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Spontaneous creation of massive spin half particles by a rotating block hole

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Abstract. The techniques of second quantization in Kerr metric for the scalar and neutrino (massless) fields are extended to the massive spin half case. The normal modes of Dirac field in Kerr metric are obtained in Chandrasekhar's representation and the field is quantized as usual by imposing equal-time anti-commutation relations. The vacuum expectation value of energy-momentum tensor is evaluated asymptotically, leading to the result that a Kerr black hole spontaneously creates, in addition to scalar and neutrino quanta, massive Dirac particles in the classical superradiant modes.

Keywords. Massive spin half field; Kerr metric; spontaneous creation; black holes.

1. Introduction

The 'quantum evaporation' of a rotating black hole, that is the spontaneous creation of field quanta in Kerr metric has engaged much attention in the last few years (Unruh 1974; Ford 1975). This and the similar phenomena of particle production from collapsing black holes (Hawking 1975) constitute important consequences of quantum field theory in curved space whose general formalism is again a subject of much current investigation. (For a review see DeWitt 1975).

For the scalar field (massless and massive), and the neutrino field, the following results are by now well-known:

(a) A scalar wave incident on a Kerr black hole is amplified in modes determined by the condition $|\omega| < |m| \Omega$ (classical superradiant modes), where ω is the frequency, m is the angular momentum projection of the wave along the axis of rotation of the black hole, and Ω is the frequency of dragging of inertial frames at the horizon. This phenomena of stimulated emission is called *superradiance* and has been exhibited for electromagnetic and gravitational waves also (Zel'dovich 1971, 1972; Misner 1972, Press and Teukolsky 1974).

(b) Second quantization of massless scalar field (Unruh 1974) and massive scalar field (Ford 1975) in Kerr metric predicts spontaneous creation of particles in the classical superradiant modes with the consequent loss in mass and angular momentum of the black hole.

(c) In contrast to the scalar case, the massless spin half field does *not* superradiate (Unruh 1973), whereas second quantization once again predicts spontaneous emission of neutrinos in the same classical superradiant modes (Unruh 1974).

A feature crucial to the above developments has been the separability of scalar (Carter 1968) and neutrino (Unruh 1973) equations in Kerr metric. The extension of the neutrino case to the massive spin half case was held up for a long time because of the absence of explicit separation of Dirac equation in Kerr metric. This separation has been exhibited only recently (Chandrasekhar 1976) and the way is now open for investigating the above results for this case.

The first of the results, namely the *absence* of superradiance for massive Dirac wave has been demonstrated recently by Lee (1977) and Martellini and Treves (1977). The connection between this and the positive-definiteness of norm in Kerr metric has also been established (Iyer and Kumar 1978), generalizing an analogous result for the massless case (Unruh 1973).

In this paper, we give the second quantization of massive spin half field in Chandrasekhar's ' separable ' representation, using techniques analogous to the neutrino case. In the next section, the normal modes of the field are obtained, and in section 3, the field is quantized, as usual, by imposing equal-time anti-commutation relations. In section 4, the vacuum expectation value of the energy-momentum tensor is evaluated asymptotically yielding the rate of spontaneous loss of mass and angular momentum of the black hole. Thus, the Kerr black hole emits, in addition to scalar and neutrino quanta, massive spin half particles in the classical superradiant modes.

2. Massive spin half field in Kerr metric: Normal modes

2.1. *The Kerr metric*

The line element for Kerr metric is given by

$$ds^2 = \left(1 - \frac{2Mr}{|\rho|^2}\right) dt^2 - \frac{|\rho|^2}{\Delta} dr^2 - |\rho|^2 d\theta^2$$

$$- \left[(r^2 + a^2) \sin^2\theta + \frac{2Mr}{|\rho|^2} a^2 \sin^4\theta \right] d\phi^2 + \frac{4Mr}{|\rho|^2} a \sin^2\theta \, d\phi dt$$

$$\Delta = r^2 + a^2 - 2Mr, \quad \rho = r + ia\cos\theta. \tag{1}$$

The axially symmetric Kerr solution represents the geometry outside of a rotating black hole of mass M and angular momentum Ma. Since the metric is stationary, it may refer either to a primordial black hole or to one much after the collapse. The t and ϕ independence of the metric leads to two independent Killing vectors:

$$\xi^{\mu}_{t} = (1, 0, 0, 0) \text{ and } \xi^{\mu}_{\phi} = (0, 0, 0, 1).$$

It is easily seen that ξ_t is null at the surfaces $r = M \pm \sqrt{M^2 - a^2\cos^2\theta}$, space-like between these surfaces and time-like over the rest of the manifold. The horizon beyond which time-like vectors in the direction of increasing r do not exist is located at

$$r_+ = M + \sqrt{M^2 - a^2}. \tag{2}$$

The region between the horizon and the ' static limit ' ($r = M + \sqrt{M^2 - a^2\cos^2\theta}$) is the *ergosphere*.

The ergosphere is a distinctive feature of Kerr black hole. It has long been known classically (Penrose 1969) that a particle falling in this region can escape back with its energy amplified, with the accompanying loss of mass and angular momentum of the black hole. This is the classical counterpart of the emission processes predicted by quantum field theory in Kerr metric.

