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Eigenvalue density for ensemble of 2-body random Hamiltonians with non-zero mean for matrix elements

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Abstract. We obtain an expression for the ensemble-averaged moments in m-particle space of 2-body random Hamiltonians with same non-zero mean for the matrix elements, in the limit of $N \to \infty$, $m \gg 2$. The eigenvalue density function can then be immediately obtained in terms of the eigenvalue density (a Gaussian when $m \gg 2$) for zero mean ensembles. The results of Monte-Carlo calculations for iso-scalar rotationally invariant 2-body ensembles have also been given.

Keywords. Random matrices; moments; Monte-Carlo calculations.

1. Introduction

New ensembles of random matrices have been recently introduced by Edwards and Jones (1976), where the mean of each matrix element is same but different from zero as in the conventional random matrix ensembles. It is well known (Wigner 1955) that when each Hamiltonian matrix element is distributed randomly with a Gaussian probability density with zero mean, the resulting eigenvalue density is a semi-circle. The dominant part of the interaction for such ensembles comes from that part of the interaction where all the particles interact simultaneously, i.e. the body-rank (k) of the interaction equals the number of particles (m) in the space. It has been however demonstrated by French and Wong (1970) and by Bohigas and Flores (1971) that the ensembles of two-body random interactions in $m > 2$ spaces give Gaussian eigenvalue density; the density goes over into a semi-circle as body-rank (k) of the interaction increases to m, the number of particles. This has also been recently proved analytically by Mon and French (1975). In all these considerations, the mean of each matrix element is taken to be zero. In the second section we deal with the ensemble-averaged moments in m-particle space, of two-body random interaction with non zero-mean for the matrix elements, in the limit of the number of the single-particle states going to infinity ($N \to \infty$) and $m \gg 2$; the number of particles m being finite. In section 3, we introduce an ensemble of ensembles and section 4 gives our results of the Monte-Carlo calculations.

2. Moments

Consider a random 2-body Hamiltonian R in the space of N single-particle states.

The value of each 2-body matrix element is distributed according to a probability density function

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\,[-(x-a)^2/2\sigma^2], \tag{1}$$

a being the mean. R can be written as $R = aU + R_0$ where R_0 is a conventional random 2-body interaction with zero mean for the matrix elements and all the matrix elements of U are unity in 2-particle space. Mon and French (1975) have shown that for a dilute ($m/N \to 0$, $N \to \infty$) system with $m \gg 2$, any two two-body operators essentially commute. Hence the p-th moment of R in m-particle space is given by $\langle R^p \rangle^m$ = (trace of R^p in m-particle space)/Dimensionality

$$= \langle (aU + R_0)^p \rangle^m,$$

$$= \left\langle R_0^p + \sum_{q=1}^{p-1} (aU)^q\, R_0^{p-q} \binom{p}{q} + a^p\, U^p \right\rangle^m, \tag{2}$$

where $\binom{p}{q}$ is a binomial coefficient. Ensemble averaged moments (indicated by a bar) of R_0 in the limit $N \to \infty$ and $m \gg 2$ but finite (dilute system) are given by (Mon and French 1975)

$$\overline{\langle R_0^p \rangle^m} = M_p^0(m) = (p-1)\,!!\,\overline{\langle R_0^2 \rangle^m}^{p/2}, \quad \text{for } p \text{ even}$$

$$= 0 \qquad \text{for } p \text{ odd.} \tag{3}$$

We now evaluate the moments of U in m-particle space for a dilute system. Here we give analytic expressions for moments up to fourth order, which lead us to a generalized result in the $N \to \infty$ limit for any order; a derivation using the arguments based on the diagrammatic methods is given in the appendix.

A m-particle basis state (for fermions) in the space defined by N single-particle states can be written as N dimensional vector (**m**) with its m components being unity

Table 1. Matrix elements of U: $\langle \mathbf{m}_\alpha | U | \mathbf{m}_\beta \rangle$

Δ	Matrix element	Dimensionality $(\Delta)^+$
0	$\binom{m}{2}$	$\binom{N}{m}$
2	$m-1$	$\binom{N}{m}\ \binom{m}{1}\ \binom{N-m}{1}$
4	1	$\binom{N}{m}\ \binom{m}{2}\ \binom{N-m}{2}$

