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Second order phase transition in two dimensional sine-Gordon field theory—Lattice model

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Abstract. Two dimensional sine-Gordon (SG) field theory on a lattice is studied using the single-site basis variational method of Drell and others. The nature of the phase transition associated with the spontaneous symmetry breakdown in a SG field system is clarified to be of second order. A generalisation is offered for a SG-type field theory in two dimensions with a potential of the form $[\cos^n(\sqrt{\lambda}/m)\varphi - 1]$.

Keywords. Sine-Gordon system; lattice formulation; variational techniques in field theory; phase transition; Simon-Griffiths theorem; spontaneous symmetry breaking.

1. Introduction

Recently Drell *et al* (DWY) (1976) have developed powerful nonperturbative variational techniques to study field theories on a discrete lattice. The lattice approach to field theory, pioneered by Wilson (1974), is nonperturbative, and permits the study of phenomena for which the usual weak coupling perturbation expansion is useless. It is also useful for the study of strong coupling theories as well as the question of quark confinement.

Phase transition is a large distance behaviour associated with symmetry breaking. Several authors (Dolan and Jackiw 1974; Kirzhnitz and Linde 1972; Weinberg 1974) have studied the role of finite temperature effects in quantum field theories from the point of view of symmetry restoration. For an N-component fermion field theory in one space—one time dimension, Dashen *et al* (1975) have shown that the symmetry breaking disappears at zero critical temperature. A two dimensional φ^4 field system undergoes a second order phase transition when the coupling constant is varied continuously, and there is no conflict with the Simon-Griffiths theorem (Chang 1976; Drell *et al* 1976; Simon and Griffiths 1973), which rules out the possibility of a first order transition for a vanishing external source. Another nonlinear model field theory that has attracted considerable interest in recent years following the discovery of extended objects with particle-like properties, is the sine-Gordon (SG) field theory. This is a theory of a scalar field in one space—one time dimension, and is characterized by a potential $[\cos(\sqrt{\lambda}/m)\varphi - 1]$ (Goldstone and Jackiw 1975; Jackiw 1977; Rajaraman 1975). Recently static, c-number solutions have been obtained in a SG type theory with a potential of the type $\cos^4(\sqrt{\lambda}/m)\varphi$ (Babu Joseph and Shenoi 1978). Coleman (1975) has studied the equivalence between the SG theory and the massive Thirring model and shown that the ground state energy in the Thirring model is unbounded from below for the coupling constant $\beta^2 > 8\pi$ in addition to the instability occuring for $\beta^2 < 0$. Using a lattice formulation of the Thirring model in one space one time dimension, Luther (1976) has elucidated the meaning of this instability and explored ways to circumvent it. In this paper we study the nature of phase transition occurring in a two dimensional SG theory formulated on a lattice by DWY single site basis variational method. The main conclusion arrived at by us is that the phase transition involved is of second order and is in accord with the Simon-Griffiths theorem. Similar remarks apply to modified SG theories characterised by a potential $[\cos^n(\sqrt{\lambda}/m)\,\varphi - 1]$.

2. Two dimensional SG field theory on a lattice

The two dimensional SG theory of a scalar field is defined by the lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\frac{\partial\varphi(x,t)}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2 - \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}}{m}\varphi\right) - 1\right]. \tag{1}$$

This gives a hamiltonian

$$H = \int dx \left[\frac{1}{2}\pi^2(x,t) + \frac{1}{2}\left(\frac{\partial\varphi(x,t)}{\partial x}\right)^2 + \frac{m^4}{\lambda}\left(\cos\left(\frac{\sqrt{\lambda}}{m}\varphi\right) - 1\right]\tag{2}$$

where $\pi(x,t) = \dot{\varphi}(x,t)$. The lagrangian in eq. (1) possesses the discrete symmetries

$$\frac{\sqrt{\lambda}}{m}\varphi \longleftrightarrow \pm\frac{\sqrt{\lambda}}{m}\varphi \pm 2n\pi$$

where $n = 0, 1, 2, 3, \ldots$. The equation of motion of the SG field is

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}\varphi\right) = 0. \tag{3}$$

There exist infinitely many discrete degenerate minima of the potential at $\varphi = \pm(m/\sqrt{\lambda})\,2\pi n$ $(n = 0, 1, 2, 3, \ldots)$ and these could lead to spontaneous symmetry breaking. In the usual way we assume that there exist 'meson' states which are energy and momentum eigenstates with the meson mass given in the lowest order by m. This model also possesses static finite energy solutions

$$\varphi_c(x) = \pm\frac{4m}{\sqrt{\lambda}}\tan^{-1}\exp\left(\pm m(x - x_0)\right) \tag{4}$$

with energy

$$E_c(\varphi_c) = \frac{8m^3}{\lambda} \tag{5}$$

These, however, represent fermions.

