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Modulational instability of ion-acoustic-waves in two electron temperature plasmas

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Abstract. The envelope properties of ion-acoustic waves in a two-electron-temperature plasma are studied. The nonlinear Schrödinger equation describing the envelope of these waves is obtained from the plasma fluid equations by employing the Krylov-Bogoliubov-Mitropolsky perturbation method. It is shown that the ion-acoustic waves can be modulationally unstable or stable depending on the ratios of the densities and the temperatures of the hot and the cold electron components. Even a small fraction of the cold electron component can drastically affect the stability of the system.

Keywords. Modulational instability; ion-acoustic waves; nonlinear-Schrödinger equation.

1. Introduction

In the linear regime, the presence of a small fraction of cold electrons in a plasma of hot electrons and cold ions is found to considerably affect the characteristics of ion-acoustic waves (IAW) (Jones *et al* 1975). Such plasmas in which the electrons can be divided into two groups having distinct temperatures, may be called two-electron-temperature (TET) plasmas. The examples of TET plasmas are: the hot cathode discharge plasmas (Oleson and Found 1949, Jones *et al* 1975), the thermonuclear plasmas which are turbulent and have high energy tail (Sudan 1973, Morales and Lee 1974), the rf produced plasma in ELMO confinement device (Krall and Trivelpiece 1973), etc. The properties of ion-acoustic solitons in a weakly nonlinear TET plasma have been studied by Goswami and Buti (1976). The envelope properties of the IAW in a TET plasma are discussed in the present paper. In section 2, Krylov-Bogoliubov-Mitropolsky (KBM) method (Bogoliubov and Mitropolsky 1961, Kakutani and Sugimoto 1974, Sharma and Buti 1976 and Buti 1977) is used to derive the nonlinear Schrödinger (NS) equation governing the envelope of these waves. The modulational instability and envelope states of the IAW for different ratios of the densities of the cold and the hot electrons and also of their temperatures are discussed in section 3.

2. Nonlinear Schrödinger equation

Consider a one-dimensional plasma in which the electrons are divided into two groups —the hot component with density n_h and temperature T_h, and the cold component with density n_l and temperature T_l. The ion-acoustic waves have phase velocity smaller than the thermal velocities of both the electron species and consequently both the electron fluids can be taken as isothermal. The propagation of IAW in this TET plasma can be described by the following fluid equations:

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nu)=0 \tag{1a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial x}=0, \tag{1b}$$

$$\frac{\partial^2 \phi}{\partial x^2}+n-n_h-n_l=0, \tag{1c}$$

$$n_h=\nu \exp(\beta\phi/\mu+\nu\beta) \tag{1d}$$

$$n_l=\mu \exp(\phi/\mu+\nu\beta), \tag{1e}$$

where $\beta=T_l/T_h$, $\mu=n_{l0}/n_0$, $\nu=n_{h0}/n_0$;
n_{l0} and n_{h0} being the initial densities of the lower and higher temperature electron components respectively and n_0 that of the ions. In eqs (1a–1e), the densities are normalized to n_0, the velocity to the effective ion-acoustic velocity

$$C_{\text{eff}}=(T_{\text{eff}}/M)^{1/2} \text{ with } T_{\text{eff}}=n_0T_hT_l\,(n_{l0}T_h+n_{h0}T_l)^{-1}$$

the lengths by the effective Debye length, $\lambda_{\text{eff}}=(T_{\text{eff}}/4\pi\, n_0e^2)^{1/2}$, the time by the inverse of the ion plasma frequency, ω_{pi}^{-1} and ϕ by T_{eff}/e. The charge neutrality condition is now expressed as $(\mu+\nu)=1$. In terms of $\alpha=n_{l0}/n_{h0}$, $\mu=\alpha/(1+\alpha)$ and $\nu=1/(1+\alpha)$. For weakly nonlinear systems, we can use the following expansions:

$$\begin{bmatrix}\phi\\ n\\ u\\ n_h\\ n_l\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\\ \nu\\ \mu\end{bmatrix}+\epsilon\begin{bmatrix}\phi_1\\ n_1\\ u_1\\ n_{h1}\\ n_{l1}\end{bmatrix}+\epsilon^2\begin{bmatrix}\phi_2\\ n_2\\ u_2\\ n_{h2}\\ n_{l2}\end{bmatrix}+\epsilon^3\ldots \tag{2}$$