2.2. *Chandrasekhar's representation*

Dirac equation in curved space is

$$\gamma^\mu \nabla_\mu \psi + i \mu \psi = 0 \tag{3}$$

where γ^μ satisfy

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}. \tag{4}$$

The covariant derivative is given by

$$\nabla_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi, \; \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi} \Gamma_\mu, \tag{5}$$

where Γ_μ are the spinor affine connections. The details of this general formalism are available in, for example, Brill and Wheeler (1957).

Equation (3) gives a conserved current

$$J^\mu = \bar{\psi}_1 \gamma^\mu \psi_2 \tag{6}$$

with

$$\triangle_\mu J^\mu = 0, \tag{7}$$

where ψ_1 and ψ_2 are any two solutions of the equation. The conserved current yields a natural time-independent inner product,

$$\langle \psi_1 | \psi_2 \rangle = \int \sqrt{-g} \; \bar{\psi}_1 \gamma^2 \psi_2 \, d^3x. \tag{8}$$

In the Kerr metric, Dirac equation is separable in the following representation:

$$\gamma^\mu = \sqrt{2} \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{9}$$

where

$$\tilde{\sigma}^\mu = -\epsilon \, (\sigma^\mu)^T \epsilon, \; \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

(T denotes transpose of the matrix).

The σ^μ matrices are given below:

$$\sigma^t = \begin{pmatrix} \frac{r^2+a^2}{2|\rho|^2} & \frac{ia\sin\theta}{\sqrt{2}\,\rho^*} \\ \frac{-ia\sin\theta}{\sqrt{2}\,\rho} & \frac{r^2+a^2}{\Delta} \end{pmatrix} \qquad \sigma^r = \begin{pmatrix} -\frac{\Delta}{2|\rho|^2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma^\theta = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}\,\rho^*} \\ -\frac{1}{\sqrt{2}\,\rho} & 0 \end{pmatrix} \qquad \sigma^\phi = \begin{pmatrix} \frac{a}{2|\rho|^2} & \frac{i\,\mathrm{cosec}\,\theta}{\sqrt{2}\,\rho^*} \\ -\frac{i\,\mathrm{cosec}\,\theta}{\sqrt{2}\,\rho} & \frac{a}{\Delta} \end{pmatrix} \tag{11}$$

This representation was given recently by Chandrasekhar (*op. cit*) in Newman-Penrose (1962) spinor formalism, which we have translated in the more familiar four-component form above. The separated form of the wave function ψ is,

$$\psi(t,r,\theta,\phi) = \exp(-i\omega t)\exp(im\phi)\left(\frac{S^-(\theta)R^-(r)}{\sqrt{2}\,\rho^*}, \frac{S^+(\theta)R^+(r)}{\sqrt{\Delta}}, -\frac{S^-(\theta)R^+(r)}{\sqrt{\Delta}}, -\frac{S^+(\theta)R^-(r)}{\sqrt{2}\,\rho}\right), \tag{12}$$

while the adjoint $\bar{\psi}$ is given by

$$\bar{\psi}(t,r,\theta,\phi) = \exp(i\omega t)\exp-(im\phi)\left(-\frac{S^{-*}(\theta)R^{+*}(r)}{\sqrt{\Delta}}, -\frac{S^{+*}(\theta)R^{-*}(r)}{\sqrt{2}\,\rho^*}, \frac{S^{-*}(\theta)R^{-*}(r)}{\sqrt{2}\,\rho}, \frac{S^{+*}(\theta)\,R^{+*}(r)}{\sqrt{\Delta}}\right). \tag{13}$$

The spinor affine connections Γ_μ are obtained from the known spin co-efficients of Newman-Penrose formalism for Kerr metric (Teukolsky 1973). Using equations (3), (5), (9) to (12) one then obtains the following set of radial and angular equations (Chandrasekhar *op. cit.*, Lee *op. cit.*).

$$\left[\frac{d}{d\theta}+\frac{\cot\theta}{2}-a\omega\sin\theta+m\,\mathrm{cosec}\,\theta\right]S^+ = (-\lambda+\mu a\cos\theta)S^-, \tag{14}$$

$$\left[\frac{d}{d\theta}+\frac{\cot\theta}{2}+a\omega\sin\theta-m\,\mathrm{cosec}\,\theta\right]S^- = (\lambda+\mu a\cos\theta)\,S^+ \tag{15}$$

$$\Delta^{1/2}\left(\frac{d}{dr}+\frac{i}{\Delta}K\right)R^+ = (\lambda-i\mu r)\,R^- \tag{16}$$

$$\Delta^{1/2}\left(\frac{d}{dr}-\frac{i}{\Delta}K\right)R^- = (\lambda+i\mu r)\,R^+ \tag{17}$$

where

$$K = (r^2 + a^2)\,\omega - ma. \tag{18}$$

In the above equations, λ stands for the separation constant and is determined by the regularity condition on $S^{\pm}(\theta)$. The angular functions satisfy the following identity:

$$\int \sin\theta\, d\theta\, [S^{+*}(\lambda')\, S^{+}(\lambda) + S^{-*}(\lambda')\, S^{-}(\lambda)] = 0, \quad \lambda \neq \lambda' \tag{19}$$

Also note $S^{\pm}(\pi - \theta)$ and $S^{\mp}(\theta)$ satisfy the same equation and λ is real.