$^+$Number of matrix elements with $\Delta = (\mathbf{m}_\alpha - \mathbf{m}_\beta)^2$

and all other components being zero. The difference between two basis states $\mathbf{m}_\alpha$ and $\mathbf{m}_\beta$ is defined by $\triangle = (\mathbf{m}_\alpha - \mathbf{m}_\beta)^2$. Then the U matrix in m-particle space has four different kinds of matrix elements, corresponding to $\triangle = 0$, 2, 4 and $\triangle > 4$. The matrix elements of U are given in table 1. The results follow simply from combinatorial considerations. For example the $\triangle = 0$ (diagonal) matrix element gets the same contribution from all the distinct 2-particle pairs of among m particles, which are $\binom{m}{2}$ in number; obviously there are $\binom{N}{m}$ diagonal matrix elements. Initial state $\mathbf{m}_\alpha$ and final state $\mathbf{m}_\beta$ with $\triangle = 2$ correspond to a particle-hole type matrix element with $(m-1)$ particles unchanged. U being a 2-particle operator, each of the $(m-1)$ particles when paired with the particle that changes orbit, contribute equally. It can be seen easily that there are $m(M-m)\binom{N}{m}$ such matrix elements. All $\triangle = 4$ matrix elements are unity and $\triangle > 4$ matrix elements are trivially zero. Table 2

Table 2. Partial matrix elements of U^2 :

$\langle \mathbf{m}_\alpha | U^2(\triangle 1, \triangle 2) | \mathbf{m}_\beta \rangle = \Sigma_\gamma \langle \mathbf{m}_\alpha | U | \mathbf{m}_\gamma \rangle \langle \mathbf{m}_\gamma | U | \mathbf{m}_\beta \rangle$ $\triangle 1$, $\triangle 2$ fixed

$\triangle$	$\triangle 1^*$	$\triangle 2^*$	Partial matrix element	Dimensionality ($\triangle$)
0	0	0	$\binom{m}{2}^2$	$\binom{N}{m}$
	2	2	$(m-1)^2 m(N-m)$	
	4	4	$\binom{m}{2}\binom{N-m}{2}$	
2	0	2	$\binom{m}{2}(m-1)$	$\binom{N}{m} m(N-m)$
	2	0	$\binom{m}{2}(m-1)$	
	2	2	$(m-1)^2(N-m-1)$	
	2	4	$(m-1)^2(N-m-1)$	
	4	2	$(m-1)^2(N-m-1)$	
	4	4	$(m-1)\binom{N-m-1}{2}$	
4	0	4	$\binom{m}{2}$	$\binom{N}{m}\binom{m}{2}\binom{N-m}{2}$
	4	0	$\binom{m}{2}$	
	2	2	$4(m-1)^2$	
	2	4	$2(m-1)(N-m-2)$	
	4	2	$2(m-1)(N-m-2)$	
	4	4	$\binom{N-m-2}{2}$	
6	2	4	$9(m-1)$	$\binom{N}{m}\binom{m}{3}\binom{N-m}{3}$
	4	2	$9(m-1)$	
	4	4	$9(N-m-3)$	
8	4	4	36	$\binom{N}{m}\binom{m}{4}\binom{N-m}{4}$

$^*\triangle 1 = (\mathbf{m}_\alpha - \mathbf{m}_\gamma)^2$ and $\triangle 2 = (\mathbf{m}_\gamma - \mathbf{m}_\beta)^2$

gives the partial (according to intermediate states) matrix elements of U^2. These also follow from simple combinatorial considerations. From these matrix elements of U and U^2 one can evaluate explicitly $\langle U \rangle^m$, $\langle U^2 \rangle^m$, $\langle U^3 \rangle^m$ and $\langle U^4 \rangle^m$.

$$\langle U \rangle^m = \binom{m}{2} \equiv A1,$$

$$\langle U^2 \rangle^m = \binom{m}{2}^2 + m\,(m-1)^2\,(N-m) + \binom{m}{2}\binom{N-m}{2} \equiv A2,$$

$$\langle U^3 \rangle^m = A1^*\,A2 + m\,(m-1)^2\,(N-m)\left[2\binom{m}{2} + 3\,(m-1)\,(N-m-1) + \binom{N-m-1}{2}\right]$$

$$+\binom{m}{2}\binom{N-m}{2}\left[2\binom{m}{2} + 4(m-1)^2 + 4(m-1)\,(N-m-2) + \binom{N-m-2}{2}\right],$$