Let ϕ, represent the monochromatic plane wave, namely

$$\phi_1=a\exp(i\,\psi)+\bar{a}\exp(-i\psi) \tag{3}$$

where a is the amplitude, $\bar{a}$ its complex conjugate and $\psi=(kx-\omega t)$ is its phase; k being the wave number and ω the frequency. The complex amplitude a is a slowly varying function of x and t; this slow variation is given by

$$\frac{\partial a}{\partial t}=\epsilon A_1(a,\bar{a})+\epsilon^2A_2(a,\bar{a})+\ldots, \tag{4}$$

$$\frac{\partial a}{\partial x}=\epsilon B_1(a, \bar{a})+\epsilon^2 B_2\ (a, \bar{a})+\ldots;$$

and their complex conjugates. The quantities A_1, B_1, A_2, B_2, ... are yet unknown and are to be determined from the conditions that the perturbation scheme envisaged by eqs (2)–(4) are free from secularities. Equations (2)–(4) can be substituted into eqs (1a)–(1e) and the equations to different orders in ϵ obtained. The ϵ-order equations yield

$$u_1=(k/\omega)\ a \exp(i\psi)+\text{C.C}$$

$$n_1=(k^2/\omega^2)\ a \exp(i\psi)+\text{C.C.}, \tag{5}$$

$$n_{hl}=\frac{\nu\beta}{(\mu+\nu\beta)}[a \exp(i\psi)+\text{C.C.}],$$

$$n_{l1}=(\mu/\mu+\nu\beta)\ [a \exp(i\psi)+\text{C.C.}].$$

Also we obtain to the same order

$$\mathscr{L}(\phi_1)=0 \tag{6}$$

where $\mathscr{L}$ is the operator defined by

$$\mathscr{L}\equiv\frac{\partial_4}{\partial x^2\partial t^2}+\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}. \tag{7}$$

From eqs (3), (6) and (7) we get the linear dispersion relation for ion-acoustic waves in a TET plasma as

$$D(k, \omega)\equiv\omega^2(1+k^2)-k^2=0. \tag{8}$$

It may be noted that the wavenumber k is normalized by the inverse of the effective Debye length λ_{eff} and the phase velocity by the effective ion-acoustic velocity C_{eff}. Thus in the linear limit the ion-acoustic waves in a TET plasma behave in the same way as in a plasma with a single electron component, with the ion-velocity and the Debye length defined by the effective temperature.

From eqs (1a)–(1e), to order ϵ^2, we can eliminate n_2, u_2, n_{h2} and n_{l2} and use eq. (4) to obtain the equations for ϕ_2, viz.,

$$\mathscr{L}(\phi_2)=2\omega^2\{3(1+k^2)^2-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}a^2 \exp(2i\psi)$$

$$-i\left(\frac{\partial D}{\partial\omega}A_1-\frac{\partial D}{\partial k}\beta_1\right)\exp(i\psi)+\text{C.C.} \tag{9}$$

The exp ($\pm i\psi$) terms on the right hand side give rise to resonant secularity in the solution for ϕ_2 due to eqs (7) and (8). The condition for the removal of this secularity is

$$A_1 + V_g B_1 = 0, \tag{10}$$

where

$$V_g = -\frac{\partial D/\partial k}{\partial D/\partial \omega} = \frac{\omega^3}{k^3},$$

is the group velocity of the ion-acoustic waves. The secular free solution of eq. (9) is now given by

$$\phi_2 = \left[\frac{1}{6k^2}\left\{3(1+k^2)^2 - \frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}\right\} a^2 \exp(2i\psi) + b(a,\bar{a})\exp(i\psi) + \text{C.C.}\right] + C_1(a,\bar{a}), \tag{11}$$

where b and C_1 are constants with respect to ψ but are functions of a and $\bar{a}$. The solution of the equations to order ϵ^2 are

$$n_2 = \left[\frac{(1+k^2)}{2k^2}\left\{(1+k^2)(1+4k^2) - \frac{(\mu+\nu\beta^2)}{3(\mu+\nu\beta)^2}\right\} a^2 \exp(2i\psi) - \frac{1}{\omega^3}(2ik^3A_1 + 2ik\omega B_1 - k^2\omega b)\exp(i\psi) + \text{C.C.}\right] + C_2(a,\bar{a}), \tag{12a}$$

$$U_2 = \left[\frac{1}{2k\omega}\left\{(1+k^2)(1+2k^2) - \frac{(\mu+\nu\beta^2)}{3(\mu+\nu\beta)^2}\right\} a^2 \exp(2i\psi) - \frac{i}{\omega^2}(kA_1 + \omega B_1 + ik\omega b)\exp(i\psi) + \text{C.C.}\right] + C_3(a,\bar{a}), \tag{12b}$$

