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Algebraic aspects of the Wigner distribution in quantum mechanics

N MUKUNDA
Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560 012

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Abstract. The algebraic structure underlying the method of the Wigner distribution in quantum mechanics and the Weyl correspondence between classical and quantum dynamical variables is analysed. The basic idea is to treat the operators acting on a Hilbert space as forming a second Hilbert space, and to make use of certain linear operators on them. The Wigner distribution is also related to the diagonal coherent state representation of quantum optics by this method.

Keywords. Wigner distribution; Weyl correspondence; Weyl representation.

1. Introduction

The Wigner distribution function was introduced into quantum mechanics more than four decades ago in a study of quantum corrections to classical statistical mechanics (Wigner 1932). It involves describing the states of a quantum mechanical system by means of c-number functions defined on the corresponding classical phase space in a certain way, in contrast to describing them via, say, Schrödinger wave functions. Several years later it became clear through the work of Groenewold and Moyal that this description of quantum states had a natural connection with a description of operators in quantum mechanics given by Weyl much earlier (Groenewold 1946; Moyal 1949; Weyl 1931 pp. 274, 275). This description of operators, known as the Weyl correspondence, amounts to a rule according to which one can set up a one-to-one correspondence between the operator dynamical variables of a quantum mechanical system and the c-number phase space functions that are the dynamical variables of the associated classical system. The connection between the Wigner distribution and the Weyl correspondence is then the following. The expectation value of an operator in a state is expressible as the phase space integral of the product of the c-number dynamical variable and the Wigner distribution function corresponding to the operator and the state respectively.

These ideas have been used by Baker to make the Wigner distribution function the basis of a system of postulates intended to give a complete description of quantum kinematics and dynamics (Baker 1958). On the other hand, it is well known that there is no natural or intrinsically preferred correspondence between the operator dynamical variables of a quantum system and the dynamical variables of the classical system that is its analogue. Every correspondence, including that of Weyl, is ultimately a convention. Several authors have therefore studied various possible alternatives to the Weyl correspondence motivated from different points of view and analysed the relationships between these alternatives. Each of these alternative correspondences naturally brings with it a replacement for the Wigner distribution function and these too have been studied systematically (see Mehta 1964; Agarwal and Wolf 1970; and other references cited therein). It must be mentioned, however, that the Weyl correspondence seems to be the most convenient one in examining the semi-classical limit of quantum mechanics; recent work of Berry studies this question in detail (Berry 1977).

As is already clear from the reference to the work of Baker, the use of the Wigner distribution gives us an exact mathematical description of the states of a quantum system and not just of their semi-classical limiting forms. This suggests that one should achieve an understanding of the Wigner distribution purely within the mathematical framework of quantum mechanics, in contrast to the tendency to view it as the result of first setting up the Weyl correspondence between classical and quantum dynamical variables. An analogy with the description of quantum mechanics at the state vector level will make this point clear. Here one views the abstract vectors of Hilbert space as primary objects and Schrödinger wavefunctions as but one way of describing them, rather than the other way around. In the same spirit it would be worthwhile to understand the Weyl correspondence and the Wigner distribution as aspects of the description of quantum mechanical operators and their mutual relationships in themselves. This is even necessary from the following point of view. When the kinematics of a quantum system is based on Cartesian position and momentum variables, we know that the spectra of these operators suffer no quantisation, so the use of the notion of a corresponding phase space is permissible (though slightly misleading). On the other hand, if the quantum kinematics is based on other kinds of operators whose spectra are quantized when compared to the possible values of their classical limits, the situation is changed entirely. It is therefore useful to understand the algebraic structure behind the Wigner distribution in a direct fashion, valid for both kinds of quantum kinematics; and allow a correspondence between quantum and classical variables to emerge in those cases where it exists.

A possible way to achieve the aims of the previous paragraph lies in viewing the linear operators acting on a given Hilbert space as forming a Hilbert space in their own right, with a suitably defined inner product. It has recently been shown elsewhere that this point of view greatly simplifies the structure and meaning of the diagonal coherent state representation in quantum optics (Mukunda and Sudarshan 1978). It involves the use of what are often called superoperators—these are linear operators acting on the space of operators on a given Hilbert space. We shall call them *operations* to convey on the one hand their mathematical similarity to operators representing physical observables, except that they do not act on state vector space, and also to convey on the other hand the fact that they do not represent observables but play a different role. It is the aim of this paper to show in detail how these methods can be used to understand the Wigner distribution from a new angle.

The contents of the paper are arranged as follows. Section 2 studies the quantum mechanics of a system with one degree of freedom described by canonical Cartesian position and linear momentum variables. Well known facts such as the unitary nature of the operator basis used in the Weyl representation of operators, and the reality of the Wigner distribution, are directly related to properties of suitable operations on operators. In section 3 the case of a quantum system described by an angle variable and its conjugate, usually a component of angular momentum, is studied. We show that the present approach reproduces a prescription for setting up the Wigner distribution in this case, suggested elsewhere (Mukunda 1978). Section 4 is concerned with the relation between the Wigner distribution and the diagonal coherent state representation. This relation has been known for some time (Mehta 1964; Agarwal and Wolf 1970); the methods of this paper supply a proper operator basis for its understanding. Section 5 contains some comments and questions.

