our earlier analysis based on well founded canonical procedure has established.

This does not necessarily prove that for the higher dimensional field theory examples considered in the references, the corresponding equivalence is not true. The analysis of our simple example, however, brings out the crucial elements involved in such equivalence proofs, and will help us throw more light on their validity for the interesting field theoretic cases.

3. Equivalence proofs in field theory

To consider the possibility of such equivalence in field theory, let us work with an example. Consider the systems mentioned in the introduction and described by the Lagrangians

$$\mathscr{L}_1 = \bar{\psi}\,(i\gamma\cdot\delta - M)\,\psi + \tfrac{1}{2}g_1^{\,2}(\bar{\psi}\psi)^2, \tag{42}$$

$$\mathscr{L}_2 = \bar{\psi}\,(i\gamma\cdot\delta - M)\,\psi - g_1\bar{\psi}\psi\phi - \tfrac{1}{2}\phi^2, \text{ and} \tag{43}$$

$$\mathscr{L}_3 = \bar{\psi}\,(i\gamma\cdot\delta - M)\,\psi - g_2\bar{\psi}\psi\phi - \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}(\partial_\mu\phi)^2. \tag{44}$$

Here $\psi=\psi(\mathbf{x}, t)$ is a Fermi field and $\phi=\phi\,(\mathbf{x}, t)$ a Bose field. Our arguments will be illustrated for these systems, but they are equally valid when adapted to the other equivalence candidates in the literature mentioned.

As stated in the introduction, we have no quarrel with the equivalence of $\mathscr{L}_1$ and $\mathscr{L}_2$. One field equation arising from $\mathscr{L}_2$ is

$$\phi = - g_1 \bar{\psi} \psi \,. \tag{45}$$

This is a constraint equation, which can be used to eliminate ϕ, and this makes $\mathscr{L}_2$ identical to $\mathscr{L}_1$. Careful analysis using the canonical procedure for quantising constrained systems will support this, as in the preceding section. So will functional integral methods by the well known procedure of integrating over the ϕ field. Of course, this equivalence is meaningful only when both theories exist, as they certainly do in (1+1) dimensions. But the interesting questions concern the equivalence of $\mathscr{L}_3$ with $\mathscr{L}_2$ and $\mathscr{L}_1$.

The example in the preceding section was deliberately chosen with a finite number of degrees of freedom. Thus we were able to claim with certainty that the system L_3 in (8) was distinct from L_2, in having two degrees of freedom versus one. For field theories, there is a field variable at every space point **x**. The systems $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$ in this section all have a continuous infinity of degrees of freedom. We cannot therefore easily claim, by counting degrees of freedom, that $\mathscr{L}_3$ is inequivalent to $\mathscr{L}_2$. Nor can we rely on the formal equations of motion for such purposes. They are for $\mathscr{L}_2$,

$$(i\,\gamma.\delta - M)\,\psi = g_1\,\psi\,\phi; \;\; \phi = - g_1\,\bar{\psi}\,\psi, \tag{46}$$

and for $\mathscr{L}_2$,

$$(i\,\gamma\cdot\delta - M)\,\psi = g_2\,\psi\,\phi; \;\; \Box\,\phi + m^2\,\phi = - g_2\,\bar{\psi}\,\psi. \tag{47}$$

Classically, these equations are *not* equivalent, no matter how you scale the fields or vary the parameters. But in quantum field theory, the presence of operators such as $\bar{\psi}(\mathbf{x}, t)\,\psi(\mathbf{x}, t)$ makes these equations ill-defined, thanks to ultraviolet divergences. Thus, in contrast to §2, the quantum field theoretic systems are much more murky. One cannot *a priori* rule out the equivalence of $\mathscr{L}_2$ and $\mathscr{L}_3$. Of course, these systems do not look superficially equivalent. The usual renormalisation analysis would say that in (3 + 1) dimensions, the system $\mathscr{L}_1 = \mathscr{L}_2$ is non-renormalisable while the Yukawa system $\mathscr{L}_3$ *is* renormalisable. Thus conventional wisdom would suggest that $\mathscr{L}_2$ is not equivalent to $\mathscr{L}_3$. Therefore the burden of proof is not on us to prove their inequivalence, but rather on those authors who claim to demonstrate their equivalence. All that we do here is to critically examine such claims and point out that the proof involves an essential and non-permissible use of infinite quantities.