$$\equiv A1^*\,A2 + m(m-1)^2\,(N-m)\,A3 + \binom{m}{2}\binom{N-m}{2}A4,$$

$$\langle U^4 \rangle^m = (A2)^2 + m(m-1)^2\,(N-m)\,(A3)^2 + \binom{m}{2}\binom{N-m}{2}(A4)^2$$

$$+\binom{m}{3}\binom{N-m}{3}[18\,(m-1) + 9\,(N-m-3)]^2$$

$$+\binom{m}{4}\binom{N-m}{4}36^2. \tag{4}$$

Taking the limit $N \to \infty$, m finite, we find that the moments of U behave as

$$\langle U \rangle^m = \binom{m}{2},$$

$$\langle U^2 \rangle^m \to \binom{m}{2}\binom{N}{2},$$

$$\langle U^3 \rangle^m \to \binom{m}{2}\binom{N}{2}^2,$$

$$\langle U^4 \rangle^m \to \binom{m}{2}\binom{N}{2}^3. \tag{5}$$

This leads us to the general result for the dilute system

$$\langle U^p \rangle^m \to \binom{m}{2}\binom{N}{2}^{p-1}. \tag{6}$$

Now we turn to $\overline{\langle U^x R_0^y \rangle^m}$

$$\overline{\langle U^x R_0^y \rangle^m} = \binom{N}{m}^{-1} \sum_{\alpha,\beta} \overline{\langle \mathbf{m}_\alpha | U^x | \mathbf{m}_\beta \rangle \langle \mathbf{m}_\beta | R_0^y | \mathbf{m}_\alpha \rangle}. \tag{7}$$

Since R_0 and U are uncorrelated, we have

$$\overline{\langle U^x R_0^y \rangle^m} = \binom{N}{m}^{-1} \sum_{\alpha,\beta} \overline{\langle \mathbf{m}_\alpha | U^x | \mathbf{m}_\beta \rangle} \, \overline{\langle \mathbf{m}_\beta | R_0^y | \mathbf{m}_\alpha \rangle}. \tag{8}$$

Using the reasoning of Mon and French (1975), we see that to go from state $\mathbf{m}_\alpha$ to state $\mathbf{m}_\beta$, the matrix elements of R_0^y must involve at least one of the two-body matrix elements of R_0 odd number of times, which therefore gives a vanishing contribution under ensemble averaging. Therefore,

$$\overline{\langle \mathbf{m}_\alpha | R_0^y | \mathbf{m}_\beta \rangle} = \delta_{\alpha\beta} \overline{\langle \mathbf{m}_\alpha | R_0^y | \mathbf{m}_\beta \rangle}, \tag{9}$$

and hence

$$\overline{\langle U^x R_0^y \rangle^m} = \binom{N}{m}^{-1} \sum_{\alpha} \langle \mathbf{m}_\alpha | U^x | \mathbf{m}_\alpha \rangle \, \overline{\langle \mathbf{m}_\alpha | R_0^y | \mathbf{m}_\alpha \rangle}. \tag{10}$$

Diagonal matrix element of U^x is a constant, and substituting its value in the limit $N \to \infty$ we get

$$\overline{\langle U^x R_0^y \rangle^m} \binom{m}{2}\binom{N}{2}^{x-1} M_y^0(m). \tag{11}$$

Substituting (6) and (11) in (2) we obtain,

$$\begin{aligned}\overline{\langle R^p \rangle^m} \equiv M_p(m) &= M_p^0(m) + \sum_{q=1}^{p-1} a^q \binom{p}{q}\binom{m}{2}\binom{N}{2}^{q-1} M_{p-q}^0 \\ &\quad + a^p \binom{m}{2}\binom{N}{2}^{p-1} \\ &= (1-r) M_p^0(m) + r \sum_{q=0}^{p} a^q \binom{p}{q}\binom{N}{2}^q M_{p-q}^0(m)\end{aligned} \tag{12}$$

where $r = \binom{m}{2}\Big/\binom{N}{2}$ and $M_0^0(m) = 1$.