2. The case of Cartesian variables

We consider a quantum system with one Cartesian degree of freedom. The basic operators $\hat{q}$ and $\hat{p}$ obey the standard Heisenberg commutation relation

$$[\hat{q}, \hat{p}] = i\hbar, \tag{1}$$

and form an irreducible set in the sense that any other operator is expressible as a function of them. We denote by $\mathcal{H}$ the Hilbert space on which $\hat{q}$ and $\hat{p}$ act. Eigenvalues of $\hat{q}$ $(\hat{p})$ will be denoted by $q, q', \ldots$ $(p, p', \ldots)$; the corresponding eigenvectors satisfy

$$\begin{aligned} \langle q' \mid q \rangle &= \delta(q'-q), \ \langle p' \mid p \rangle = \delta(p'-p); \\ \langle q \mid p \rangle &= (2\pi\hbar)^{-1/2} \exp(iq\,p/\hbar). \end{aligned} \tag{2}$$

The finite unitary transformations generated by $\hat{q}$ and $\hat{p}$ act on the eigenvectors of $\hat{p}$ and $\hat{q}$ respectively in this manner:

$$\exp(i\sigma\hat{q}) \mid p\rangle = \mid p + \hbar\sigma\rangle, \quad \exp(-i\tau\hat{p}) \mid q\rangle = \mid q + \hbar\tau\rangle. \tag{3}$$

Given the Hilbert space $\mathcal{H}$ we construct a second Hilbert space $\mathscr{K}$ in this way: the elements of $\mathscr{K}$ are all linear operators $A, B, \ldots$ on $\mathcal{H}$ of Hilbert-Schmidt class and the inner product in $\mathscr{K}$ is defined as

$$(A, B) = \mathrm{Tr}\,(A^{\dagger} B). \tag{4}$$

An operator A is of Hilbert-Schmidt class precisely when (A, A) is finite. It is intuitively obviously that $\mathscr{K}$ must be very similar to the Hilbert space one would use to describe the states of a quantum system with two degrees of freedom, because $\mathscr{K}$ is essentially the tensor product of $\mathcal{H}$ with itself. We exploit this fact in this way. We define four operations, that is, linear operators on $\mathscr{K}$, through their effects on a general element A in $\mathscr{K}$:

$$\begin{aligned} \omega(\hat{q})\,A &= \tfrac{1}{2}\{\hat{q}, A\}, \quad \omega(\hat{p})\,A = \tfrac{1}{2}\{\hat{p}, A\}, \\ \Omega(\hat{q})\,A &= [\hat{q}, A], \qquad \Omega(\hat{p})\,A = [\hat{p}, A]. \end{aligned} \tag{5}$$

(Curly brackets denote anticommutators). All of these operations are hermitian with respect to the scalar product (4). With the help of eq. (1) it is now easy to show that $\omega(\hat{q})$, $\Omega(\hat{p})$ form a canonical pair while $\omega(\hat{p})$ and $-\Omega(\hat{q})$ form another independent canonical pair. Thus one finds:

$$[\omega(\hat{q}), \Omega(\hat{p})] = [\omega(\hat{p}), -\Omega(\hat{q})] = i\hbar, \tag{6}$$

with all other commutators vanishing. We shall view the ω as 'positions' and the Ω as 'momenta'.

A basis for $\mathscr{H}$ can be constructed using the simultaneous 'eigenvectors' of the two commuting operations $\Omega(\hat{p})$ and $\Omega(\hat{q})$. If we let σ and τ denote eigenvalues of these operations, with $-\infty<\sigma, \tau<\infty$, the corresponding 'eigenvectors' turn out to be the 'displacement operators'

$$D(\sigma, \tau) = \exp(i\sigma\hat{q} - i\tau\hat{p}). \tag{7}$$

One has, in fact,

$$\Omega(\hat{p})\, D(\sigma, \tau) = \hbar\sigma\, D(\sigma, \tau),$$

$$\Omega(\hat{q})\, D(\sigma, \tau) = \hbar\tau\, D(\sigma, \tau). \tag{8}$$

These relations have been noted a long time ago in Weyl (1931 p. 275). A formal but elementary argument shows that these operators can be used as a basis for the space of operators (For a rigorous discussion, see Pool 1966; see also Agarwal and Wolf 1970). We present it here only because a similar argument will be needed in the next section. We begin with the following obvious expression for the projection operator onto an eigenvector $|q\rangle$ of $\hat{q}$:

$$|q\rangle\langle q| = \delta(q - \hat{q}) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\sigma \exp(i\sigma(\hat{q} - q)). \tag{9}$$

If we now take account of eq. (3) and apply a suitable exponential in $\hat{p}$, we get

$$|q'\rangle\langle q| = \exp(-i(q'-q)\hat{p}/\hbar)\, |q\rangle\langle q|$$

$$= (2\pi)^{-1} \int_{-\infty}^{\infty} d\sigma \exp\left(-i\sigma(q'+q)/2\right) D(\sigma, (q'-q)/\hbar), \tag{10}$$

which shows that the operators $D(\sigma, \tau)$ do form a basis in terms of which a general operator can be expanded. That they in fact form an ortho-normal basis (apart from factors) for $\mathscr{H}$ follows from a straightforward calculation that shows:

$$\left(D(\sigma', \tau'), D(\sigma, \tau)\right) = (2\pi/\hbar)\, \delta(\sigma' - \sigma)\, \delta(\tau' - \tau). \tag{11}$$

As a result, a general operator A in $\mathscr{H}$ possesses an expansion of the form

$$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma \, d\tau \, a(\sigma, \tau) \, D(\sigma, \tau) \tag{12}$$

the ' Weyl weight ' $a(\sigma, \tau)$ being expressible as

$$a(\sigma, \tau) = (\hbar/2\pi) \big(D(\sigma, \tau), A\big). \tag{13}$$

The inner product (4) in $\mathscr{H}$, when written in terms of Weyl weights, appears as

$$(A, B) = (2\pi/\hbar) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma \, d\tau \, a(\sigma, \tau)^* \, b(\sigma, \tau), \tag{14}$$

so that the Hilbert-Schmidt property for A is the same as square integrability of its Weyl weight.