These proofs, as stated earlier, calculate the Generating functional

$$\exp\,[iW[J, \bar{J}, \eta] \equiv \int D[\phi]\,D[\bar{\psi}]\,D[\psi]$$
$$\exp\left[i \int dxdt\,(\mathscr{L} + \bar{J}\,\psi + \bar{\psi}\,J + \eta\phi)\right]$$

for the cases $\mathscr{L}_2$ and $\mathscr{L}_3$. We will merely quote the final result, since it has been worked out in Eguchi (1976) and is also a straightforward generalisation of the steps in our § 2. For both $\mathscr{L}_2$ and $\mathscr{L}_3$, one can write exp $[i\, W_{2,3}\, (J\,\bar{J},\, \eta)]$ in the common form

$$\int D[\phi_R] \exp\left[i \int dt\, d\mathbf{x}\, (\tfrac{1}{2}\, (\partial_\mu\, \phi_R)^2 - \tfrac{1}{2}\, \mu^2\, \phi^2{}_R + \eta_R\, \phi_R) + i\, \tilde{V}[g_R \phi_R\, J, \bar{J}]\right] \tag{48}$$

where, for the $\mathscr{L}_2$, ϕ, η and the coupling g_1 are scaled by

$$\phi_R(\mathbf{x},\, t)/\phi(\mathbf{x},\, t) = (g_1{}^2 I_2)^{1/2} = (g_1/g_{1R}) = \eta(\mathbf{x},\, t)/\eta_R(\mathbf{x},\, t) \tag{49}$$

while for $\mathscr{L}_3$,

$$(\phi_R/\phi) = (g_2{}^2 I_2 + 1)^{1/2} = g_2/g_{2R} = \eta/\eta_R, \tag{50}$$

with $\mu^2/2 \equiv m^2/2 - g_2{}^2 I_0/(1 + g_2{}^2 I_2) = (\tfrac{1}{2} - g_1{}^2 I_0)/(g_1{}^2 I_2)$.

The functional $\tilde{V}\, [g_R\, \phi_R,\, J,\, \bar{J}]$ once again depends on ϕ_R in the combination $g_R\, \phi_R$. These relations look very similar to (37) to (39), but the constants I_0, I_2,... are here obtained from the Taylor expansion

$$I(p_\mu) = I_0 + \tfrac{1}{2} p_\mu\, p^\mu I_2 + \ldots \tag{51}$$

of $$I(p_\mu) = i \int [d^D k/(2\pi)^D \;\mathrm{Tr}\, \{[1/(p + k)\cdot\gamma - M] \cdot [1/(k\cdot\gamma - M)]\}. \tag{52}$$

Thus $$I_0 = i \int [d^D k/(2\pi)^D]\; (1/(k^2 - M^2), \tag{53}$$

while $$I_2 = -\, i \int d^D k/(2\pi)^D\; 1/(k^2 - M^2)^2. \tag{54}$$

We have omitted unimportant numerical constants, and D-1 is the space dimensionality. Equation (52) is nothing but the generalisation of (28) to include $D-1$ space-dimensions and the fact that ψ, unlike the variable x in §2, is a fermion field. The discussion now divides into examining two cases, (i) when I_2 is finite and (ii) when I_2 diverges. When I_2 is finite, we can repeat the argument used in §2, namely, that the value of g_{1R} in (49) can never equal g_{2R} in (50) for any finite choice of the parameters g_1, g_2 etc. Then, notwithstanding the fact that exp (iW_2) and exp (iW_3) can be cast in the common form in (48) the two systems $\mathscr{L}_2$ and $\mathscr{L}_3$ cannot be equivalent. An example of such a case is when the space dimensionality $(D\, -\, 1)$ is unity for our model. Then I_2, given in (54) involves a finite integral, an i the $(\bar{\psi}\, \psi)^2$ theory is not equivalent to the Yukawa theory in (1+1) dimensions.