$\rho_0(x)$ is the ensemble averaged, m-particle eigenvalue density for two-body random ensemble with zero mean. Then $M_p^0(m) = \int \rho_0(x)\, x^p\, dx$. Observe that the first term in (12) corresponds to the density function $\rho_0(x)$, which is a Gaussian for $m \gg 2$; $(1 - r)$ fraction of the total strength goes into this, while the remaining fraction of the strength (r) is also distributed according to a Gaussian density function which is same as ρ_0 but translated by an amount $a\binom{N}{2}$,

$$\rho(x) = (1 - r)\ \rho_0(x) + r\ \rho_0\left(x - a\binom{N}{2}\right). \tag{13}$$

This can be very easily verified by evaluating the moments of $\rho(x)$. Thus $\rho(x)$ is a sum of two Gaussians, both having the same width but different centroids. With $a \to 0$, we end up with the moments $M_p^0(m)$ and corresponding density $\rho_0(x)$; with width of each matrix element $\sigma \to 0$, we obtain two peaks (δ functions) separated by $a*\binom{N}{2}$. Also observe that, $\binom{m}{2}\Big/\binom{N}{2}$ fraction of the total strength splits away from the main part of the distribution, the total number of states being equal to $\binom{N}{m}$.

The results given above can be readily generalized for k-body interactions, still maintaining $m \gg k$. Equation (12) goes into

$$\overline{\langle R^p \rangle^m} = M_p(m) = (1 - r)\, M_p^0(m) + r \sum_{q=0}^{p} a^q \binom{p}{q}\binom{N}{k}^q M_{p-q}^0\ (m) \tag{14}$$

with $r = \binom{m}{k}\Big/\binom{N}{k}$, and $M_p(m)$ and $M_p^0(m)$ are the moments of the eigenvalue density function generated by k-body interactions ($M_p^0(m)$ are Gaussian moments for $m \gg k$). The eigenvalue density function then becomes

$$\rho(x) = (1 - r)\ \rho_0(x) + r\ \rho_0\left(x - a\binom{N}{k}\right). \tag{15}$$

3. Mixture of ensembles

As an interesting extension of this, consider an ensemble of ensembles; the recipe for such a construction is as follows: (i) First construct an ensemble of interactions, where each matrix element is distributed randomly with the probability density function given by (1); such an ensemble is denoted by ens (a), (ii) Now, consider an ensemble of ens (a), where a is distributed randomly according to prescribed probability density function $s(a)$. Then, the moments of the doubly ensemble-averaged eigenvalue density are given by

$$M_p(s{:}m) = \left\{1 - \frac{\binom{m}{k}}{\binom{N}{k}}\right\} M_p^0(m)$$

$$+\frac{\binom{m}{k}}{\binom{N}{k}}\sum_q \binom{p}{q}\binom{N}{k}^q M^0_{p-q}(m)\int s(a)\,a^q\,dq$$

$$=\left\{1-\frac{\binom{m}{k}}{\binom{N}{k}}\right\} M_p{}^0(m)+\frac{\binom{m}{k}}{\binom{N}{k}}\sum_q \binom{p}{q}\binom{N}{k}^q M^0_{p-q}(m)\,m_q(s), \tag{16}$$

where $m_q(s)$ is the q-th moment of $s(a)$, and $M_q{}^0(m)$ is the q-th moment of eigenvalue density for a conventional k-body interaction ensemble. The eigenvalue density $\rho_s(x)$, corresponding to these moments is given by

$$\rho_s(x)=\left\{1-\frac{\binom{m}{k}}{\binom{N}{k}}\right\}\rho_0(x)+\frac{\binom{m}{k}}{\binom{N}{k}}\int_{-\infty}^{\infty}\rho_0\left(x-y\binom{N}{k}\right)s(y)\,dy. \tag{17}$$

Second term in eq. (17) is a convolution of $\rho_0(x)$ with $s(y)$. If $s(y)$ is a Gaussian and $\rho_0(x)$ is also a Gaussian, since $m \gg k$, then the second term becomes a convolution of a Gaussian with a Gaussian, the result of which is again a Gaussian.

4. Monte-Carlo calculations

In previous sections we gave results for $N\to\infty$ limit and for general Hamiltonian without considerations of the obvious physical symmetries. In general, the 2-body Hamiltonian is isoscalar and invariant under spatial rotations, so that it preserves J and T. In this section, we give results of the Monte-Carlo calculations for 4-particles in s-d shell, with $J=2$, $T=0$; the dimensionality of the Hamiltonian matrix being 56×56. Selection of this particular case depended mainly on the available computer facilities. The interaction in s-d shell, is defined by 63 two-body matrix elements in JT formalism and the 3 single-particle energies which have been taken to be zero. The two-body matrix elements were generated randomly using Gaussian-random

Table 3. Average properties of eigenvalue density and spacing distribution

a	mean	total width	skew-ness	excess	mean spacing	λ	Cen 1+	Cen 2+	width of each Gaussian
0·0	− 0·164	4·98	−0·02	−0·54	0·56	—	—	—	—
−1·5	− 9·20	7·77	−0·26	−0·58	0·58	0·36	−4·71	−17·11	4·99
−2·25	−13·66	10·17	−0·37	−0·59	0·75	0·33	−8·07	−25·24	6·21
+1·0	5·83	6·29	+0·12	−0·58	0·46	0·43	1·79	11·24	4·21

+Cen 1 and +Cen 2 are the centroids of two Gaussians.