While the properties of the displacement operators $D(\sigma, \tau)$ recounted above are not particularly new, the value of regarding them as ' eigenvectors ' of the operations $\Omega(\hat{q})$, $\Omega(\hat{p})$ must be clear. This viewpoint in fact leads immediately to two other familiar properties of these operators. The circumstance that the Ω's are defined through the process of *commutation* shows immediately that the product of two displacement operators $D(\sigma, \tau)$ and $D(\sigma', \tau')$ is an ' eigenvector ' of $\Omega(\hat{p})$ and $\Omega(\hat{q})$ with eigenvalues $\sigma+\sigma'$, $\tau+\tau'$ respectively and so can differ at most by a numerical factor from $D(\sigma+\sigma', \tau+\tau')$. So the form of the multiplication law

$$D(\sigma', \tau') \, D(\sigma, \tau) = \exp\big(i\hbar \, (\sigma'\tau - \tau'\sigma)/2\big) \, D(\sigma' + \sigma, \tau' + \tau) \tag{15}$$

which is actually the Weyl form of eq. (1), gets related to the form of the Ω's. By the same token, namely using again the presence of commutators in the definitions of $\Omega(\hat{p})$ and $\Omega(\hat{q})$, one sees that for any σ and τ, $D(\sigma, \tau)^\dagger \, D(\sigma, \tau)$ is an ' eigenvector ' of $\Omega(\hat{p})$ and $\Omega(\hat{q})$ with zero as eigenvalue for both; but $D(0, 0)$, by the defining eqs. (8), commutes with $\hat{q}$ as well as $\hat{p}$ and so is a multiple of the unit operator. Thus the unitarity property of the displacement operators is directly related to their being ' eigenvectors ' of operations defined through commutation. (In this connection see the discussion of Schwinger (1960) on unitary operator bases for finite dimensional spaces).

The basis $D(\sigma, \tau)$ for $\mathscr{H}$ is mathematically similar to a basis of ' momentum eigenfunctions ' in quantum mechanics. We get an alternative basis of ' position eigenfunctions' for $\mathscr{H}$ by finding the simultaneous 'eigenvectors' of $\omega(\hat{q})$ and $\omega(\hat{p})$. Let us write $W(q, p)$ for these:

$$\omega(\hat{q}) \, W(q, p) = qW(q, p), \ \omega(\hat{p}) \, W(q, p) = pW(q, p). \tag{16}$$

Since the ω are defined through *anticommutation*, $W(q, p)^\dagger$ obeys these same equations, so the essential hermiticity of $W(q, p)$ follows. If we use eq. (2) twice over, we get the inner product of $W(q, p)$ with $D(\sigma, \tau)$. With suitable choice of numerical factors we have

$$\big(W(p, p),\, D(\sigma, \tau)\big) = (2\pi\hbar)^{-1} \exp(i\sigma q - i\tau p). \tag{17}$$

This determines that the elementary exponential on the right is essentially the Weyl weight of $W(q, p)$, so using eqs (13) and (14) we get:

$$W(q, p) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\sigma\, d\tau \exp(-i\sigma q + i\tau p)\, D(\sigma, \tau),$$

$$\big(W(q', p'),\, W(q, p)\big) = (2\pi\hbar)^{-1}\, \delta(q' - q)\, \delta(p' - p). \tag{18}$$

We shall call the basis $W(q,p)$ for $\mathscr{H}$, the Wigner operator basis, as contrasted with the Weyl basis $D(\sigma, \tau)$. (The operators $W(q, p)$ are an instance of Δ-operators of Agarwal and Wolf 1970). The hermiticity of the former and the unitarity of the latter reflect the modes of construction of the ω and the Ω. If we expand an operator A belonging to $\mathscr{H}$ in the form

$$A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, dp\, A(q, p)\, W(q, p),$$

$$A(q, p) = 2\pi\hbar\, (W(q, p), A), \tag{19}$$

we may call $A(q, p)$, which is a c-number function of two independent variables, the Wigner representative of A. It is the double Fourier transform of the Weyl weight of A. Hermitian operators have real Wigner representatives. We reserve the term 'Wigner distribution' for the case when A happens to be a density operator ρ describing a general statistical state. Density operators are hermitian non-negative elements of $\mathscr{H}$ with unit trace; for them we include a numerical constant in the expansion (19) and write

$$\rho = 2\pi\hbar \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, dp\, \rho(q, p)\, W(q, p). \tag{20}$$