Passage from the continuum field theory to the lattice model (Banks *et al* 1976, Drell *et al* 1976, Kogut *et al* 1975, Wilson 1974) is carried out in the following manner. The continuous space is replaced by a discrete lattice of linear dimension L with a lattice spacing $1/\Lambda$ defined so that there are $2N+1$ points on a side. Instead of the continuous coordinate x we introduce a discrete label j for each site in the lattice such that $j=0, \pm 1, \pm 2, \ldots, \pm N$. One dimensional volume integration is replaced by a summation according to

$$\int dx \to \frac{1}{\Lambda} \sum_j . \tag{6}$$

Further correspondence formulae which will be required later are

$$\begin{aligned} \pi(x) \to \pi_j &= \sum_{k=-k_{\max}}^{+k_{\max}} \exp(ikj/\Lambda)\, \pi(k) \\ \pi(k) &= \frac{1}{2N+1} \sum_j \pi_j \exp(-ikj/\Lambda) \\ \varphi(x) \to \varphi_j &= \sum_{k=-k_{\max}}^{+k_{\max}} \exp(ikj/\Lambda)\, \varphi(k) \\ \varphi(k) &= \frac{1}{2N+1} \sum_j \varphi_j \exp(-ikj/\Lambda) \end{aligned} \tag{7}$$

where $k=2\pi n/L$ ($n=0, \pm 1, \pm 2, \ldots$) so that $k_{\max}=2\pi N/L$.

The hamiltonian, formulated on the lattice reads

$$H = \frac{1}{\Lambda} \sum_j \left[\frac{1}{2}\pi_j^2 + \frac{1}{2}(\partial_j \varphi_j)^2 + \frac{m^4}{\lambda}\left(\cos\left(\frac{\sqrt{\lambda}}{m} \varphi_j \right) - 1 \right) \right]. \tag{8}$$

The DWY definition of the gradient has the advantage that it does not lead to a doubling of the field degrees of freedom in the fermionic case. Following them we consider an arbitrary function f_j so that

$$f_j = \sum_k \exp(ikj/\Lambda)\, f(k). \tag{9}$$

The gradient operator ∂_j is defined to satisfy

$$\begin{aligned} \partial_l f_l &= \sum_k ik \exp(ikl/\Lambda)\, f(k) \\ &= \sum_{l'} f_{l'} \left[\frac{1}{2N+1} \sum_k \exp\left(ik(l-l')/\Lambda\right) ik \right]. \end{aligned} \tag{10}$$

Accordingly we have

$$\int dx \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 \to \frac{1}{\Lambda}\sum_j \frac{1}{2}(\partial_j\varphi_j)^2$$

$$= \frac{L}{2}\sum_k k^2\,\varphi(k)\,\varphi(-k)$$

$$= \frac{1}{\Lambda}\sum_{j,j'} \frac{1}{2}\Lambda^2\,\varphi_j\varphi_{j'}\,D(j-j') \tag{11}$$

where

$$D(j-j') = \frac{1}{2N+1}\sum_k \frac{k^2}{\Lambda^2}\exp\,[ik\,(j-j')/\Lambda]. \tag{12}$$

DWY give the explicit results

$$D(j) = \begin{cases} \dfrac{4N(N+1)}{(2N+1)^2}\,\dfrac{\pi^2}{3} \xrightarrow[N\to\infty]{} \dfrac{\pi^2}{3} & \text{for } j=0 \\[2ex] \dfrac{(2\pi)^2\,(-1)^j}{2\,(2N+1)^2}\,\dfrac{\cos\,(\pi j/2N+1)}{\sin\,(\pi j/2N+1)^2} \xrightarrow[N\to\infty]{} \dfrac{2(-1)^j}{j^2} & \text{for } j\neq 0. \end{cases}$$