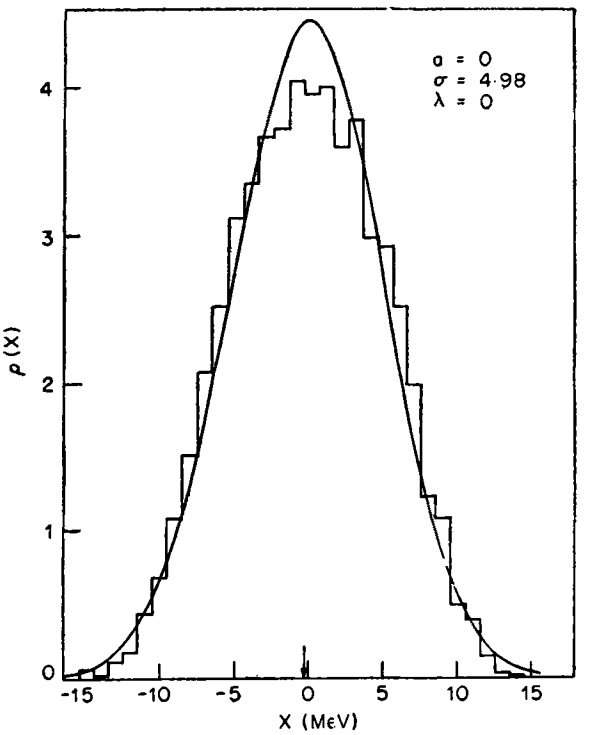
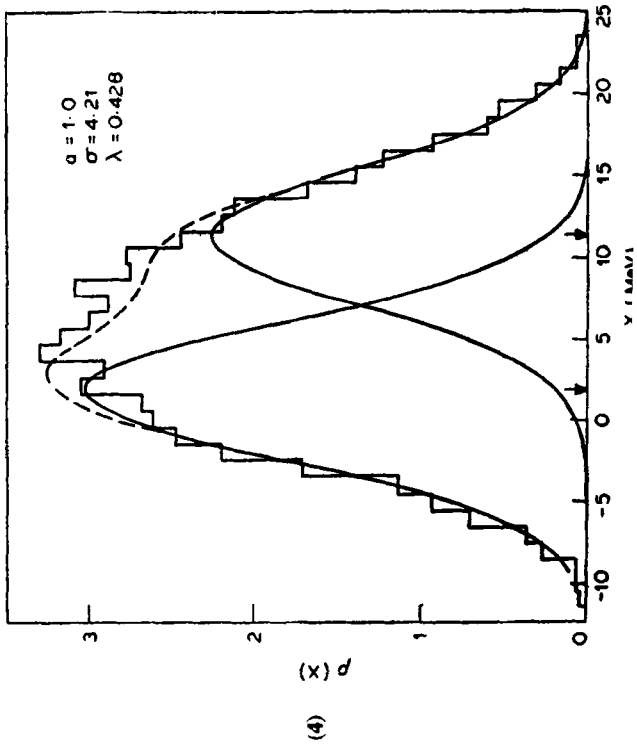
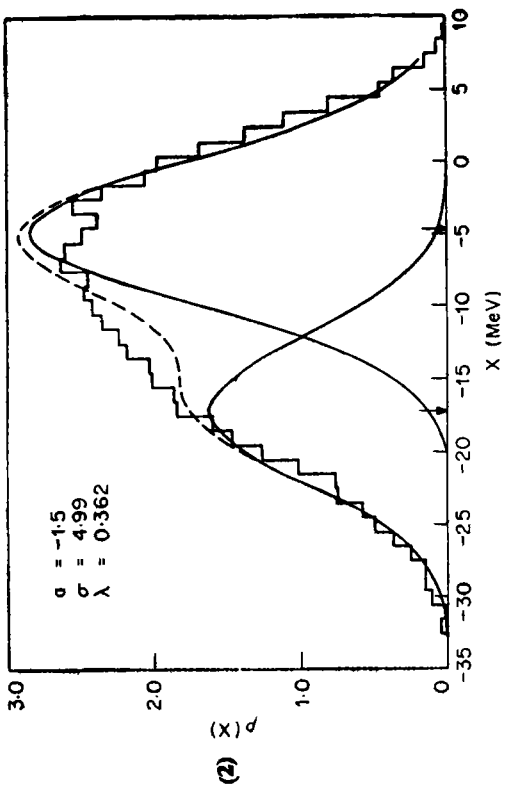
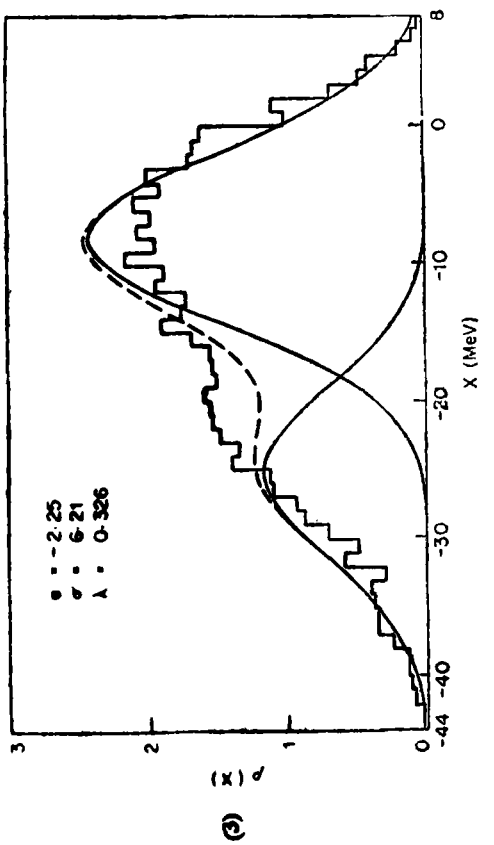