The real c-number function $\rho(q, p)$ is the Wigner distribution corresponding to the state ρ. The pure state condition is

$$(\rho, \rho) = 2\pi\hbar \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, dp\, (\rho(q, p))^2 = 1; \tag{21}$$

for a general state this expression is less than unity. Combining eqs (18), (19) and (20), the expectation value of an observable A in the state ρ takes the form

$$\langle A \rangle = (\rho, A) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dq\, dp\, \rho(q, p)\, A(q, p). \tag{22}$$

All the essential elements of the usual descriptions of the Weyl correspondence and Wigner distribution are contained in the equations developed above. The former is really the expansion (19) if we think of $A(q, p)$ as a classical dynamical variable expressed as a function of classical phase space variables q, p; then A is the quantum operator determined by it via Weyl's rule. However we view $A(q, p)$ as the expansion coefficient of A corresponding to using the Wigner basis for $\mathscr{K}$. In any case, eq. (19) makes the statement of Weyl's correspondence very concise. A parallel interpretation for $\rho(q, p)$ exists. For the case where ρ is a pure state corresponding to the Schrödinger wave function $\psi(q)$, one easily checks that $\rho(q, p)$ is the expected expression

$$\rho(q, p) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\tau\, \psi(q+\hbar\tau/2)^* \,\psi(q-\hbar\tau/2) \exp(i\tau p). \tag{23}$$

The product law for the elements of the Wigner basis is naturally more complex in form than eq. (15), since anticommutation with a fixed operator is not a derivation. In any case, a product of two W's can be expanded linearly in terms of W's as:

$$W(q', p')\, W(q, p) = (\pi\hbar)^{-2} \exp(2i(q'p - p'q)/\hbar) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq''\, dp'' \exp\left(\frac{2i(q''(p'-p)-p''(q'-q))}{\hbar}\right) W(q'', p''). \tag{24}$$

Using this, one can show that if the commutator of two operators A and B is the operator C, then the Wigner representative of C is just the Moyal bracket (Moyal 1949) of the representatives of A and B; we omit the details.

The final point to be mentioned in this section is the classical limit of the operations $\omega(\hat{q})$, $\omega(\hat{p})$ and of the Wigner basis elements $W(q, p)$. Let us in this limit continue to use the symbols $\hat{q}$, $\hat{p}$, these denoting classical dynamical variables, while the letters q, p stand for numbers. It is clear that $\omega(\hat{q})$ and $\omega(\hat{p})$ tend to the operations of multiplying a general function $f(\hat{q}, \hat{p})$ by $\hat{q}$ and by $\hat{p}$ respectively. One therefore expects $W(q, p)$ to tend, in this limit, to a product of two delta functions in the variables (or expressions) $\hat{q}-q$ and $\hat{p}-p$. Guided by this expectation, we can show that for finite $\hbar$ the operator $W(q, p)$ as set up in eq. (18) can be cast into the following rather formal but suggestive form:

$$W(q, p) = \delta\left(\hat{q} - q + i\frac{\hbar}{2}\frac{\partial}{\partial p}\right) \delta(\hat{p} - p). \tag{25}$$

The operator of partial differentiation with respect to p occurring inside the first delta function is meant to act on the p present in the second delta function; of course the order of the two delta functions must be preserved because $\hat{q}$ and $\hat{p}$ do not commute. In spite of being formal, this expression exhibits the classical limit very clearly.

To summarise this section, we have shown that Wigner representatives $A(q, p)$ and Wigner distributions $\rho(q, p)$ are as natural mathematical descriptions of observables A and states ρ, as Schrödinger wavefunctions $\psi(q)$ are of abstract state vectors $|\psi\rangle$. So also the expression (22) for the expectation of A in the state ρ in the form of a 'phase space average' is exactly like the expression of the inner product $\langle\phi|\psi\rangle$ of two vectors as an integral over the product of their wavefunctions. In all this, no specific reference need be made to the corresponding classical system.

3. The case of angle variables

Let us consider next a canonical pair of operators $\hat{\theta}$, $\hat{M}$, where the eigenvalues θ of the former lie in the range $-\pi$ to π while those of the latter are quantised and consist of the set $m\hbar$, $m=0, \pm 1, \pm 2, \ldots$. The operator relationship between $\hat{\theta}$ and $\hat{M}$ is no longer expressible via a simple commutation rule such as (1). Instead we set down directly equations similar to (2) for the eigenvectors $|\theta\rangle$, $|m\rangle$ of $\hat{\theta}$, $\hat{M}$ respectively:

$$\langle\theta'|\theta\rangle = \delta(\theta'-\theta), \quad -\pi < \theta,\ \theta' < \pi;$$

$$\langle m'|m\rangle = \delta_{m',m}, \quad m', m = 0, \pm 1, \pm 2, \ldots;$$

$$\langle\theta|m\rangle = (2\pi)^{-1/2} \exp(im\theta). \tag{26}$$

The vectors $|\theta\rangle$ and the vectors $|m\rangle$ give two orthonormal bases for the Hilbert space $\mathcal{H}$ on which $\hat{\theta}$, $\hat{M}$ act. As replacement for eq. (3) we have:

$$\exp(in\hat{\theta})\,|m\rangle = |m+n\rangle, \quad \exp(-i\tau\hat{M}/\hbar)\,|\theta\rangle = |[\theta+\tau]\rangle,$$

$$n = 0, \pm 1, \pm 2, \ldots, \quad -\pi < \tau < \pi. \tag{27}$$

It will suffice for our purposes to restrict n and τ in this way; and the expression $[\theta+\tau]$ stands for the value modulo 2π of $\theta+\tau$, chosen in the range $-\pi$ to π.