Clearly then, in order to proceed further with an equivalence proof along these lines, I_2 must necessarily diverge. In more familiar language, this amounts to the wave function renormalisation constant $Z \equiv 1/(g^2 I_2 + 1)$ being zero. The equivalence candidates in the literature do come under this category. In our example, when the dimension $D=4$, the integral I_2 diverges. Superficially, it would seem that now the 'renormalised' couplings

$$g_{1R} = (g_1/(g_1{}^2 I_2)^{1/2} \quad \text{and} \quad g_{2R} = g_2/(g_2{}^2 I_2 + 1)^{1/2},$$

will be equal. Combining this with the fact that both exp (iW_2) and exp (iW_3) have the same form (48), the two theories $\mathscr{L}_2$ and $\mathscr{L}_3$ would appear equivalent. This is the crux of Eguchi's claim of equivalence as well as that of other papers with the same theme.

However, even when I_2 diverges, this proof of equivalence is not valid. As everyone knows one must be very careful in dealing with divergent quantities, or else one may end up with wrong or paradoxical results. If one were willing to be casual about using infinite quantities, a much simpler 'proof' of the equivalence of $\mathscr{L}_2$ and $\mathscr{L}_3$ could be proposed. Take $\mathscr{L}_3$ and rescale ϕ into $\tilde{\phi} = m\,\phi$ and g_2 into $\tilde{g}_2 = g_2/m$. Then $\mathscr{L}_3$ has the form $\bar{\psi}\,(i\gamma\cdot\delta - M)\,\psi - \tilde{g}_2\,\bar{\psi}\,\psi\,\tilde{\phi} - \frac{1}{2}\,\tilde{\phi}^2 + 1/2m^2\,(\partial_\mu\,\tilde{\phi})^2$. Then as $m \to \infty$, if we drop the last term, we would end up with the Lagrangian $\mathscr{L}_2$, which in turn is equivalent to $\mathscr{L}_1$. The well known catch in this argument is that no matter how large m is, there will always be field configurations with $(\partial_\mu\,\tilde{\phi})^2$ much larger than $m^2\,\tilde{\phi}^2$ and the kinetic energy term cannot be neglected. In the language of perturbation theory, in any divergent loop integral, involving ϕ-propagators one cannot replace

$$\int \frac{d^4p}{(p^2-m^2)\,(\ldots)} \text{ by } \int \frac{d^4p}{(-m^2)\,(\ldots)},$$

for any m^2 however large.

Let us examine the Eguchi proof to see if a similar non-permissible step is involved, when I_2 is divergent. Indeed, if we put $I_2 = \infty$ in (49) and (50), the renormalised coupling constants g_{1R} and g_{2R} both vanish! Substituting into (48), we would conclude that both $\mathscr{L}_2$ and $\mathscr{L}_3$ correspond to a set of non-interacting Fermi and Bose fields for all finite values of the original couplings g_1 and g_2! Alternately, if we tried to keep g_{2R} real and non-zero, then upon inverting the relation

$$(g_{2R})^2 = (g_2)^2/[(g_2)^2\, I_2 + 1]\,, \text{ we have } (g_2)^2 = (g_{2R})^2/[1 - I_2\,(g_{2R})^2]\,.$$

