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Envelope-soliton propagation for three interacting coherent excitations in a dispersive medium

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Abstract. An initial value problem is set up to describe propagation of a low-frequency wave-field interacting with two almost transparent wave-fields in a dispersive medium. With no linear loss, perfect phase-matching, and equal group velocities for the two high-frequency wave-packets, it is shown how the solution of the above problem can evolve to well-known soliton solutions of the sine-Gordon equation. Other attempts for solving the more general problem in which all the group velocities are different are also discussed.

Keywords. Solitons; solitary waves; nonlinear classical electrodynamics; interacting classical fields; interacting excitations.

1. Introduction

In recent years, there has been a revival of interest in investigating pulse-like solitary wave propagation in different nonlinear dispersive media (Zakharov and Shabat 1971; Scott *et al* 1973; Lamb 1971; Armstrong *et al* 1970; Bers *et al* 1976; Zakharov and Manakov 1976; Makhankov 1978). In particular, the case of coherent-optical pulse propagation in a resonant two-level medium has been considered extensively by Lamb (1973; 1974; 1976). Armstrong *et al* (1970) had found that in the slowly-varying envelope approximation (SVEA), the propagation of a low frequency wave (ω_1) interacting with two other (almost transparent) waves (ω_2, ω_3, $\omega_3=\omega_1+\omega_2$) of group velocities $V_2 \simeq V_3 \equiv u$ can be described by equations similar to those of the resonant two-level system. Their method of obtaining localized travelling wave solutions was applied later to a specific problem in plasma physics by Nozaki and Taniuti (1973). In this paper, we show that the complete initial value problem can be set up for describing such propagations of interacting waves, and their evolution to soliton solutions can be obtained by using standard methods in the field. For definiteness, we consider here only the case of interacting electromagnetic excitations (individually longitudinal or transverse) but similar results follow for other types of three interacting classical fields in physics. Even in this limited sense of electromagnetic waves, our approach is general enough to deal with the interaction of any three wave-packets which may individually represent either an ionic mode (phonon), an electronic plasma mode, a transverse photon mode, or any of the possible collective coherent excitations of a charged system. In this way, it uses a common framework for different processes, instead of individually considering stimulated Raman process (Steudel 1977), etc. Moreover, note that we are interested in finding the complete solution of the initial value problem instead of just the possible particular soliton solutions, already obtained for the first time by Armstrong *et al* (1970). In the end, we will discuss the mathematical difficulties associated with the solution of the initial value problem in the general case in which all the group velocities are different, and for which, in reality, only a numerical method is available at present.

2. Three interacting electromagnetic excitations

In the absence of linear damping and nonlinear interaction between the three waves in a 'homogeneous' medium, we can describe these electromagnetic modes by the electric fields $\mathbf{e}_i\,\mathscr{E}_i\,(\mathbf{r}, t) = \mathbf{e}_i\, A_i \exp(-i\,\omega_i\, t + i\,\mathbf{q}\cdot\mathbf{r})$; $i = 1, 2, 3$; where $\mathbf{e}_i$ are unit polarization vectors. The transverse and the longitudinal mode frequencies are determined by the relations $\epsilon_{1t}(\mathbf{q}, \omega) = c^2 q^2/\omega^2$ and $\epsilon_{1l}(\mathbf{q}, \omega) = 0$, respectively. The linear damping rates are determined by $\Gamma_t = [\Omega^2\, \epsilon_{2t}(\mathbf{q}, \Omega)/(\partial/\partial\,\Omega)\, \Omega^2 \epsilon_{1t}(\mathbf{q}, \Omega)]_{\Omega=\omega}$ and $\Gamma_l = [\epsilon_2(\mathbf{q}, \Omega)/(\partial/\partial\,\Omega)\ \epsilon_1(\mathbf{q}, \Omega)]_{\omega}$, and the group velocities are determined by $\mathbf{V}_i = (\partial\omega/\partial\mathbf{q})_i$. Here ϵ_{1t} and ϵ_{2t} are the real and imaginary parts of the transverse dielectric function, etc., In the presence of linear damping and nonlinear polarization given by $\mathbf{e}_1 \cdot \mathbf{P}_{1\mathrm{NL}} = \mathbf{e}_1 \cdot \chi_{\mathrm{NL}} : \mathbf{e}_2\,\mathbf{e}_3\, \mathscr{E}_2{}^*\, \mathscr{E}_3 \equiv \chi_{\mathrm{NL}}\, \mathscr{E}_2{}^*\, \mathscr{E}_3$; $\mathbf{e}_2 \cdot \mathbf{P2}_{\mathrm{NL}} = \chi_{\mathrm{NL}}\, \mathscr{E}_1{}^*\, \mathscr{E}_3$; $\mathbf{e}_3 \cdot \mathbf{P}_{3\mathrm{NL}} = \chi^*_{\mathrm{NL}}\, \mathscr{E}_1\, \mathscr{E}_2$, for propagation in an effective x-direction, Maxwell's equations lead to the following coupled equations (in SVEA) for space-time variations of the amplitudes A_i:

$$\left(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x} + \Gamma_1\right) A_1 = i\delta_1\, \chi_{\mathrm{NL}}\, A_3\, A_2{}^* \exp(i\triangle qx) \tag{1a}$$

$$\left(\frac{\partial}{\partial t} + V_2 \frac{\partial}{\partial x} + \Gamma_2\right) A_2 = i\delta_2\, \chi_{\mathrm{NL}}\, A_3\, A_1{}^* \exp(i\triangle qx) \tag{1b}$$

$$\left(\frac{\partial}{\partial t} + V_3 \frac{\partial}{\partial x} + \Gamma_3\right) A_3 = i\delta_3\, \chi^*_{\mathrm{NL}}\, A_1\, A_2 \exp(-i\triangle qx). \tag{1c}$$

Here, $\triangle q = q_{3x} - q_{2x} - q_{1x}$ is the phase mismatch, $V_i = \mathbf{V}_i \cdot \hat{x}$ are x-components of group velocities, and new frequencies δ_i are given by $\delta_t = 2\pi\omega^2/qc\ [(\partial/\partial\Omega)\,\Omega\,(\epsilon_{rt})^{1/2}]_{\omega}$, for transverse waves, and $\delta_l = [4\pi/(\partial/\partial\Omega)\epsilon_{1l}(\mathbf{q}, \Omega)]_{\omega}$, for longitudinal waves.

Now consider the situation in which $V_1 \simeq V_2 \equiv u$ and $\Gamma_2 \simeq \Gamma_3 \simeq 0$. Using the new variables

$$\xi = x - ut;\quad t' = t;\quad p = -i\,|\chi_{\mathrm{NL}}|\, A_3\, A_2{}^* \tag{2}$$

$$W = (4\delta_2\, \delta_3)^{-1/2}\, [\delta_2\, |A_3|^2 - \delta_3\, |A_2|^2];\quad W(t' \to -\infty) \equiv W_{\mathrm{in}} = \text{constant} \tag{3}$$

$$E_1 = A_1 \exp(-i\triangle qx)\, (\chi^*_{\mathrm{NL}}/|\chi_{\mathrm{NL}}|);\ (\delta_2\delta_3)^{1/2}\, |\chi_{\mathrm{NL}}|^2\, |W_{\mathrm{in}}| = \nu_{\mathrm{NL}} \tag{4}$$

$$P = p/|\chi_{\mathrm{NL}}\, W_{\mathrm{in}}|;\ N = W/|W_{\mathrm{in}}|;\ E = E_1/|\chi_{\mathrm{NL}}\, W_{\mathrm{in}}| \tag{5}$$