Returning to the inner product of (8), since $(-g)^{1/2}$ and γ^t are time-independent for our case, we have:

$$0 = \int (-g)^{1/2}\, i\frac{\partial \bar{\psi}_1}{\partial t}\gamma^t \psi_2\, d^3x + \int (-g)^{1/2}\, \bar{\psi}_1 \gamma^t\, i\frac{\partial \psi_2}{\partial t}\, d^3x$$

$$= -\langle \psi_2 | H | \psi_1 \rangle^* + \langle \psi_1 | H | \psi_2 \rangle,$$

which proves the hermiticity of the Hamiltonian with respect to the inner product. Using the explicit separable form given by eqs (9) to (13), the positive-definiteness of norm for the massive case has also been recently demonstrated (Iyer and Kumar 1978). Thus the energy eigenvalues must necessarily be real, and the possibility of unstable growing modes is excluded.

2.3. *Solution set*

To develop second quantization in Kerr metric, the first task is to obtain an orthonormal set of solutions in terms of which an arbitrary field solution may be expanded. For a given ω, m and λ there are two linearly independent solutions characterized by their behaviour at horizon and at infinity. From the radial eqs (16) and (17), the following second order equations in $R^{\pm}$ are obtained:

$$\triangle \frac{d^2R^+}{dr^2} + \left[r - M + \frac{i\mu\triangle}{\lambda - i\mu r}\right]\frac{dR^+}{dr}$$
$$+ \left[\frac{K^2 - iK(r-M)}{\triangle} + 2i\omega r - \frac{\mu K}{\lambda - i\mu r} - \mu^2 r^2 - \lambda^2)\right] R^+ = 0, \tag{20}$$

$$\triangle \frac{d^2R^-}{dr^2} + \left[(r - M - \frac{i\mu\triangle}{\lambda + i\mu r}\right]\frac{dR^-}{dr}$$
$$+ \left[\frac{K^2 + iK(r-m)}{\triangle} - 2i\omega r - \frac{\mu K}{\lambda + i\mu r} - \mu^2 r^2 - \lambda^2\right] R^- = 0. \tag{21}$$

Introducing the usual variable r' defined by

$$\frac{dr'}{dr} = \frac{r^2 + a^2}{\triangle}, \quad r \to \infty\ r' \to \infty, \quad r \to r_+\ r' \to -\infty, \tag{22}$$

the asymptotic behaviour of $R^{\pm}$ can be readily obtained:

$$R^{+}(r) \underset{r\to\infty}{\longrightarrow} \exp(\pm ik)\left[r' - \frac{\mu^2 M}{k^2}\ln r\right], \tag{23}$$

where $k = \omega\left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2}$. (24)

$R^{-}(r)$ has the same asymptotic behaviour.

The behaviour of $R^{\pm}$ near the horizon is given by

$$R^{+}(r) \underset{r\to r_+}{\longrightarrow} \exp(-i\tilde{\omega}r'), \quad \Delta^{1/2}\exp(i\tilde{\omega}r'), \tag{25}$$

$$R^{-}(r) \underset{r\to r_+}{\longrightarrow} \Delta^{1/2}\exp(-i\tilde{\omega}r'), \quad \exp(i\tilde{\omega}r'), \tag{26}$$

where

$$\tilde{\omega} = \omega - m\Omega, \quad \Omega = \frac{a}{2Mr_+}. \tag{27}$$

For a particular solution $R^{\pm}$, the corresponding $R^{\mp}$ is determined from the coupled radial equations. We choose the two linearly independent solutions characterized as below:

Type I

$$R_{\mathrm{I}}^{+} \underset{r\to\infty}{\longrightarrow} N_{\mathrm{I}}\left[\exp(-ia) - \frac{\omega}{\mu}\left[1 - \left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2}\right] B_{\mathrm{I}}\exp(ia)\right],$$

$$R_{\mathrm{I}}^{-} \underset{r\to\infty}{\longrightarrow} N_{\mathrm{I}}\left[-\frac{\omega}{\mu}\left[1 - \left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2}\right]\exp(-ia) + B_{\mathrm{I}}\exp(ia)\right],$$

$$R_{\mathrm{I}}^{+} \underset{r\to r_+}{\longrightarrow} N_{\mathrm{I}}\, A_{\mathrm{I}}\exp(-i\tilde{\omega}r'),$$

$$R_{\mathrm{I}}^{-} \underset{r\to r_+}{\longrightarrow} N_{\mathrm{I}}\, \beta\, \Delta^{1/2} A_{\mathrm{I}}\exp(-i\tilde{\omega}r'). \tag{28}$$