The second Hilbert space $\mathscr{K}$ is defined again as the space of Hilbert-Schmidt operators on $\mathcal{H}$. We first construct an orthonormal basis for $\mathscr{K}$ similar to the displacement operators of the Cartesian case. For this let us define the set of unitary operators

$$Z(n,\tau) = \exp(in\hat{\theta}) \exp(-i\tau\hat{M}/\hbar) \exp(-in\tau/2),$$

$$n = 0, \pm 1, \pm 2, \ldots, \quad -\pi < \tau < \pi. \tag{28}$$

The phase factor has been included so that we have the relation [easily checked using eq (27)].

$$Z(n,\tau)^{\dagger} = Z(-n,-\tau). \tag{29}$$

First we show, using arguments similar to those occurring in eqs (9) and (10), that these operators do form a basis for $\mathscr{K}$. If we take the expression

$$|m\rangle\langle m| = \delta_{m\hbar,\,\hat{M}} = (2\pi)^{-1}\int_{-\pi}^{\pi} d\tau \exp(im\tau)\exp(-i\hat{M}\tau/\hbar) \tag{30}$$

for the projection operator onto an eigenvector of $\hat{M}$, and then apply an exponential in $\hat{\theta}$ and use eq. (27), we get

$$\begin{aligned} |m'\rangle\langle m| &= \exp\left(i(m'-m)\hat{\theta}\right)|m\rangle\langle m| \\ &= (2\pi)^{-1}\int_{-\pi}^{\pi} d\tau \exp\left(i(m'+m)\tau/2\right) Z(m'-m,\tau). \end{aligned} \tag{31}$$

So the completeness of $Z(n, \tau)$ is established. Next an easy calculation establishes that these operators in fact form an orthonormal basis for $\mathscr{K}$; with the same definition, (4), for the scalar product, we have

$$\left(Z(n',\tau'), Z(n,\tau)\right) = 2\pi\delta_{n',\,n}\,\delta(\tau'-\tau). \tag{32}$$

It will be clear that the present case has been treated differently from the Cartesian one. After setting up the space $\mathscr{K}$, we have not first defined four operations on $\mathscr{K}$ and then diagonalised two of them to arrive at the basis $Z(n, \tau)$ for $\mathscr{K}$. Rather, this basis has been directly set up. The reason is that we have no simple replacement for eq. (1) in the angle case, and so no simple way of defining operations on $\mathscr{K}$ through the laws of commutation and anticommutation with $\hat{\theta}$ and $\hat{M}$. But now that the basis $Z(n, \tau)$ for $\mathscr{K}$ is defined, we exploit the fact that n runs over the spectrum of $\hat{M}/\hbar$ and τ over that of $\hat{\theta}$: we define two operations $\mathscr{N}$, $\mathfrak{T}$, i.e., two linear operators on $\mathscr{K}$, by setting

$$\mathscr{N}\, Z(n,\tau) = n\, Z(n,\tau), \quad \mathfrak{T}\, Z(n,\tau) = \tau\, Z(n,\tau). \tag{33}$$

Thus *by definition* $\mathscr{N}$ and $\mathfrak{T}$ are hermitian, commute with each other, and have $Z(n, \tau)$ for their simultaneous 'eigenvectors'. In principle eq. (33) does specify the action of $\mathscr{N}$ and $\mathfrak{T}$ on general elements A in $\mathscr{K}$.

Comparing now eq. (33) with the relation between eigenvectors of $\hat{\theta}$ and $\hat{M}$ in $\mathfrak{H}$ given in eq. (26), we see that we can set up conjugates Θ, $\mathscr{M}$ to $\mathscr{N}$ and $\mathfrak{T}$ respectively: the pairs Θ, $\mathscr{N}$ and $\mathfrak{T}$, $\mathscr{M}$ behave each like $\hat{\theta}$, $\hat{M}$ and commute with each other. The precise definitions of Θ, $\mathscr{M}$ are patterned after eq. (27) and are:

$$\begin{aligned} &\exp(in'\Theta)\, Z(n,\tau) = Z(n'+n,\tau), \\ &\exp(-i\tau'\mathscr{M})\, Z(n,\tau) = Z(n,[\tau'+\tau]), \\ &n' = 0, \pm 1, \pm 2, \ldots, \quad -\pi < \tau' < \pi. \end{aligned} \tag{34}$$

The vanishing of the commutators between Θ and $\mathfrak{T}$, Θ and $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M}$ is clear from eqs (33) and (34). Θ and $\mathcal{M}$ are also obviously hermitian.

We have reproduced the same pattern as existed in the Cartesian case, only somewhat indirectly. While $\mathcal{H}$ supports one angle-angular momentum pair $\hat{\theta}$—$\hat{M}$, $\mathcal{K}$ supports two such independent pairs $\Theta-\mathcal{N}$, $\mathfrak{T}-\mathcal{M}$. The definition of these four operations on $\mathcal{K}$ is essentially unique, once the structure of $\mathcal{H}$ is given in terms of eigenvectors of $\hat{\theta}$ and $\hat{M}$.