Figure 1. Histogram of the ensemble-averaged-eigenvalue density, normalized to 56 (the number of states with $J=2$, $T=0$ for four particles in *s-d* shell). The mean of the two-body matrix elements, $a=0$. A Gaussian has been fitted to the eigenvalue density. Flatness of the histogram at the top is due to large negative excess, indicating that the number of particles is not very much larger than the body rank of the interaction.

variate. The first four moments of the 6300 random numbers generated (100 interactions) are mean $= -0{\cdot}0273$, width $= 1{\cdot}009$, skewness $= 0{\cdot}0$ and excess $= -0{\cdot}106$. The random interactions with nonzero mean for matrix elements were generated by adding a constant number to all the matrix elements corresponding to zero mean interactions. The mean values for the 2-body matrix elements used here are $a = 0{\cdot}0$, $+1{\cdot}0$, $-1{\cdot}5$, $-2{\cdot}25$. In table 3, we present values of various ensemble averaged quantities, related to the eigenvalue density functions and the spacing distributions. In figures 1 to 4, we give the ensemble-averaged eigenvalue density functions, normalized to 56 for various values of a. Note that the density functions except for $a=0{\cdot}0$ can be expressed as a sum of two Gaussians as expected. However, (i) the centroids of neither of the Gaussians is zero; (ii) the widths of the two Gaussians, though assumed equal to one-another, are not same as the width of $a = 0{\cdot}0$ eigenvalue density function; (iii) the strength-splitting parameter λ (though approximately constant) varies with respect to a. It should be borne in mind that though the Monte-Carlo results given here are at variance with the analytical results given in section 3 they do not constitute a serious contradiction, because in the Monte-Carlo calculations presented here N (the number of single-particle states) is restricted to 24, and in fact, N_{eff}, the effective value of N defined ahead, is much smaller due to symmetry and Pauli-principle constraints. Also, m, the number of particles is very small and







Figures 2-4. Histograms of the ensemble averaged eigenvalue density, normalized to 56. A sum of two Gaussians with equal widths, has been fitted via calculation of moments of the eigenvalue density. The values of various parameters are given in table 3.