Type II

$$R_{\mathrm{II}}^{+} \underset{r\to\infty}{\longrightarrow} N_{\mathrm{II}}\left[-\frac{\omega}{\mu}\left[1 - \left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2}\right] B_{\mathrm{II}}\exp(ia)\right],$$

$$R_{\mathrm{II}}^{-} \underset{r\to\infty}{\longrightarrow} N_{\mathrm{II}}\, B_{\mathrm{II}}\exp(ia),$$

$$R_{\mathrm{II}}^{+} \underset{r\to r_+}{\longrightarrow} N_{\mathrm{II}}\, [\beta^{*}\Delta^{1/2}\exp(i\tilde{\omega}r') + A_{\mathrm{II}}\exp(-i\tilde{\omega}r')],$$

$$R_{\mathrm{II}}^{-} \underset{r\to r_+}{\longrightarrow} N_{\mathrm{II}}\, [\exp(i\tilde{\omega}r') + \beta\Delta^{1/2} A_{\mathrm{II}}\exp(-i\tilde{\omega}r')], \tag{29}$$

$$a = \omega \left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2} \left(r' - \frac{\mu^2 M}{\omega^2 - \mu^2} \ln r\right),$$

$$\beta = \frac{(\lambda + i\mu r_+)}{(r_+ - M) - 2i\tilde{\omega}\,(r_+^2 + a^2)},$$

$$N_{\mathrm{I}} = \left[2\pi \frac{\omega^2}{\mu^2} \left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2} \left(1 - \frac{\mu^2}{\omega^2}\right)^{1/2}\right],$$

$$N_{\mathrm{II}} = \pi^{-1/2}.$$

The significance of these solutions can be understood by construction of appropriate wave-packets. Consider, for example, the solution R_{I}^+ and construct wave-packets given by

$$R_{nj}^+(r, t) \sim \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} \exp(-2\pi i n \epsilon^{-1} k)\, R_{\mathrm{I}}^+(r) \exp(-i\omega t)\, dk \tag{30}$$

where n and j are integers. Such wave-packets have been considered by Hawking (1975) and Ford (1975) in connection with the scalar field solutions. This wave-packet has a momentum spread of ϵ. The labels n and j refer to the mean position (at $t=0$) and the mean momentum, respectively. The dominant contribution comes from the points of stationary phase. Thus at infinity, the outgoing (ingoing) piece $\exp(ikr')$ ($\exp(-ikr')$) of R_{I}^+ is localized when $t = +\infty$ ($-\infty$). Similarly, at the horizon the outgoing (ingoing) piece $\exp(i\tilde{\omega}r')$ ($\exp(-i\tilde{\omega}r')$) is localized when $t = -\infty$ ($+\infty$). Type-I solution, therefore, describes an ingoing wave at infinity in the remote past and an outgoing wave at infinity plus an ingoing wave at the horizon in the distant future. Type-II solution describes an outgoing wave at the horizon in the remote past and an outgoing wave at infinity plus an ingoing wave at the horizon in the distant future. Clearly, Type-II solution is unphysical but is necessary for obtaining a complete set.

Finally, we note that from the radial equations (16) and (17), we have, for any two solutions with the same ω, m and λ,

$$\frac{d}{dr}\,[R_1^{+*} R_2^+ - R_1^{-*} R_2^-] = 0, \tag{31}$$

and

$$\frac{d}{dr}\,[R_1^+ R_2^- - R_1^- R_2^+] = 0. \tag{32}$$

Using (28) and (29) for the behaviour of the solutions at infinity and the horizon, the above equations yield the following 'Wronskian relations':

$$1 - |B_{\mathrm{I}}|^2 = \pi N_{\mathrm{I}}^2\, |A_{\mathrm{I}}|^2, \tag{33a}$$

$$|B_{\mathrm{II}}|^2 = \pi N_{\mathrm{I}}^2\, (1 - |A_{\mathrm{II}}|^2), \tag{33b}$$

$$B_{\mathrm{I}}^* B_{\mathrm{II}} + \pi N_{\mathrm{I}}^2 A_{\mathrm{I}}^* A_{\mathrm{II}} = 0, \tag{33c}$$

$$\pi N_{\mathrm{I}}^2 A_{\mathrm{I}} = B_{\mathrm{II}}. \tag{33d}$$

2.4. *Orthonormality*

The basis set is now completely characterized by ω, m, λ and η:

$$\psi(\omega, m, \lambda, \eta; x) = \exp(-i\omega t) \exp(im\phi) \left(\frac{S^- R^-}{\sqrt{2}\,\rho^*}, \frac{S^+ R^+}{\sqrt{\Delta}}, -\frac{S^- R^+}{\sqrt{\Delta}}, -\frac{S^+ R^-}{\sqrt{2}\,\rho}\right)^T, \tag{34}$$

where $S^\pm \equiv S^\pm(\omega, m, \lambda; \theta)$, $R^\pm \equiv R^\pm(\omega, m, \lambda, \eta; r)$ where η refers to Type-I or II. We next establish orthogonality of this set with respect to the inner product defined by (8). The orthogonality in ω, of course, follows from the hermiticity of the Hamiltonian. Similarly, the orthogonality in m follows trivially from the ϕ dependence of the above solution. Consider next the inner product of two solutions with the same ω and m, but different λ and η. Using (8) and the explicit separated forms of wave-functions, we find,