The expansion of an operator A on $\mathcal{H}$ in terms of $Z(n, \tau)$,

$$A = \sum_n \int_{-\pi}^{\pi} d\tau\, a(n, \tau)\, Z(n, \tau), \tag{35}$$

is the Weyl representation of A in the present case. We can call the c-number function $a(n, \tau)$ the Weyl weight of A and easily express the scalar product in $\mathcal{K}$ in terms of these weights. To define Wigner representatives for operators on $\mathcal{H}$, we must use the orthonormal basis for $\mathcal{K}$ made up of simultaneous 'eigenvectors' of Θ and $\mathcal{M}$. Let us write $W(\theta, m)$ for them, using the same variables as in general eigenvalues of $\hat{\theta}$ and $\hat{M}$:

$$\begin{aligned} &\Theta\, W(\theta, m) = \theta W(\theta, m), \\ &\mathcal{M}\, W(\theta, m) = m W(\theta, m), \\ &m = 0, \pm 1, \pm 2, \ldots, -\pi < \theta < \pi. \end{aligned} \tag{36}$$

A double use of eq. (26) determines $W(\theta, m)$:

$$\big(W(\theta, m), Z(n, \tau)\big) = (2\pi)^{-1} \exp\,(in\theta - i\tau m); \tag{37}$$

this then fixes the normalisation of these operators to be

$$\big(W(\theta', m'), W(\theta, m)\big) = (2\pi)^{-1}\, \delta(\theta' - \theta)\, \delta_{m', m}. \tag{38}$$

As in section 2, we call the basis $W(\theta, m)$ the Wigner basis for the space $\mathcal{K}$. The explicit form of $W(\theta, m)$ is given by

$$W(\theta, m) = (2\pi)^{-2} \sum_n \int_{-\pi}^{\pi} d\tau \exp\,(-in\theta + i\tau m)\, Z(n, \tau). \tag{39}$$

We can see using eq. (29) that this is a basis of hermitian operators, contrasted with the unitary Weyl basis $Z(n, \tau)$. A real physical observable A possesses an expansion

$$\begin{aligned} &A = \sum_m \int_{-\pi}^{\pi} d\theta\, A(\theta, m)\, W(\theta, m), \\ &A(\theta, m) = 2\pi\, \big(W(\theta, m), A\big), \end{aligned} \tag{40}$$

with a real Wigner representative $A(\theta, m)$. Conversely one can view this as the Weyl correspondence in the present case, the c-number function $A(\theta, m)$ being taken into the operator A. The Wigner distribution $\rho(\theta, m)$ for a density operator ρ is defined with a different numerical factor,

$$\rho(\theta, m) = \big(W(\theta, m), \rho\big), \tag{41}$$

so that the expectation of A in the state ρ takes the form

$$\langle A\rangle = (\rho, A) = \sum_m \int_{-\pi}^{\pi} d\theta\, A(\theta, m)\, \rho(\theta, m). \tag{42}$$

The essential features of the Cartesian case have thus been recovered.

One can ask whether the operations $\Theta, \mathcal{N}, \mathfrak{T}$ and $\mathcal{M}$ can be given more explicit forms than eqs (33) and (34). This is in fact possible for $\mathcal{N}$ and $\mathcal{M}$. For example we have from eq. (27) the result

$$[\hat{M}, \exp(in\hat{\theta})] = n\hbar \exp(in\hat{\theta}) \tag{43}$$

which then leads to

$$[\hat{M}, Z(n, \tau)] = n\hbar\, Z(n, \tau). \tag{44}$$

On comparing this with the definition of $\mathcal{N}$ in eq. (33), we see that $\mathcal{N}$ is essentially the operation of commutation with $\hat{M}$. So for any operator A we have

$$\mathcal{N} A = [\hat{M}, A]/\hbar. \tag{45}$$

Similarly in the case of $\mathcal{M}$ we can make use of eqs (28), (34) and (44), to show:

$$\begin{aligned}\mathcal{M} Z(n, \tau) &= i\frac{\partial}{\partial\tau} Z(n, \tau)\\ &= \tfrac{1}{2}\{\hat{M}, Z(n, \tau)\}/\hbar,\end{aligned} \tag{46}$$

so for any operator A we have

$$\mathcal{M} A = \{\hat{M}, A\}/2\hbar. \tag{47}$$

Unfortunately, analogous simple expressions for Θ and $\mathfrak{T}$ in terms of $\hat{\theta}$ do not exist. For $\mathfrak{T}$ one can go as far as the following on the basis of eqs (33) and (43)

$$\begin{aligned}\exp(im\mathfrak{T})\, Z(n, \tau) &= \exp(im\tau)\, Z(n, \tau)\\ &= \exp(in\hat{\theta}) \exp\big(-i\tau(\hat{M} - m\hbar)/\hbar\big) \exp(-in\tau/2)\\ &= \exp(im\hat{\theta})\, Z(n, \tau) \exp(-im\hat{\theta}).\end{aligned} \tag{48}$$

For a general operator A, what this gives is the basic relation

$$\exp(i\mathfrak{T})\,A = \exp(i\hat{\theta})\,A\,\exp(-i\hat{\theta}) \tag{49}$$

and all positive and negative integral powers thereof. In the case of Θ we can similarly show on the basis of eqs (34) and (43) that

$$\exp(in'\Theta)A = \exp(in'\hat{\theta}/2)\;A\,\exp(in'\hat{\theta}/2) \tag{50}$$

holds for even integral n' only, while for odd n', nothing nearly as simple obtains. These problems in getting explicit forms for $\mathfrak{T}$ and Θ reflect the differences in the kinematics of a Cartesian canonical pair, and an angle-angular momentum pair.