Equation (1) leads to the new set of equations:

$$\partial P/\partial t' = -2\,\nu_{NL}\, NE;\quad \partial N/\partial t' = \nu_{NL}\,(PE^* + P^*E) \tag{6}$$

$$\partial E/\partial t' + (V_1 - u)\,\partial E/\partial \xi = -\,\delta_1 P - \Gamma_1 E - i\, V_1\,\triangle q E \tag{7}$$

where we have taken all δ's positive, as a particular case. Because Bloch equations (6) imply $N^2 + |P|^2 = \text{const} = N_{in}^2 = 1$, since $N(-\infty) \equiv N_{in} = \pm 1$, $|P(-\infty)| = 0$, they can be transformed into a new form

$$\partial\phi/\partial t' = \nu_{NL}\, E + \nu_{NL}\, \phi^2\, E^*, \tag{8}$$

where, $$\phi = \frac{P}{1-N};\quad P = \frac{2\phi}{1+\phi^*\phi},\quad N = \frac{\phi\phi^*-1}{\phi\phi^*+1}. \tag{9}$$

The general initial value problem in our case is defined by (7)–(9). When $\triangle q \neq 0$, E is complex, in general. However, for $\triangle q = 0$, one can assume $E = E^*$. In such a case, (8) and (9) give

$$\phi = -\,N_{in}\,[\tan(\sigma/2)]^{-N_{in}};\quad \sigma = 2\,\nu_{NL} \int_{-\infty}^{t'} dt'\, E(t');$$

$$P = -\,N_{in}\,\sin\sigma;\quad N = N_{in}\,\cos\sigma, \tag{10}$$

so that the propagation eq. (7) can be written as

$$\frac{\partial^2\sigma}{\partial X\,\partial T} = N_{in}\,\sin\sigma - (\Gamma_1/\delta_1)\,(\partial\sigma/\partial T) \tag{11}$$

where $$T = 2\,\nu_{NL}\,[t - \xi/(V_1 - u)];\quad X = \xi\,\delta_1\,(V_1 - u)^{-1} \tag{12}$$

In the absence of damping term Γ_1, (11) is the well known sine-Gordon equation. Its solutions have already been discussed by many authors (Lamb 1971, 1974), using a Bäcklund transformation, or a similarity transformation. The corresponding initial value problem has been considered by Ablowitz *et al* (1973). Note that $N_{in} = +1$ corresponds to the situation in which initially at time $t = -\infty$, the wave at the highest frequency ω_3 fills the space, with $A_2(-\infty) = 0$; whereas $N_{in} = -1$ corresponds to the presence of the wave of frequency ω_2, with $A_3(-\infty) = 0$. However, in our problem it has also to be noted that the relative signs of the group velocities $V_i \equiv \mathbf{V}_i \cdot \hat{x}$ can be either positive or negative. To avoid confusion, note that Armstrong *et al* (1970) had assumed all group velocities positive only. Only when the wave 1 is moving in the direction opposite to the waves 2 and 3, or when $|V_1| > |u|$ if they are moving in the same direction, $t = -\infty$ implies $T \to -\infty$; otherwise $t \to +\infty$ implies $T \to -\infty$. We assume that we are dealing with the former case; this is so, e.g., in the case of the backward stimulated Raman or Brillouin process, or when $V_1 > u$ in the forward processes. Then, for $N_{in} = +1$, one has the well-known π-pulse solution for the wave 1. If one also retains the damping term in (11), this leads to a steady state saturated parametric amplifier (Armstrong *et al* 1970). When $N_{\text{in}}=-1$ and $\Gamma_1=0$, the initial value problem (11) can be reduced to the solution of a linear eigen-value problem whose potential is related to the solution σ and which can be constructed by using the inverse scattering method. We will discuss this solution in the next section.

3. Solution of the initial-value problem

In the absence of damping, i.e. when $\Gamma_1 \ll \delta_1 |\chi_{\text{NL}}|\,|W_{\text{in}}|^{1/2}$, the initial value problem defined by (10)–(12) can be written as

$$\frac{\partial^2\sigma}{\partial X \partial T} = N_{\text{in}} \sin\sigma \tag{13}$$

$$T = 2\nu_{\text{NL}}\left(t - \frac{x-ut}{V_1-u}\right); \qquad X = \frac{(x-ut)\,\delta_1}{V_1-u} \tag{14}$$