$$\begin{aligned}&\langle\psi(\omega, m, \lambda_1, \eta_1) \mid \psi(\omega, m, \lambda_2, \eta_2)\rangle \\ &= \frac{1}{2}\int dr\, d\Omega \left\{\frac{r^2+a^2}{\Delta}\left(S^{+*}(\lambda_1) S^+(\lambda_2) + S^{-*}(\lambda_1) S^-(\lambda_2)\right)\left(R^{+*}(\lambda_1, \eta_1)\right.\right. \\ &\quad \left. R^+(\lambda_2, \eta_2) + R^{-*}(\lambda_1, \eta_1) R^-(\lambda_2, \eta_2)\right) + \frac{ia \sin\theta}{\sqrt{\Delta}}\left(S^{-*}(\lambda_1) S^+(\lambda_2)\right. \\ &\quad \left.\left. + S^{+*}(\lambda_1) S^-(\lambda_2)\right)\left(R^{+*}(\lambda_1, \eta_1) R^-(\lambda_2, \eta_2) - R^{-*}(\lambda_1, \eta_1) R^+(\lambda_2, \eta_2)\right)\right\}\end{aligned} \tag{35}$$

(The integration over r is from r_+ to ∞).

First, take $\eta_1 = \eta_2$. The first term above does not contribute for $\lambda_1 \neq \lambda_2$ as a consequence of the identity, eq. (19). To deal with the second term, we recall that the inner product is time-independent; we may, therefore, consider the radial packets in the integrand at $t = -\infty$, which, for Type-I (II), are localized at infinity (horizon). Using (28) and (29), one then finds that the second term gives vanishing contribution for both types.

Next, take $\lambda_1 = \lambda_2$ and $\eta_1 \neq \eta_2$, and as before, evaluate the inner product at $t = -\infty$. Both the terms of (35) are now evidently zero, because the two types of radial packets are localized at different regions and, therefore, have no overlap. The orthogonality of the solution set in all the labels is thus established. The angular functions $S^\pm(\theta)$ are normalized as:

$$\int d\Omega\,(|S^+|^2 + |S^-|^2) = 1 \tag{36}$$

and the overall constants $N_{\rm I}$ and $N_{\rm II}$ appearing in the radial solutions (28) and (29) have been so arranged that the set is orthonormal:

$$\langle \psi(\omega_1, m_1, \lambda_1, \eta_1) \mid \psi(\omega_2, m_2, \lambda_2, \eta_2) \rangle = \delta(\omega_1 - \omega_2)\, \delta_{m_1 m_2}\, \delta_{\lambda_1 \lambda_2}\, \delta_{\eta_1 \eta_2} \quad (37)$$

3. Dirac field quantization in Kerr metric

An arbitrary solution of the field equation may be expanded in terms of the orthonormal set constructed above. In analogy to the scalar and neutrino case (Unruh 1974), we have,

$$\Psi(x) = \sum_m \int_{\kappa>0} d\omega \sum_{\lambda, \eta} [a(\omega, m, \lambda, \eta)\, \psi(\omega, m, \lambda, \eta; x) + b^\dagger(\omega, m, \lambda, \eta)$$

$$\psi(-\omega, -m, \lambda, \eta; x)] \quad (38)$$

where

$$\kappa > 0 \quad \text{if} \quad \eta = \mathrm{I} \quad \omega > 0$$

$$\eta = \mathrm{II} \quad \tilde{\omega} > 0$$

$$< 0 \quad \text{otherwise.} \quad (39)$$

Inverting (38) using the orthonormality of the modes, we get,

$$a(\omega, m, \lambda, \eta) = \langle \psi(\omega, m, \lambda, \eta) \mid \Psi \rangle \quad ; \quad \kappa > 0, \quad (40a)$$

$$b^\dagger(\omega, m, \lambda, \eta) = \langle \psi(-\omega, -m, \lambda, \eta) \mid \Psi \rangle \; ; \quad \kappa > 0. \quad (40b)$$

From the time-independence of inner product, a and $b^\dagger$ are constant in time. Substitution of (40) back in (38) yields the ' closure ' relation:

$$\sum_m \int_{\kappa>0} d\omega \sum_{\lambda, \eta} (-g)^{1/2} [\psi(\omega, m, \lambda, \eta; x)\, \bar{\psi}(\omega, m, \lambda, \eta; x')$$

$$+ \psi(-\omega, -m, \lambda, \eta; x)\, \bar{\psi}(-\omega, -m, \lambda, \eta; x')]\, \gamma^t$$

$$= \delta(r - r')\, \delta(\theta - \theta')\, \delta(\phi - \phi') \equiv \delta^3(\bar{x} - \bar{x}'). \quad (41)$$

Consider next the Lagrangian density of the field:

$$\mathscr{L}(x) = (-g)^{1/2} (i \bar{\Psi}\, \gamma^\mu \nabla_\mu \Psi - \mu \bar{\Psi} \Psi) \quad (42)$$

from which the conjugate momentum of the field is

$$\Pi(x) = \frac{\partial \mathscr{L}}{\partial (\partial \Psi / \partial t)} = i (-g)^{1/2}\, \bar{\Psi}\, \gamma^t. \quad (43)$$

The momentum conjugate to $\bar{\Psi}$ is zero.