4. Application to coherent states

In this section we wish to bring out the connection between the Wigner representation for operators and the diagonal coherent state representation, using the techniques of section 2. (For simplicity, $\hbar$ will now be set equal to unity). Let us begin by recalling the form of the latter representation. In terms of the operators $\hat{q}$, $\hat{p}$ the annihilation operator $\hat{a}$ is defined by

$$\hat{a} = (\hat{q} + i\,\hat{p})/\sqrt{2}. \tag{51}$$

For each complex number z as eigenvalue, $\hat{a}$ possesses a normalised eigenvector $\chi(z)$. (We use this notation in place of the customary $|\,z\,\rangle$). The $\chi(z)$ form a nonorthonormal overcomplete set in $\mathcal{H}$. Their overcompleteness allows every operator A in K to be represented in the form

$$A = \frac{1}{\pi}\int d^2z\;\varphi(z)\;\chi(z)\;\chi(z)^\dagger, \tag{52}$$

with φ a c-number weight function which must be interpreted in general as a distribution. This is the diagonal coherent state representation; with the understanding that φ may be a distribution, it essentially says that the elements $\chi(z)\;\chi(z)^\dagger$ form a basis for $\mathscr{K}$. (For a thorough discussion, see Klauder and Sudarshan 1968, Chapter 8).

It has been shown elsewhere (Mukunda and Sudarshan 1978) that the representation (52) is mathematically identical to the representation of a vector $|\,\psi\,\rangle$ in $\mathcal{H}$ as an integral over a subset of eigenstates of $\hat{a}$ in the form:

$$|\,\psi\,\rangle = \int_{-\infty}^{\infty} v(r)\;\chi(ir)\;dr. \tag{53}$$

This representation too generally requires a distribution $v(r)$. The connection between (52) and (53) is achieved by defining two operations $\mathscr{A}_1$, $\mathscr{A}_2$ on $\mathscr{K}$ by the equations

$$\mathscr{A}_1 A = (\hat{a}\,A - A\,\hat{a}^\dagger)/\sqrt{2},\quad \mathscr{A}_2 A = -\,i\,(\hat{a}A + A\hat{a}^\dagger)/\sqrt{2}. \tag{54}$$

These operations $\mathscr{A}_j$ and their adjoints $\mathscr{A}_j^\dagger$ obey the same commutation relations as do $\hat{a}$ and $\hat{a}^\dagger$, except that they refer to two degrees of freedom. This is just as expected for the space $\mathscr{K}$. (Note that $\mathscr{A}_1$ and $\mathscr{A}_2$ are *not* formed in the manner of eq. (51) from the two canonical pairs $\omega(\hat{q})$, $\Omega(\hat{p})$ and $\omega(\hat{p})$, $\Omega(\hat{q})$ of section 2; the normalisation factors involved are slightly different). The elements of $\mathscr{K}$ which are simultaneous 'eigenvectors' of $\mathscr{A}_1$ and $\mathscr{A}_2$ are $\chi(z_1)\,\chi(z_2)^\dagger$:

$$\mathscr{A}_1\,\chi(z_1)\,\chi(z_2)^\dagger = \frac{(z_1 - z_2{}^*)}{\sqrt{2}}\,\chi(z_1)\,\chi(z_2)^\dagger,$$

$$\mathscr{A}_2\,\chi(z_1)\,\chi(z_2)^\dagger = -\,i\,\frac{(z_1 + z_2{}^*)}{\sqrt{2}}\,\chi(z_1)\,\chi(z_2)^\dagger. \tag{55}$$

It is obvious now that the 'eigenvectors' of $\mathscr{A}_j$ used in the representation (52) for elements of $\mathscr{K}$ correspond to pure imaginary eigenvalues $i(\sqrt{2})\,\mathrm{Im}\,z$, $-\,i(\sqrt{2})\,\mathrm{Re}z$ for $\mathscr{A}_1$, $\mathscr{A}_2$; hence the connection with (53).

To relate (52) to the Wigner representation, we therefore first establish certain connections between vectors at the level of $\mathscr{H}$. Formally one can see that $\hat{a}$ is related to $\hat{p}$ by a similarity transformation:

$$\hat{a} = S(i\hat{p}/\sqrt{2})S^{-1}, \quad S = \exp\left(-\tfrac{1}{2}\,\hat{q}^2\right). \tag{56}$$

The operator S is hermitian and bounded; its domain is $\mathscr{H}$ while its range R is a proper subset of $\mathscr{H}$. By applying S to the (nonnormalisable) eigenvectors $|\,p\,\rangle$ of $\hat{p}$ we produce the normalisable non-orthonormal eigenvectors of $\hat{a}$ with pure imaginary eigenvalues: in detail one finds

$$\chi(ip/\sqrt{2}) = \sqrt{2}\,\pi^{1/4}\,S\,|\,p\,\rangle. \tag{57}$$

This relation can be checked by working out the Schrödinger wave functions of both sides. It must be noted that only this subset of eigenvectors of $\hat{a}$ is produced in this manner. If $|\varphi\rangle$ is a vector in $\mathscr{H}$, with a square integrable momentum space wavefunction $\tilde{\varphi}\,(p)$,

$$|\,\varphi\,\rangle = \int_{-\infty}^{\infty} dp\,\tilde{\varphi}\,(p)\,|\,p\,\rangle, \tag{58}$$

and we apply S to it, the result is a general vector in R. From eq. (57) it has the representation

$$S\,|\,\varphi\,\rangle = [(1/\sqrt{2})\,\pi^{1/4}]\int_{-\infty}^{\infty} dp\,\tilde{\varphi}\,(p)\,\chi\left(\frac{ip}{\sqrt{2}}\right), \tag{59}$$

with a square integrable weight function $\tilde{\varphi}$. Thus the following precise statement can be made concerning the expansion (53): if $|\psi\rangle$ belongs to the range R of the operator S, we have the expansion with square-integrable weight v, while v has to be a distribution if $|\psi\rangle$ is outside of R.