$$N_{\text{in}} = W_{\text{in}}/|W_{\text{in}}| = \pm 1; \;\; \nu_{\text{NL}} = (\delta_2\,\delta_3)^{1/2}\,|\chi_{\text{NL}}|^2\,|W_{\text{in}}| \tag{15}$$

$$E = \partial\sigma/\partial T. \tag{16}$$

Now, for definiteness, let us consider the case in which $N_{\text{in}}=-1$, and $t \to +\infty$ and $-\infty$ imply $T \to +\infty$ and $-\infty$, respectively. This problem can be solved by the following inverse-scattering method developed by Zakharov and Shabat (1972) and used by Ablowitz *et al* (1973) for the sine-Gordon case. One has to introduce a complex spinor

$$v = \begin{bmatrix} v_1(X,T) \\ v_2(X,T) \end{bmatrix}, \tag{17}$$

which satisfies the following linear partial differential equations:

$$\partial v_1/\partial X + im\, v_1 = h(X,T)\, v_2, \tag{18a}$$

$$\partial v_2/\partial X - im\, v_2 = -h(X,T)\, v_1, \tag{18b}$$

$$\partial v_1/\partial T = -(i/4m)\,(v_1 \cos\sigma + v_2 \sin\sigma), \tag{19a}$$

$$\partial v_2/\partial T = -(i/4m)\,(v_1 \sin\sigma - v_2 \cos\sigma). \tag{19b}$$

Here, σ satisfies the sine-Gordon equation (13), with $N_{\text{in}}=-1$, and the 'potential'

$$h(X,T) = -\frac{1}{2}\frac{\partial\sigma(X,T)}{\partial X}, \tag{20}$$

in which $\sigma \to 0$ or multiple of 2π, as $X \to \pm\infty$. By cross-differentiation of (18) and (19) one can indeed show that the eigenvalues m are independent of X and

T. Thus, if the initial value of the 'potential' h is known at a given T, say $T=0$, i.e., $t = x/V_1$, the eigenvalues m, valid for all T, and scattering solutions corresponding to $X \to \pm \infty$, for $T = 0$, can be obtained by solving (18). By using (19) one can then find the scattering data at any T, since it enters only as a parameter. The 'potential' $h(X, T)$, and hence $\sigma(X, T)$ and $E(X, T)$, can then be constructed by solving the inverse problem represented the Marchenko integral equation.

At $T=0$, in terms of a linear combination of the complete set of two linearly independent solutions given by

$$v(X,m) = \begin{bmatrix} v_1(X,m) \\ v_2(X,m) \end{bmatrix}, \quad \bar{v}(X,m) = \begin{bmatrix} v_2^*(X,m^*) \\ -v_1^*(X,m^*) \end{bmatrix}, \tag{21}$$

any solution of (18), for a fixed m, can now be expanded. In particular, for real m (in general $m = m' + im''$), one can introduce the scattering data as the coefficients of the expansion of the Jost function

$$\Phi(X, m') = a_0(m')\bar{v} + b_0(m')v, \tag{22}$$

which has the asymptotic form

$$\Phi \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \exp(-im'X) \text{ as } X \to -\infty, \tag{23}$$

where v and $\bar{v}$ are assumed to have the asymptotic forms

$$v \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \exp(im'X), \quad \bar{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \exp(-im'X) \text{ as } X \to +\infty. \tag{24}$$

The scattering coefficients a_0 and b_0 can be analytically continued to the upper-half m-plane, Im $m>0$, if the potential $h(X, 0)$ is well behaved. Then the zeroes of $a_0(m)$ in the upper half-plane give the discrete eigenvalues $m_j (j=1, 2, \ldots N)$, and at these values one has

$$\Phi_j = c_{j0}\, v(X, m_j): \quad c_{j0} \equiv b_0(m_j). \tag{25}$$

Equations (19) then lead to the following T-dependence of these scattering coefficients:

$$a_T(m') = a_0(m'), \quad b_T(m') = b_0(m') \exp(iT/2m') \tag{26a}$$

$$c_{jT} = c_{j0} \exp(iT/2m_j). \tag{26b}$$

In terms of these coefficients, the complete solution of the initial value problem is obtained by solving the Marchenko integral equation problem:

$$K_T(X, Y) = B_T^*(X + Y) - \int_X^\infty dk \int_X^\infty dZ\, B_T^*(Y + Z) \times B_T(k + Z)\, K_T(X, k) \tag{27}$$

$$B_T(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dm' \,\frac{b_0(m')}{a_0(m')}\exp\,[\,i\,(m'X + T/2m')]$$

$$-\,i\sum\nolimits_{j=1}^{N} c_{jo}\exp\,[i\,(m_j\,X + T/2m_j)] \tag{28}$$

$$h(X,T) = -\frac{1}{2}\frac{\partial\sigma}{\partial X} = -\,2K_T\,(X,X) \tag{29}$$

$$\sigma(X,T) = \sigma(T,T) + 4\int_T^X dY\,K_T(Y,Y);\quad E(X,T) = \frac{\partial\sigma}{\partial T}. \tag{30}$$

The solution $K_T(X, X)$ of eq. (27) contains development due to two different parts: the part associated with discrete eigenvalues m_j which gives rise to soliton solutions, if any, and the part associated with the continuous eigenvalues (the first term in (28)) which gives rise to a background. However, it can be shown that the contribution due to the background part falls off as $1/\sqrt{T}$, for any X, so that in the asymptotic limit the discrete part, if it exists, dominates. Although, it is necessary to keep the background part to assure vanishing of the non-causal contributions from the discrete part, as long as we retain the contributions only in the causal region, we can ignore the background part. In such a case, one can solve the Marchenko problem to obtain

$$h(X,T) = -\frac{1}{2}\frac{\partial\sigma}{\partial X} = -\left[\frac{d^2}{dX^2}\{\ln\,[\det\,(I+AA^*)]\}\right]^{1/2} \tag{31}$$

where the matrix elements of the $N\times N$ matrix A are given by

$$A_{ij} = \frac{(c_{j0}c_{j0}{}^*)^{1/2}}{(m_i - m_j{}^*)}\exp\left[\frac{i}{4}\left(\frac{1}{m_i} - \frac{1}{m_j{}^*}\right)T\right]\exp\,[i(m_i - m_j{}^*)\,X]. \tag{32}$$

Note that the above *truncated* solution may satisfy the given starting initial condition $h(X, T=0) = h(X,0)$ only after a possible translation of axes, for each A_{ij}.

From the preceding analysis it is quite clear that the crucial problem one has to solve for investigating the evolution of the initial value of E to N- soliton solutions at large T is to obtain the discrete eigenvalues m_j of (18), and the corresponding coefficients c_{j0}. For complete solution at any T, the scattering coefficients $a_0(m')$ and $b_0(m')$ for continuous eignvalues are also needed. This eigenvalue problem can be rewritten in the Schrödinger forms (at T=0)

$$\left(-\frac{d^2}{dX^2} - h^2 - i\frac{dh}{dX}\right)\psi = m^2\psi, \tag{33}$$

$$\left(-\frac{d^2}{dX^2} - h^2 + i\frac{dh}{dX}\right)\bar{\psi} = m^2\bar{\psi}, \tag{34}$$

where we have introduced new variables

$$\psi = v_1 + iv_2; \quad \bar{\psi} = v_1 - iv_2. \tag{35}$$

In the Scrödinger eq. (33) or (34), the potential $-h^2 \mp i(dh/dX)$ is complex, but its real part is always attractive, since $h(X) \equiv h(X,0)$ is a real function. In the absence of the term $\mp i(dh/dX)$ in the potential, the discrete eigenvalues m_j correspond to the bound states of the potential $-h^2(X)$, with m_j^2 purely negative, i.e. m_j always purely imaginary (Im $m>0$). However, with the complex potential, the discrete eigenvalues can either be purely imaginary or arise as complex conjugate pairs m_j, $-m_j^*$. To be specific, let us consider, as an example, the most familiar initial condition on $h(X)$ given by