Quantization of the field is effected by imposing the equal-time anti-commutation relations (ACR):

$$[\pi^a(x), \pi^b(x')]_{+\atop t=t'} = [\Psi^a(x), \Psi^b(x')]_{+\atop t=t'} = 0$$

$$[\pi^a(x), \Psi^b(x')]_{+\atop t=t'} = i\delta^3(\bar{x}-\bar{x}')\,\delta^{ab}. \tag{44}$$

From (40) and the t-independence of a, $a^\dagger$, we have,

$$[a(\omega, m, \lambda, \eta), a^\dagger(\omega', m', \lambda', \eta')]_+$$

$$= -i\int d^3x\,d^3x'\,\sqrt{-g(x)}\,(\bar{\psi}(\omega, m, \lambda, \eta; x)\,\gamma^t(x))^a$$

$$\psi^b(\omega', m', \lambda', \eta'; x')\;[\Pi^b(x'), \Psi^a(x)]_{+\atop t=t'}$$

$$= \int d^3x\,\sqrt{-g(x)}\;\bar{\psi}(\omega, m, \lambda, \eta; x)\,\gamma^t\,\psi(\omega', m', \lambda', \eta'; x)$$

using (44). The orthonormality of the modes, eq (37), then gives:

$$[a(\omega, m, \lambda, \eta), a^\dagger(\omega', m', \lambda', \eta')]_+ = \delta(\omega-\omega')\,\delta_{mm'}\,\delta_{\lambda\lambda'}\,\delta_{\eta\eta'}, \kappa > 0. \tag{45a}$$

Similarly,

$$[b(\omega, m, \lambda, \eta), b^\dagger(\omega', m', \lambda', \eta')]_+ = \delta(\omega-\omega')\,\delta_{mm'}\,\delta_{\lambda\lambda'}\,\delta_{\eta\eta'}, \kappa > 0 \tag{45b}$$

All other anti-commutators are zero.

The vacuum is defined by

$$a(\omega, m, \lambda, \eta)\,|\,0\,\rangle = b(\omega, m, \lambda, \eta)\,|\,0\,\rangle = 0;\ \kappa > 0. \tag{46}$$

Some remarks may be made about the field expansion, eq. (38), and the choice of the vacuum state. The introduction of κ in (38) that distinguishes Type-I from Type-II can be motivated by first considering the scalar case. It is known (Unruh 1974) that the norm of scalar field modes $\phi(\omega, m, \lambda, \eta; x)$ in Kerr metric is related to κ, that is, Type-I mode has +ve (−ve) norm for $\omega>0$ ($\omega<0$) whereas Type-II mode has +ve (−ve) norm for $\tilde{\omega}>0$ ($\tilde{\omega}<0$). Thus for Type-II mode, which in the remote past originates out of the horizon, the effective frequency is $\tilde{\omega}$ and not ω. (This is related to the fact that the horizon rotates with angular velocity Ω with respect to the observer at infinity, see Hawking 1975). For Type-I mode, on the other hand, which in the remote past originates from infinity, the frequency is ω as usual, since the Kerr metric is asymptotically flat.

For the spin half case, however, all modes have positive-definite norm but we still choose to consider $\tilde{\omega}$ and not ω as the effective frequency at the horizon. This is made plausible by Unruh's calculation of energy density at infinity and at the horizon. It turns out that the energy density is +ve (−ve) at infinity for $\omega>0$ ($\omega<0$) and

+ve (−ve) at the horizon for $\tilde{\omega}>0$ ($\tilde{\omega}<0$). This then supports the interpretation of a and b in (38) as the annihilation operators. The choice of the vacuum given by (46), therefore, corresponds to no particles at infinity and no particles coming out at the black hole from the past horizon, with the negative energy sea $\omega<0$ ($\tilde{\omega}<0$) completely filled for an observer at infinity (horizon).

Vacuum expectation value of energy-momentum tensor

The energy-momentum tensor for Dirac field in curved space is given by:

$$T_{\mu\nu} = \frac{i}{4}[\bar{\Psi}\, \gamma_\mu \nabla_\nu \Psi + \bar{\Psi}\, \gamma_\nu \nabla_\mu \Psi - (\nabla_\mu \bar{\Psi})\gamma_\nu \Psi - (\nabla_\nu \bar{\Psi})\gamma_\mu \Psi)]. \tag{47}$$

To determine the rate of spontaneous creation of field quanta, consider the vacuum expectation value of $T_{\mu\nu}$. Using (38), (45) and (46), we have,

$$\begin{aligned}
&\langle 0 | T_{\mu\nu} | 0 \rangle \\
&= \frac{i}{4} \sum_m \int_{\kappa>0} d\omega \sum_{\lambda,\eta} [\bar{\psi}(-\omega, -m, \lambda, \eta)\, \gamma_\mu \nabla^\nu \psi(-\omega, -m, \lambda, \eta) \\
&\quad + \psi(-\omega, -m, \lambda, \eta)\, \gamma_\nu \nabla_\mu \psi(-\omega, -m, \lambda, \eta) \\
&\quad - (\nabla_\mu \bar{\psi}(-\omega, -m, \lambda, \eta))\, \gamma_\nu \psi(-\omega, -m, \lambda, \eta) \\
&\quad - (\nabla_\nu \psi(-\omega, -m, \lambda, \eta))\, \gamma_\mu \psi(-\omega, -m, \lambda, \eta)].
\end{aligned} \tag{48}$$