is not much larger than the body rank of the interaction ($m=4$, $k=2$). We can use the strength splitting parameter λ to define the effective single-particle states as

$$\lambda = \binom{m}{k} \Big/ \binom{N_{\text{eff}}}{k} \tag{18}$$

using $\lambda = 0{\cdot}36$ for $a = -1{\cdot}5$, we obtain $N_{\text{eff}} \simeq 6$. Thus asymptotic limit $N \to \infty$ is far from reached in the present calculations. We therefore think that a large scale Monte-Carlo calculation with large N, m is warranted. In figure 5 we have plotted the difference between the two centroids as a function of the mean of the matrix elements. A straight line through origin has been fitted to the data; the deviations from the straight line are not very large. The nearest-neighbour spacing distributions, in figures 6 to 9, have been compared with Wigner distribution with corresponding mean spacings. It is remarkable to see that the shape of the spacing distribution is independent of a; the mean spacing however increases with $|a|$. Also, the spacing

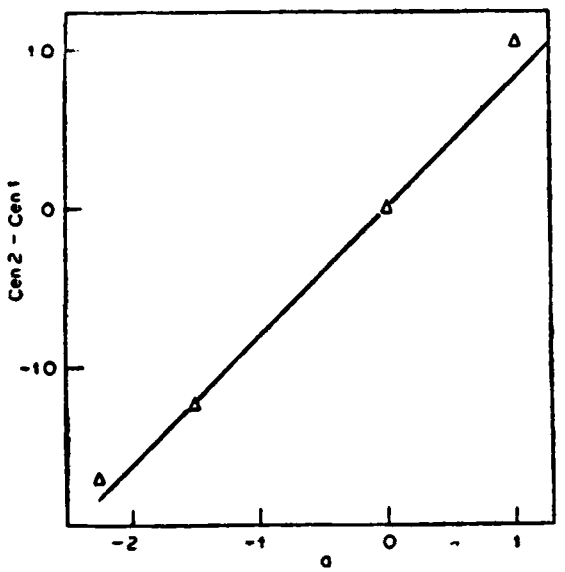


Figure 5. Difference between the centroids of two Gaussians vs a, the mean of the two-body matrix element.

distribution does not give any indication of superposition of two different classes of levels; this is clearly indicated by the absence of increases probability of finding very small nearest-neighbour spacings.

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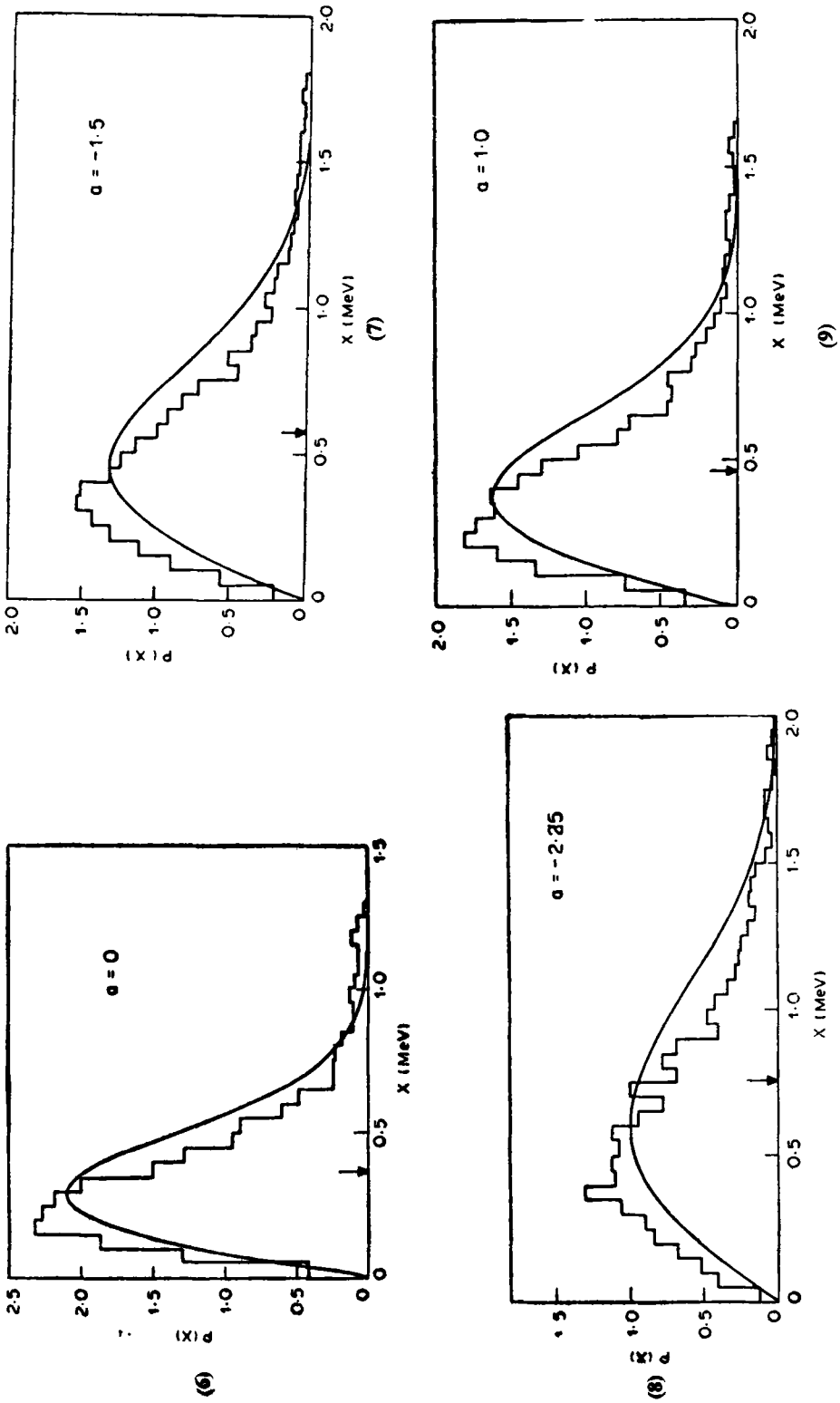