$$n_{h2} = \left[\frac{\nu\beta\, a^2}{6k^2(\mu+\nu\beta)}\left\{3(1+k^2)^2 + \frac{3\beta k^2}{(\mu+\nu\beta)} - \frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}\right\}\exp(2i\psi) + \frac{\nu\beta}{(\mu+\nu\beta)} b(a,\bar{a})(\exp i\psi) + \text{C.C.}\right] + C_4(a,\bar{a}) \tag{12c}$$

and

$$n_{l2} = \left[\frac{\mu a^2}{6k^2(\mu+\nu\beta)}\left\{3(1+k^2)^2 + \frac{3k^2}{(\mu+\nu\beta)} - \frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}\right\}\exp(2i\psi) + \frac{\mu}{(\mu+\nu\beta)} b(a,\bar{a})\exp(i\psi) + \text{C.C.}\right] + C_5(a,\bar{a}), \tag{12d}$$

where the constants of integration C_1–C_5 can be determined from the conditions for the removal of secularities in higher orders. They are found to be

$$C_1 = \frac{a\bar{a}}{(V_g^2-1)}\left[3+k^2-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2(1+k^2)^3}\right]+\tilde{C}_1, \tag{13a}$$

$$C_2 = \frac{a\bar{a}}{(V_g^2-1)}\left[3+k^2-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}\right]+\tilde{C}_2, \tag{13b}$$

$$C_3 = \frac{a\bar{a}V_g}{(V_g^2-1)}\left[1+k^2-\frac{2}{V_g^2}-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}\right]+\tilde{C}_3, \tag{13c}$$

$$C_4 = \left(\frac{\nu\beta}{\mu+\nu\beta}\right)\left(\frac{aa}{V_g^2-1}\right)\left[3+k^2-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2(1+k^2)^3}+\frac{\beta(V_g^2-1)}{(\mu+\nu\beta)}\right]+\tilde{C}_4, \tag{13d}$$

and

$$C_5 = \left(\frac{\mu}{\mu+\nu\beta}\right)\left(\frac{a\bar{a}}{V_g^2-1}\right)\left[3+k^2-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2(1+k^2)^3}+\frac{(V_g^2-1)}{(\mu+\nu\beta)}\right]+\tilde{C}_5, \tag{13e}$$

where $\tilde{C}_1$ to $\tilde{C}_5$ are absolute constants to be determined from the initial conditions.

Now in eqs (1a)–(1e), to order ϵ^3, n_3, u_3, n_{h3} and n_{l3} can be eliminated to give an equation in ϕ_3. On using the expressions for ϕ_2, n_2, u_2, n_{h2}, n_{l2}, ϕ_1, n_1, u_1, n_{h1} and n_{l1} given above, the condition for the removal of resonant secularity in the equation for ϕ_3 is found to be

$$i(A_2+V_g\beta_2)+\tfrac{1}{2}\frac{dV_g}{dk}\left(B_1\frac{\partial B_1}{\partial a}+\bar{B}_1\frac{\partial B_1}{\partial \bar{a}}\right)+Q\,|\bar{a}|^2\,a+Ra=0, \tag{14}$$

where

$$Q=(\omega^3/4k^4)\,(3+3k^2+k^4)^{-1}\,\chi(k^2,\mu,\nu) \tag{15}$$

and

$$R=-\left(\frac{\omega}{2k^2}\right)\left\{k^2\left(C_2+\frac{2k}{\omega}\tilde{C}_3\right)-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}\omega^2\tilde{C}_1\right\}. \tag{16}$$

The function $\chi(\mu, \nu, k^2)$ appearing in eq. (15) is given by

$$\chi(\mu,\nu,k^2)=\sigma_1+\sigma_2k^2+\sigma_3k^4+\sigma_4k^6-\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}k^8-k^{10},$$

with

$$\sigma_1=9-6\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}+\frac{(\mu+\nu\beta^2)^2}{(\mu+\nu\beta)^4},$$

$$\sigma_2=30-22\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}+3\frac{(\mu+\nu\beta^3)}{(\mu+\nu\beta)^3}-\frac{(\mu+\nu\beta^2)^2}{(\mu+\nu\beta)^4},$$

$$\sigma_3=35-28\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}+3\frac{(\mu+\nu\beta^3)}{(\mu+\nu\beta)^3}-\frac{(\mu+\nu\beta^2)^2}{3(\mu+\nu\beta)^4}$$

and

$$\sigma_4=15-14\frac{(\mu+\nu\beta^2)}{(\mu+\nu\beta)^2}+\frac{(\mu+\nu\beta^3)}{(\mu+\nu\beta)^3}.$$