The integrand in the above equation can be evaluated by writing every quantity explicitly in Chandrasekhar's representation. This evaluation can be made simpler by going back to the two-component form and translating the σ matrices and affine connections to the null Kinnersley tetrad and the Newman-Penrose spin co-efficients available in the literature. A lengthy but straightforward calculation then yields for the asymptotically leading term of $\langle 0 | T_{rt} | 0 \rangle$.

$$\begin{aligned}
r^2 \langle 0 | T_{rt} | 0 \rangle \xrightarrow[r\to\infty]{} &\Big[\sum_m \int_{k>0} d\omega \sum_{\lambda,\eta} \frac{i}{4\sqrt{2}} | N(\eta) |^2 \times \\
&\Big\{ i\omega (| S^+ |^2 + | S^- |^2)(| R^+ |^2 - | R^- |^2) + \\
&| S^- |^2 \Big(R^{-*} \frac{dR^-}{dr} - R^+ \frac{dR^{+*}}{dr} \Big) + | S^+ |^2 \Big(R^{+*} \frac{dR^+}{dr} - R^- \frac{dR^{-*}}{dr} \Big) \Big\} \Big] + \text{c.c.}
\end{aligned}$$

where

$$S^\pm \equiv S^\pm (-\omega, -m, \lambda; \theta), \; R^\pm \equiv R^\pm (-\omega, -m, \lambda, \eta; r). \tag{49}$$

Using the asymptotic form of radial solutions (28) and (29), (49) becomes:

$$r^2 \langle 0 | T_{rt} | 0 \rangle$$

$$\xrightarrow[r \to \infty]{} \frac{1}{\sqrt{2}\,\pi} \Big[\sum_m \int_{\omega > 0} d\omega\, \omega \sum_\lambda (|S^+|^2 + |S^-|^2)(|B_{\rm I}|^2 - 1)$$

$$+ \sum_m \int_{\tilde{\omega} > 0} d\omega\, \omega \sum_\lambda (|S^+|^2 + |S^-|^2)$$

$$\left(1 - \frac{\omega^2}{\mu^2} \left(1 - \sqrt{1 - \frac{\mu^2}{\omega^2}}\right)^2\right) |B_{\rm II}|^2 \Big]. \tag{50}$$

Next, employing the ' Wronskian ' relations [eqs (33d) and (33a)] we get,

$$r^2 \langle 0 | T_{rt} | 0 \rangle$$

$$\xrightarrow[r \to \infty]{} - \frac{1}{\sqrt{2}\pi} \Big[\sum_m \int_{\omega > 0} d\omega\, \omega \sum_\lambda (|S^+|^2 + |S^-|^2)\,(1 - |B_{\rm I}|^2)$$

$$- \sum_m \int_{\tilde{\omega} > 0} d\omega\, \omega \sum_\lambda (|S^+|^2 + |S^-|^2)(1 - |B_{\rm I}|^2) \Big]. \tag{51}$$

Clearly, the rhs of (51) is nonvanishing only in the range $\omega\tilde{\omega} < 0$ (classical superradiant modes). Thus,

$$r^2 \langle 0 | T_{rt} | 0 \rangle \xrightarrow[r \to \infty]{} - \frac{1}{\sqrt{2}\pi} \sum_m \int_{\omega \tilde{\omega} < 0} d\omega\, |\omega|$$

$$\sum_\lambda (|S^+|^2 + |S^-|^2)(1 - |B_{\rm I}|^2). \tag{52}$$

From the vanishing divergence of $T^{\mu\nu}$,

$$T^{\mu\nu}_{;\mu} = 0, \tag{53}$$

the *outgoing* energy flux across a surface at infinity is:

$$\frac{dE}{dt} = \int_{r \to \infty} (-g)^{1/2}\, d\theta\, d\phi\, \langle 0 | T^{rt} | 0 \rangle. \tag{54}$$

The rate of spontaneous loss of mass of the black hole is, therefore, given by

$$\frac{dM}{dt} = - \frac{dE}{dt} = \int_{r \to \infty} d\Omega\, r^2 \langle 0 | T_{rt} | 0 \rangle, \tag{55}$$

$$= -\frac{1}{\sqrt{2}\pi} \sum_m \int_{\omega\tilde{\omega}<0} d\omega\,|\,\omega\,| \sum_\lambda (1 - |\,B_{\mathrm{I}}(\omega, m, \lambda)\,|^2) \tag{56a}$$

$$= -\frac{1}{\sqrt{2}\pi}\Big[\sum_{m>0} \int_0^{m\Omega} d\omega\,|\,\omega\,| \sum_\lambda (1 - |\,B_{\mathrm{I}}(\omega, m, \lambda)\,|^2),$$

$$+ \sum_{m<0} \int_{m\Omega}^{0} d\omega\,|\,\omega\,| \sum_\lambda (1 - |\,B_{\mathrm{I}}(\omega, m, \lambda)\,|^2\Big]. \tag{56b}$$

From the ' Wronskian relation ' (33a), it then follows that

$$dM/dt < 0 \tag{57}$$

which proves the existence of spontaneous creation of massive spin half quanta in the classical superradiant modes.