$$h(X) \equiv h(X,0) = -\frac{1}{2}\frac{\partial\sigma(X,0)}{\partial X} = \beta \operatorname{sech} \alpha X \tag{36}$$

where without any loss of generality we take $\alpha>0$. With the substitution

$$Z = \tanh \alpha X + \frac{i\alpha}{\beta} \operatorname{sech} \alpha X \tag{37a}$$

$$\frac{\partial}{\partial X} = \alpha\left(\operatorname{sech}^2 \alpha X - i\frac{\alpha}{\beta} \operatorname{sech} \alpha X \tanh \alpha X\right)\frac{\partial}{\partial Z} \equiv \frac{\alpha}{2} f(Z)\frac{\partial}{\partial Z}. \tag{37b}$$

Equation (33) can be transformed into the form

$$\left[\frac{\partial}{\partial Z} f(Z)\frac{\partial}{\partial Z} + \frac{2\beta^2}{\alpha^2} + \frac{4m^2}{\alpha^2 f(Z)}\right)\psi(Z) = 0. \tag{38}$$

Note that for $X\to+\infty$, $Z\to+1$, and when $X\to-\infty$, $Z\to-1$. In general, $f(Z)$ is an involved function of Z, but when $\beta=\pm\alpha$, $f(Z)=(1-Z^2)$. Then, the bound state solutions which remain finite at $Z=\pm1$ can be written for a more general problem (Landau and Lifschitz 1965)

$$\left[\frac{\partial}{\partial Z}(1-Z^2)\frac{\partial}{\partial Z} + s(s+1) - \frac{\epsilon}{1-Z^2}\right]\psi = 0; \quad \epsilon \equiv -\frac{4n^2}{\alpha^2}, \tag{39}$$

in the hypergeometric form

$$\psi_j \equiv \psi_{n+1} = (1-Z^2)^{\epsilon/2} F[\epsilon - s, \epsilon + s + 1, \epsilon + 1, (1-Z)/2] \tag{40}$$

$$s - \epsilon = n, \; n = 0, 1, 2, \ldots; \; n < s \tag{41}$$

so that F becomes a polynomial of degree n. There are only a finite number of bound states for which $m_j^2<0$ and $\epsilon>0$, i.e. $n<s$. Since in our problem (38), $s=1$

for $\beta = \pm\alpha$, we have only one discrete eigenvalue, with $n=0$, $\epsilon=s=1$, i.e.

$$m_1 = i\alpha/2 \tag{42}$$

which is purely imaginary. In this case, the eigen function is of the form

$$\psi_1 = \psi_{n=0} = (1-Z^2)^{1/2} = \sqrt{2}\,(\mathrm{sech}^2\, \alpha X \mp i\, \mathrm{sech}\, \alpha X \tanh \alpha X)^{1/2};$$

$$\beta = \pm\, \alpha. \tag{43}$$

A similar result follows for (34), with the same eigenvalue. Thus for $\beta=\pm\alpha$ in the initial value for $h(X)$ given by (36), we have only one purely imaginary discrete level, with $m_1=i\alpha/2$. For such an initial value, (31) and (32) lead to the soliton solution

$$h(X,T) = -\frac{1}{\alpha}\frac{\partial\sigma}{\partial X} = \frac{[\,|c_0|/\alpha]\; 2\alpha e^{\theta}}{1+[\,|c_0|^2/\alpha^2]\, e^{2\theta}}; \quad \theta \equiv \frac{T}{\alpha} - \alpha X \tag{44a}$$

$$\sigma = 4 \tan^{-1} \{\exp\, [\theta + \ln(|c_0|/\alpha)]\}. \tag{44b}$$

If we ignore the phase-factor $\ln(|c_0|/\alpha)$, which is equivalent to only a translation of the axes, the above solution leads to the evolved electric field of the low frequency wave in the form

$$E(X, T) = \frac{\partial\sigma}{\partial T} = \frac{2}{\alpha}\, \mathrm{sech}\, [(T/\alpha)-\alpha X]$$