Rescaling φ_j and π_j according to

$$\varphi_j = x_j \tag{13}$$

$$\pi_j = \Lambda p_j$$

the hamiltonian in eq. (8) is rewritten as

$$H = \Lambda \sum_j \left[\tfrac{1}{2}P_j^2 + \tfrac{1}{2}D\,(0)\,x_j^2 + \gamma\left(\cos\,(\beta x_j) - 1\right)\right]$$

$$+ \Lambda \sum_{j_1\neq j_2}\left[\tfrac{1}{2}D\,(j_1 - j_2)\,x_{j_1}x_{j_2}\right] \tag{14}$$

where $\beta = \sqrt{\lambda}/m$ and $\gamma = m^4/\lambda\Lambda^2$.

The energy of the field system in one of the vacua corresponding to spontaneous breakdown of symmetry satisfies the eigenvalue equation

$$H\,|\,\psi\,\rangle = E\,|\,\psi\,\rangle. \tag{15}$$

This state $|\psi\rangle$ will hereafter be referred to as the ground state of the system. In order to perform a variational calculation we construct $|\psi\rangle$ as the product of eigenstates at each site j

$$|\psi\rangle = \prod_j |\psi_j\rangle \tag{16}$$

where correlations between different sites are ignored. This approximation yields a 'single-site basis.' The trial states $|\psi_j\rangle$ are constructed by introducing creation and annihilation operators through the field variables x_j and p_j:

$$\begin{aligned} x_j &= \frac{1}{2\omega_j}(a_j + a_j^\dagger) \\ ip_j &= \left(\frac{\omega_j}{2}\right)(a_j - a_j^\dagger) \end{aligned} \tag{17}$$

where ω_j is some parameter. The annihilation operator destroys the vaccum $|0_j\rangle$ at the site j in the sense

$$a_j|0_j\rangle = 0. \tag{18}$$

For the meson sector of the Hilbert space of the SG field system we postulate the commutation relations (Aoyama and Kodama 1976)

$$[a_j, a_{j'}^\dagger] = \delta_{jj'}. \tag{19}$$

We define

$$|n_j\rangle = \frac{1}{\sqrt{n_j!}}(a_j^+)^{n_j}|0_j\rangle \tag{20}$$

and consider $|\psi_j\rangle$ to be a linear superposition

$$|\psi_j\rangle = \sum_{n_j=0}^{\infty} C^j_{n_j}|n_j\rangle \tag{21}$$

satisfying

$$\langle\psi_j|\psi_{j'}\rangle = \delta_{jj'}. \tag{22}$$

The ground state energy expectation value using the trial state $|\psi\rangle$ is

$$E = \Lambda\Big[\sum_j \langle\psi_j|\tfrac{1}{2}p_j^2 + \tfrac{1}{2}D(0)\,x_j^2 + \gamma\left(\cos(\beta x_j) - 1\right)|\psi_j\rangle + \tfrac{1}{2}\sum_{j_1\neq j_2} D(j_1 - j_2)\,\langle\psi_{j_1}|x_{j_1}|\psi_{j_1}\rangle\langle\psi_{j_2}|x_{j_2}|\psi_{j_2}\rangle\Big]. \tag{23}$$

Assuming translational invariance of $|\psi_j\rangle$ we have

$$\langle \psi_j | x_j^n | \psi_j \rangle \rightarrow \langle \psi | x^n | \psi \rangle$$
$$\langle \psi_j | p_j^2 | \psi_j \rangle \rightarrow \langle \psi | p^2 | \psi \rangle \tag{24}$$

Again from eq. (12) it follows that

$$\sum_{j_1} D(j_1 - j_2) = 0$$
$$\sum_{j_1 \neq j_2} D(j_1 - j_2) = -\sum_{j} D(0) = -(2N+1)\, D(0). \tag{25}$$

Applying these in eq. (23)

$$E = (2N+1)\, \Lambda\, [\, \langle \psi | \tfrac{1}{2} p^2 + \tfrac{1}{2} D(0)\, x^2 + \gamma (\cos(\beta x) - 1) | \psi \rangle$$
$$- \tfrac{1}{2} D(0) \langle \psi | x | \psi \rangle^2]. \tag{26}$$