We apply this analysis now to $\mathscr{H}$. The Wigner basis elements $W(q, p)$ are simultaneous 'eigenvectors' of $\omega(\hat{q})$, $\omega(\hat{p})$ with eigenvalues q, p. Thus the elements

$$\exp\left(-\tfrac{1}{4}(\Omega(\hat{q}))^2 - \tfrac{1}{4}(\Omega(\hat{p}))^2\right) W(q,p) \tag{60}$$

will be simultaneous 'eigenvectors' of the operations

$$\begin{aligned} &\exp\left(-\tfrac{1}{4}(\Omega(\hat{p}))^2\right)\omega(\hat{q})\exp\left(\tfrac{1}{4}(\Omega(\hat{p}))^2\right), \\ &\exp\left(-\tfrac{1}{4}(\Omega(\hat{q}))^2\right)\omega(\hat{p})\exp\left(\tfrac{1}{4}(\Omega(\hat{q}))^2\right), \end{aligned} \tag{61}$$

with eigenvalues q, p respectively. If we evaluate these similarity transforms, we find that the two operations in question are

$$\omega(\hat{q}) + \frac{i}{2}\Omega(\hat{p}),\ \omega(\hat{p}) - \frac{i}{2}\Omega(\hat{q}). \tag{62}$$

Acting on an operator A in $\mathscr{H}$ these are:

$$\begin{aligned} \omega(\hat{q})A + \frac{i}{2}\Omega(\hat{p})A &= \frac{1}{2}\left\{\hat{q}, A\right\} + \frac{i}{2}\left[\hat{p}, A\right] = i\,\mathscr{A}_2 A, \\ \omega(\hat{p})A - \frac{i}{2}\Omega(\hat{q})A &= \frac{1}{2}\left\{\hat{p}, A\right\} - \frac{i}{2}\left[\hat{q}, A\right] = -i\mathscr{A}_1 A. \end{aligned} \tag{63}$$

This means that the operator (60) is a simultaneous 'eigenvector' of $\mathscr{A}_1$ and $\mathscr{A}_2$ with eigenvalues ip, $-iq$. We have thus shown that the members of $\mathscr{H}$ figuring in the diagonal coherent state representation (52) are related to the members of the Wigner basis by a hermitian, bounded operation $\mathscr{S}$:

$$\begin{aligned} &\chi(z)\,\chi(z)^{\dagger} = 2\pi\,\mathscr{S}\,W(q,p), \\ &\mathscr{S} = \exp\left(-\tfrac{1}{4}(\Omega(\hat{q}))^2 - \tfrac{1}{4}(\Omega(\hat{p}))^2\right),\ z = (q+ip)/\sqrt{2}. \end{aligned} \tag{64}$$

For any operator A in $\mathscr{H}$ which lies in the range of $\mathscr{S}$, we have the representation (52) with a square integrable weight $\varphi(z)$ which is the Wigner representative of $\mathscr{S}^{-1}A$. Operators A outside the range of $\mathscr{S}$ call for distributions as weights. As stated in the introduction, a relation between the two weight functions $A(q, p)$, $\varphi(z)$ appearing in the two expansions (19), (52) has been known for a long time. The relation (64) between the projection onto a coherent state $\chi(z)$ and a Wigner basis element $W(q, p)$, stated in the language of operations, gives a concise expression to this result.

5. Concluding remarks

We hope to have brought out the essential mathematical structure underlying the Weyl-Wigner method in quantum mechanics. It has been shown that the idea of dealing with the second Hilbert space $\mathcal{H}$, leads to a natural understanding of these methods entirely within the structure of quantum mechanics and is rich enough to deal with the case of Cartesian variables as well as angle variables. The work of section 3 gives some idea about how one may deal with other kinematic situations in quantum mechanics, where the basic operator relationships may be governed by some other group such as $R(3)$ or the homogeneous Lorentz group.

In connection with the case treated in section 3, we must emphasize the fact that the Wigner representatives and distributions are not c-number functions on a ' corresponding classical phase space '. The latter would be described by two continuous real variables, one an angle and the other capable of assuming all real values. The functions occurring in eqs (40) and (41) are c-number functions of two arguments, one an angle and the other unbounded but quantised. It therefore appears that the extension of his correspondence made by Weyl to the cyclic case is not appropriate, and must be changed in the light of the results of section 3 (Weyl 1931, p. 276). Similar modifications would also be needed in the case of quantum kinematics based on groups like $R(3)$.

A final remark is the following: the possible connection between classical canonical transformations and quantum unitary transformations is an often-discussed question. While there is no intrinsic correspondence between these two sets of transformations, it seems that problems of this nature can be discussed profitably using the techniques of this paper.

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