Figures 6-9. Histograms of the probability distribution for the nearest neighbour spacing. Mean spacing is denoted by an arrow, solid-continuous line is a Wigner surmise.

Appendix

V is a 2-body operator, then V^p is a sum of q-body operators; $q = 2, \ldots, 2p$. The average of the q-body part of V^p in m-particle space with N-single particle states can be written as (French 1967)

$$\langle V^p\ (q\text{-body})\rangle^m = \binom{m}{q}\langle V^p\ (q\text{-body})\rangle^q$$

$$= \binom{m}{q}\binom{N}{q}^{-1}\{\text{Trace of } V^p\ (q\text{-body}) \text{ in } q\text{-particle space}\}$$

$$= \binom{m}{q}\binom{N}{q}^{-1}\{\text{Trace of } V^p\ (q\text{-body}) \text{ in } N\text{-particle space}\}$$

$$= \binom{m}{q}\binom{N}{q}^{-1} *\text{Vaccum expectation value of } V^p\ (q\text{-body}). \quad \text{(A1)}$$

The vacuum expectation value can be written in terms of closed vacuum diagrams (Goldstone diagrams: Goldstone 1957). Such diagrams have been used extensively by Goode and Koltun (1972, 1975). From formula A1, we see that in the limit $N \to \infty$ with finite m, the terms with minimum power of N in the denominator will dominate. Hence if V is a 2-body operator only the 2-body parts of V^p will survive under averaging in the limit $N \to \infty$.

$$\therefore \quad \langle V^p\rangle^m \cong \binom{m}{2}\langle V^p\ (2\text{-body})\rangle^2$$

$$= \binom{m}{2}\binom{N}{2}^{-1} \text{Tr } V^p\ (2\text{-body}) \text{ in 2-particle space. Now, here } U \text{ is}$$

a constant operator and therefore one can very easily see that for $N \to \infty$ we have,

$$\text{Tr } U^p\ (2\text{-body}) \text{ in 2-particle space} = \langle\!\langle\ U^p\ (2\text{-body})\ \rangle\!\rangle^2$$

$$= \sum_i \langle i\,|\,U^p\ (2\text{-body})\,|\,i\rangle$$

$$= \sum_i \langle i\,|\,U|\,j\rangle\langle j\,|\,U|\,k\rangle\langle k|\ \ldots\ |n\rangle\langle n\,|\,U|\,i\rangle$$

$j, k, \ldots n$
p-indices.

Each index goes over $\binom{N}{2}$ 2-particle states, and each matrix element equals unity.

Hence,

$$\langle\!\langle\ U^p\ (2\text{-body})^2\ \rangle\!\rangle = \binom{N}{2}^p. \quad \text{(A2)}$$

Combining A1 and A2, we get

$$\langle U^p\rangle^m = \binom{m}{2}\binom{N}{2}^{p-1}. \quad \text{(A3)}$$

In general, if U is a k-body operator, we get

$$\langle U^p\rangle^m = \binom{m}{k}\binom{N}{k}^{p-1}. \quad \text{(A4)}$$

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