In the DWY variational method a source term Jx is introduced by defining an operator $\bar{H}(J)$ as

$$\bar{H}(J) = \tfrac{1}{2} p^2 + \tfrac{1}{2} D(0)\, x^2 + \gamma (\cos(\beta x) - 1) - Jx. \tag{27}$$

We calculate the expectation value of $\bar{H}(J)$ in a trial state $|\psi_0\rangle = \exp(-i\langle x\rangle p\, |\, 0\rangle$ which is a displaced gaussian formed from eqs (17) and (18) with $\langle x\rangle = \langle \psi_0 | x | \psi_0\rangle$. Thus

$$\Gamma(J) = \langle \psi_0 | \bar{H}(J) | \psi_0 \rangle$$
$$= \frac{\omega^2}{8} + \frac{D(0)}{8\omega^2} + \frac{D(0)}{2} \langle x\rangle^2 + (\gamma \cos(\beta\langle x\rangle))\, K(\omega) - \gamma - J\langle x\rangle \tag{28}$$

where

$$K(\omega) = 1 - \frac{\beta^2}{2!} \frac{1}{(2\omega)^2} + \frac{\beta^4}{4!} \frac{3}{(2\omega)^4} + \ldots .$$

It immediately follows that

$$\frac{\partial \Gamma(J)}{\partial J} = - \langle \psi_0 | x | \psi_0 \rangle = - x(J). \tag{29}$$

For small J, invoking parity considerations to yield a nonvanishing vacuum expectation value $\langle x\rangle$, we try an expansion

$$x(J) = \sum_{r=0}^{\infty} C_{2r+1} J^{2r+1} \tag{30}$$

which gives

$$\Gamma(J) = \Gamma(0) - \tfrac{1}{2} C_1 J^2 - \tfrac{1}{4} C_3 J^4 + \ldots. \tag{31}$$

For $J \ll 1$, this can be inverted and we obtain

$$J(x) = \frac{1}{C_1} x + \frac{C_3}{C_1^4} x^3 + \ldots. \tag{32}$$

From eqs (26) and (27),

$$\frac{E(J)}{(2N+1)\,\Lambda} = E(J) = \Gamma(J) + Jx(J) - \tfrac{1}{2} D(0)\, x^2(J) \tag{33}$$

or

$$Ex(J) = \Gamma(0) - \frac{\eta x^2}{2C_1} + \frac{C_3}{C_1^4}(\tfrac{1}{4} + \eta)\, x^4 \ldots \tag{34}$$

where

$$\eta = C_1 D(0) - 1. \tag{35}$$

Let us expand $\omega(J)$ as a power series for small J

$$\omega(J) = \sum_{s=0}^{\infty} \omega_{2s} J^{2s}. \tag{36}$$

The variational conditions are

$$\frac{\partial \Gamma(J)}{\partial \omega} = 0 \tag{37}$$

and

$$\frac{\partial \Gamma(J)}{\partial \langle x \rangle} = 0. \tag{38}$$

From these conditions we obtain

$$\frac{\omega_0{}^4 - D(0)}{\gamma} = -\beta^2 \tag{39}$$

$$C_1 = \frac{1}{(\omega_0)^4} > 0$$

$$\omega_2 = \frac{\gamma\beta^4}{8(\omega_0)^{11}}$$

$$C_3 = \frac{\gamma\beta^4}{2(\omega_0)^{16}}\left[\frac{\gamma\beta^4}{16(\omega_0)^6} - \frac{1}{3}\right] \tag{40}$$

where it has been assumed that $\beta \ll \omega_0$.
In eq. (34) $Ex(J)$ will be a minimum for

$$\text{(i) } 0 < \eta \ll 1 \quad \text{and} \quad \text{(ii) } C_3 < 0.$$

The value of x corresponding to the minimum of $Ex\,(J)$, denoted by x_c, is given by

$$x_c^2 = \frac{\eta C_1^3}{|C_3|}(1-4\eta). \tag{41}$$

From eqs (35) and (40),

$$\omega_0 = [D(0)]^{1/4}\left[1 - \frac{\eta}{4} + O(\eta^2)\right]. \tag{42}$$

Hence from eq. (39)

$$\gamma\beta^2 = m^2 = (1+2\eta)\,\eta\, D(0) + O(\eta^3) \tag{43}$$

Since $C_3 < 0$,

$$\gamma\beta^4 < \frac{16}{3}(\omega_0)^6$$

or

$$\lambda < \frac{16}{3}(D\,(0))^{6/4}. \tag{44}$$

Hence we have established the condition for $0 < x_c \ll 1$, which may be stated as

$$m^2 > m_{cr}^2 = 0$$

$$\lambda < \lambda_{cr} \xrightarrow[N\to\infty]{} \frac{16}{3}\left(\frac{\pi^2}{3}\right)^{6/4}. \tag{45}$$