The calculation of outgoing angular momentum flow across a surface at infinity requires the vacuum expectation value of $T_{r\phi}$. Following the same procedure as above and retaining the leading term asymptotically, the final result, after another lengthy calculation, is:

$$\frac{dJ}{dt} = -\frac{1}{2\sqrt{2}\pi}\int d\Omega \Big\{ \sum_{m>0} \int_0^{m\Omega} d\omega \sum_\lambda (1 - |\,B_{\mathrm{I}}(\omega, m, \lambda)\,|^2) \times$$

$$[\,|\,S^+|^2 (\,|\,m\,| + |\,\omega\,|\,a \sin^2\theta + \tfrac{1}{2}\cos\theta) + |\,S^-|^2 (\,|\,m\,| + |\,\omega\,|\,a \sin^2\theta$$

$$- \tfrac{1}{2}\cos\theta)] + \sum_{m<0} \int_{m\Omega}^{0} d\omega \sum_\lambda (1 - |\,B_{\mathrm{I}}(\omega, m, \lambda)\,|^2)$$

$$\times\ [\,|\,S^+|^2 (\,|\,m\,| + |\,\omega\,|\,a \sin^2\theta - \tfrac{1}{2}\cos\theta)$$

$$+ |\,S^-|^2 (\,|\,m\,| + |\,\omega\,|\,a \sin^2\theta + \tfrac{1}{2}\cos\theta)] \Big\} \tag{58}$$

where J is the angular momentum of the black hole. Equation (58) is not as simple as the one for the loss of mass of the black hole. This is not surprising, since the normal modes are eigenstates of energy and the Z-component of orbital angular momentum but *not* of the total angular momentum. Using the 'Wronskian relation' (33a) again, (58) immediately gives,

$$dJ/dt < 0, \tag{59}$$

which shows that spontaneous creation of massive spin half quanta entails a steady loss of rotational energy of the Kerr black hole.

The extension of the above results to the Kerr-Newman (charged-rotating black hole) metric is completely straightforward. The additional term in $T_{\mu\nu}$ for this case is,

$$-\tfrac{1}{2}\, e\, [\,\bar{\Psi}\,\gamma_\mu A_\nu \Psi + \bar{\Psi}\,\gamma_\nu A_\mu \Psi].$$

This term does not contribute to the leading order asymptotically, and the form of the final equations (56), (58), therefore, remains unchanged. The mode coefficients $B_I(\omega, m, \lambda)$ will, however, be different numerically from the Kerr case because of the additional parameter Q (charge) in the radial equations.

It should be noted that the particle creation obtained here is only in the super-radiant modes and vanishes in the zero rotation limit ($a \to 0$). Our calculation refers to what is called an 'eternal' black hole and not a black hole formed through collapse which, as we know, will give Hawking radiation in all modes even in the zero rotation limit. Unruh (1976) has shown that the collapse problem involving a time-dependent metric may be equivalently handled by the corresponding stationary metric with appropriate boundary condition on the past horizon. This essentially involves the so-called ξ-prescription wherein positive frequency is defined via a null-vector field which is of Killing type on the horizon. On the other hand, the present calculation has made use of the so-called η-prescription, wherein the positive frequency is defined with respect to the Killing vector $\partial/\partial t$, which is time-like in the exterior region. Thus, the vacuum employed here is different from the vacuum in the equivalent problem of a collapsing star and the radiation obtained is due to rotation and not the collapse of the black hole.

A final remark: In curved space, there is no *a priori* reason to expect zero energy density for the vacuum because a strong gravitational field, like a strong electric field, can conceivably produce pairs. Normal ordering is, therefore, not a proper procedure here. The covariant regularization of $T_{\mu\nu}$ in curved space is a much-discussed topic currently (e.g. Christensen 1976), but is outside the scope of this work.

5. Conclusions

The spontaneous creation of electron-positron pairs (and other massive fermion pairs) by a rotating black hole was qualitatively indicated in an earlier work (Iyer and Kumar 1977), where it was shown that a strong angular momentum coupling of the field with the spin of the black hole could lead to negative energy orbits with respect to an observer at infinity. In this paper, this emission has been shown to follow as a direct consequence of field quantization in Kerr metric. Estimates of the loss rates of mass and angular momentum can be made by numerical evaluation of the coefficients appearing in the asymptotic behaviour of the normal modes. Because of the rest-mass energy μ required in this case, the emission of massless quanta (neutrinos, photons) is generally expected to dominate over that of the massive case unless the 'temperature' of the black hole (which is inversely proportional to the mass M) is much more than μ (Hawking 1975). Thus the massive particle emission discussed here will not qualitatively change the result (Page 1976) that on the time-scale of the age of the universe, these quantal processes are significant only for very tiny black holes ($M \leqslant 10^{16}$ g).

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