As $I_2 \to +\infty$, $g_2{}^2 \to -1/\infty$, leading to an imaginary coupling (non-Hermitian Hamiltonian) for the original Yukawa system. These are just some of the paradoxical results one can 'derive' if one rescaled fields by divergent factors as done by Eguchi. The reader is reminded that even in the time-honoured theory of quantum electrodynamics, where divergences are removed by the renormalisation prescription, one does not literally rescale fields by divergent constants. The renormalisation prescription is, strictly speaking, a procedure for removing divergences from S-matrix elements. The prescription acts 'as if' 'bare' fields were replaced by 'renormalised' fields, but if such rescaling of fields by infinite factors were taken literally several conceptual problems arise. In fact, careful textbooks, such as Bjorken and Drell (1965) and Bogoliubov and Shirkov (1959) avoid the concepts of 'bare' and 'renormalised' fields in discussing renormalisation. The latter book discusses this point in some detail (in § 31.4) pointing out paradoxes that can arise if one took such rescaling literally. The fact that setting Z (which corresponds in our case to $(g^2 I_2^2+1)^{-1}$ equal to zero leads to $g_R=0$ has also been pointed out by others (see for instance Lurie 1968), and is a symptom of the difficulties involved in enforcing $Z=0$ in a local field theory.

Returning to the Eguchi-type of proof, if one rescaled fields as per (48) and (49) and used a divergent I_2, one would end up with the result that $g_R=0$ for both theories $\mathscr{L}_2$ and $\mathscr{L}_3$! Alternately, if one went back to the original unscaled field ϕ, then we have

$$\exp(iW_2) \sim \int D[\phi] \exp\left\{ i \int d_x^D \left(\tfrac{1}{2} g_1^2 I_2 (\partial_\mu \phi)^2 + (g_1^2 I_0 - \tfrac{1}{2}) \phi^2 + \eta \phi\right) + i \tilde{V}[g_1 \phi, J, \bar{J}] \right\}, \tag{55}$$

while

$$\exp(iW_3) \sim \int D[\phi] \exp\left\{ i \int d_x^D \left[(\tfrac{1}{2} g_2^2 I_2 + \tfrac{1}{2}) (\partial_\mu \phi)^2 + (g_2^2 I_0 - \tfrac{1}{2} m^2) \phi^2 + \eta \phi\right] + i \tilde{V}[g_2 \phi, J, \bar{J}] \right\}. \tag{56}$$

If one substituted divergent I_2 and I_0 into the integrand, the exponent becomes meaningless. One could try, as suggested in the last paragraph of Rajasekaran and Srinivasan (1977), to put a finite ultraviolet cut off Λ into the momentum integrals contained in I_0 and I_2. This would render the integrands in (55) and (56) meaningful. But, aside from the fact that a finite Λ is inconsistent with local field theories (within which framework these proofs are suggested), we have already seen that when I_2 is finite, the systems $\mathscr{L}_2$ and $\mathscr{L}_3$ are not equivalent anyway because g_{1R} cannot equal g_{2R}.

Finally, one could try to take the limit $\Lambda \to \infty$ outside the functional integrals in (55) and (56). Then, for any finite Λ, however large, the two systems are not equivalent. There is no obvious reason why they would become equivalent as $\Lambda \to \infty$. In this context remember that as Λ becomes larger and larger, even though $I_2 (\partial_\mu \phi)^2 \gg (\partial_\mu \phi)^2$, nevertheless, the $(\partial_\mu \phi)^2$ term cannot be ignored as compared to $I_2 (\partial_\mu \phi)^2$ in (56). These terms occur inside an oscillatory exponential, where a finite term in the phase cannot be ignored even if larger terms are present.

We conclude by emphasizing the precise nature of our result. We have *not* shown, nor do we claim that the pairs of field theories considered are *inequivalent*. They may well turn out to be equivalent. But the burden of proof of equivalence rests on those who claim it. It is not the purpose of this article to claim or to prove the opposite. Rather it is to point out that the proof offered by Eguchi—simple, elegant and superficially correct—is in fact not valid.

Indeed, responsible speculations along similar lines have been around for a long time before this recent spate of papers. (See for instance the work of Bjorken 1963), Lurie and Mcfarlane 1964; see also Kikkawa 1976). What is needed is a conclusive proof one way or another. Perhaps a much more careful analysis, using functional methods, or other non-perturbative techniques, may yield a definitive result. It is our hope that our analysis of the Eguchi work will help to stimulate a well founded proof one way or the other. Such a proof in the future, while very welcome, would not negate the objections we have raised against the Eguchi proof.

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