Choosing the parameters m and λ corresponding to the critical values given in the last equation, it follows that

$$\langle x \rangle = x_c \propto \left[\frac{\eta D(0)}{\lambda}\right]^{1/2} \propto \left[\frac{(m^2 - m_{cr}^2)}{\lambda}\right]^{1/2} \tag{46}$$

Note that eq. (44) supports the assumption made earlier, namely $\beta \ll \omega_0$.

Thus we show that x_c can be made arbitrarily small by appropriate choice of m and λ without encountering a false minimum at $\langle x \rangle = 0$. This implies that a transition from the $\langle x \rangle = 0$ phase to the $\langle x \rangle \neq 0$ phase can occur by a smooth variation of the constants, and the critical point corresponds to $m_{cr} = 0$ and $\lambda_{cr} = \frac{16}{3}(\pi^2/3)^{6/4}$. This is not a first order phase transition which is distinguished by a discontinuous change from one phase to another. Since this is a second order phase transition there is no contradiction with the Simon-Griffiths theorem.

3. Sine-Gordon type theories

Consider a SG potential of the type [$\cos^n (\sqrt{\lambda}/m) \varphi - 1$] where n is a positive integer. Proceeding as in section 2, we get

$$\frac{\omega_0^4 - D(0)}{\gamma} = -\beta^2 n \tag{47}$$

$$C_1 = \frac{1}{(\omega_0)^4}$$

$$\omega_2 = \frac{\gamma\beta^4}{8(\omega_0)^{11}} (n(3n-2)) \tag{48}$$

$$C_3 = \frac{\gamma\beta^4}{2(\omega_8)^{16}} \left[\frac{\gamma\beta^4}{16(\omega_0)^6} n(3n-2) - \frac{1}{3}\right] n(3n-2).$$

At the end of a long calculation similar to that of section 2 we arrive at the following result

$$m^2 = \frac{1}{n}[D(0)\eta + D(0)\, 2\eta^2 + O(\eta^3)] > m_{cr}^2 = 0 \tag{49}$$

$$\lambda = \frac{1}{n(3n-2)} \frac{16}{3} (D(0))^{6/4} [1 - \frac{6}{4}\eta + \ldots]$$

$$< \lambda_{cr} = \frac{1}{n(3n-2)} \frac{16}{3} (D(0))^{6/4}. \tag{50}$$

We therefore, conclude that a generalised SG field theory with a potential [$\cos^n(\sqrt{\lambda}/m)\varphi - 1$] undergoes a phase transition of the kind discussed in section 2. The constants m_{cr} and λ_{cr} for a SG-type theory characterised by a potential $\cos^n(\sqrt{\lambda}/m)\varphi$ (n a positive integer) are the same as those given in eqs (49) and (50).

4. Conclusions

In nonlinear scalar field theories such as φ^4 and SG theories a phase transition corresponds to spontaneous breaking of the symmetry and this causes the emergence of a soliton sector populated by fermions. In the SG field theory studied in the present work this phase transition has been shown to be of second order and a clash with the Simon-Griffiths theorem is avoided. Similar conclusions are drawn in respect of modified SG theories. Whether or not there is a soliton sector in a modified SG theory based on a potential [$\cos^n (\sqrt{\lambda}/m)\varphi - 1$] where n is a positive integer, has not been established except in the case $n=4$ (Babu Joseph and Shenoi 1978). Comparing the result obtained herein regarding the nature of the phase transition occurring in a modified SG theory with similar results in φ^4 and SG theories we may be tempted to speculate about the possible existence of a soliton sector in a modified SG theory. There seems to be a link between the nature of the phase transition and emergence of the soliton sector in a nonlinear scalar field theory.

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