$$\equiv \left(1-\frac{u}{V_p}\right)\frac{1}{\nu_{\mathrm{NL}}\,\tau_p}\, \mathrm{sech}\, [(t-x/V_p)/\tau_p] \tag{45}$$

where,
$$\frac{1}{\tau_p} \equiv \frac{2\nu_{\mathrm{NL}}}{\alpha}\left(1+\frac{u}{V_1-u}\right) + \frac{u\delta_1\alpha}{V_1-u} \tag{46a}$$

$$\left(1-\frac{V_1}{V_p}\right)\left(\frac{u}{V_p}-1\right) = 2\nu_{\mathrm{NL}}\delta_1\tau_p^2;\; V_2=V_3\equiv u. \tag{46b}$$

Thus the fractional decrease in the pulse velocity V_p of the low-frequency wave from the group velocity V_1 is essentially determined by the parameter $\delta_1\delta_3|\chi_{\mathrm{NL}}|^2 |A_2(-\infty)|^2\, \tau_p^2$, where τ_p is its pulse-width. Note that in this case

$$\sigma(T\to\infty)=2\pi, \quad W(T\to\infty)=W_{\mathrm{in}}\equiv W(-\infty), \tag{47}$$

so that there is no net energy exchange between the two high-frequency waves and the low-frequency wave. The low-frequency pulse is equivalent to the 2π-pulse of the self-induced transparency. Also, using (1b), (1c), (2)–(5), one finds that the other wave packets evolve as

$$|A_2|^2 = 2(\delta_2/\delta_3)^{1/2}\, |W(-\infty)|\, \tanh^2\theta \tag{48a}$$

$$A_2 = -A_2(-\infty)\tanh\theta = i\,|A_2(-\infty)|\,\tanh\theta \tag{48b}$$

$$A_3 = (\delta_3/\delta_2)^{1/2}\,|A_2(-\infty)|\,\text{sech}\,\theta \tag{48c}$$

$$\theta = \frac{T}{a} - aX \equiv \frac{t}{\tau_p} - \frac{x}{V_p\tau_p}. \tag{48d}$$

Contrary to what has been attributed by Hirota (1976), the above 2π-soliton solution was first obtained by Armstrong *et al* (1970) as a special case of their more general solutions, and not by Nozaki and Taniuti (1973).

It is clear from our analysis of the eigenvalue eq. (33) and (34) that to obtain the above soliton solution it is not necessary to have the exact initial form of $h(X,0)$ given by (36). Any initial arbitrary shape of h which leads to only one bound state for the eigenvalue problem [(33) and (34)] will evolve into the 2π-soliton solution given above.

We conclude this section by noting that by solving the linear eigenvalue problem [(33) and (34)] for any given initial condition $h(X,T=0)$, the complete time-evolution of the pulse profile can be determined.

4. Discussions

We have presented in this paper the method of obtaining the time-evolution of a low frequency wave-packet of group velocity V_1 interacting with two other wave-packets having equal group velocities $V_2 = V_3$. Apart from assuming $V_2 = V_3$, we also ignored the linear damping of these waves, to allow us the soliton-like solutions at large t. In reality, we should put back the damping term, at least for the low-frequency wave packet of centre frequency ω_1. The disintegration of the soliton-structure due to this damping can, however, be investigated by perturbation methods, without any severe difficulty. The main drawback of our approach is the assumption of equal group velocities $V_2 = V_3$. This prohibits us to see how other solitary-wave like solutions obtained by Armstrong *et al* (1970) evolve from the initial conditions. In the last few years, there has been several attempts (Zakharov and Manakov 1973, 1976) to solve this difficult problem, with $V_1 \neq V_2 \neq V_3$, using new transformations and the inverse-scattering method. But at present it is fair to say that apart from obtaining some trivial solutions, the analytical problem of the time-evolution of initial values to the interesting general solutions obtained by Armstrong *et al* is still unsolved. In this sense the progress has not b en made for the general case much beyond our earlier work. However, there exists a semi-analytic method developed by Bers *et al* (1976) to investigate this problem numerically.

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