The slow variations of the amplitude a with respect to space and time are governed by the conditions (10) and (14). On defining the new space and time variables,

$$t_2=\epsilon t_1,\ t_1=\epsilon t,\ x_2=\epsilon x_1,\ x_1=\epsilon x,$$

we can rewrite eq. (10) as

$$\frac{\partial a}{\partial t_1}+V_g\frac{\partial a}{\partial x_1}=0. \tag{17}$$

This equation indicates that in the slow scale t_1 and x_1, the amplitude a propagates with the group velocity V_g without any change of form. Equation (14) can now be written as

$$i\left(\frac{\partial a}{\partial t_2}+V_g\frac{\partial a}{\partial x_2}\right)+P\frac{\partial^2 a}{\partial x_1{}^2}+Q\,|\,a\,|^2a+Ra=0, \tag{18}$$

where

$$P=\tfrac{1}{2}\frac{dV_g}{dk}=-\frac{3\omega^5}{2k^4}.$$

On using the coordinate transformations

$$\xi=\epsilon(x-V_gt)=x_1-V_gt_1;\ \tau=\epsilon^2t=\epsilon t_1=t_2.$$

eq. (18) reduces to

$$i\frac{\partial a}{\partial\tau}+P\frac{\partial^2 a}{\partial\xi^2}+Q\,|\,a\,|^2\,a+Ra=0. \tag{19}$$

This is the nonlinear Schrödinger equation governing the envelope of the ion-acoustic waves in a two-electron-temperature plasma. In the limit $\mu\to 0$, i.e., in the absence of the cold electron component; or in the limit $\beta\to 1$, i.e., the two electron components have the same temperature and thus reduce to a single component, the above expressions for P, Q and R reduce to those obtained by Kakutani and Sugimoto (1974).

3. Discussion

The nonlinear Schrödinger equation, eq. (19), governs the evolution of the envelope of the plane ion-acoustic waves. This equation is characterized by the dispersive term with coefficient P and the nonlinear term with coefficient Q. We may note that P in this TET plasma is similar to the one in a one electron component plasma except for the difference that k in this case is normalized to λ_{eff}. Q, however, is multiplied by a factor χ which in the limit $\mu=0$, $\nu=\beta=1$ (corresponding to one electron component plasma) reduces to the expression of Kakutani and Sugimoto (1974). We know that the IAW are unstable against long wavelength perturbations if $PQ>0$ and stable otherwise (Hasegawa 1975). In our case

$$PQ = -(3\omega^8/8k^8)(3 + 3k^2 + k^4)^{-1}\chi(k^2, \alpha, \beta);$$

so $PQ > 0$ only if $\chi < 0$. For different values of α and β, the critical wavenumber k_c^2 ($k > k_c$ unstable) for modulational instability can then be obtained from the equation

$$(k_c^2, \alpha, \beta) = 0.$$

Since it is not possible to solve this equation for k_c analytically, the critical values k_c^2 are obtained by numerical computation for a range of values of α and β. The variations of k_c^2 with α for fixed values of β are shown in figure 1. The variations of k_c^2 with β for given values of α are shown in figure 2. If the cold electron component is absent, i.e., $\alpha=0$, or the temperature of the two components are equal, i.e., $\beta=1$, we get the known value $k_c^2=2{\cdot}163$ for a plasma consisting of cold ions and hot isothermal electrons (Kakutani and Sugimoto 1974). It is clear from these two figures that the presence of the cold electron component reduces the critical wavenumber for modulational instability i.e, the unstable region (in k space) increases. Thus IAW wavelengths longer than $k=147$ can also become modulationally unstable. For $k>1{\cdot}47$, one knows that it is more appropriate to consider the ion plasma mode rather than the ion-acoustic mode. The presence of cold electrons thus makes the IAW of *realistic wavelengths* ($k < 1$) modulationally unstable.

Moreover figure 1 shows that the variation of k_c^2 with α is more prominent for smaller values of β and at $\beta=1$ there is no variation as expected. Similarly figure 2 shows that the variation of k_c^2 with β is more prominent for smaller values of α. These two figures clearly indicate the range of parameters one should choose to avoid this particular instability.

These observations are suggestive of the following physical mechanism responsible for this behaviour. In a plasma with cold ions and hot electrons, a perturbation in the ion density is accompanied by a perturbation in the electron density. Since the electrons are hot, the pressure gradient gives rise to an electric field and this provides the restoring force on the ions. If a small fraction of the electrons are cold, they will contribute little to the restoring electric field and consequently the ion-acoustic speed will be reduced, as was observed experimentally (Jones *et al* 1975). This effect is manifested in the nonlinear state rather strongly, as is shown here. Since even a small fraction of cold electrons change the envelope characteristics of the IAW, it is important to take into account the existence of the cold electron component in the study of ion-acoustic waves.

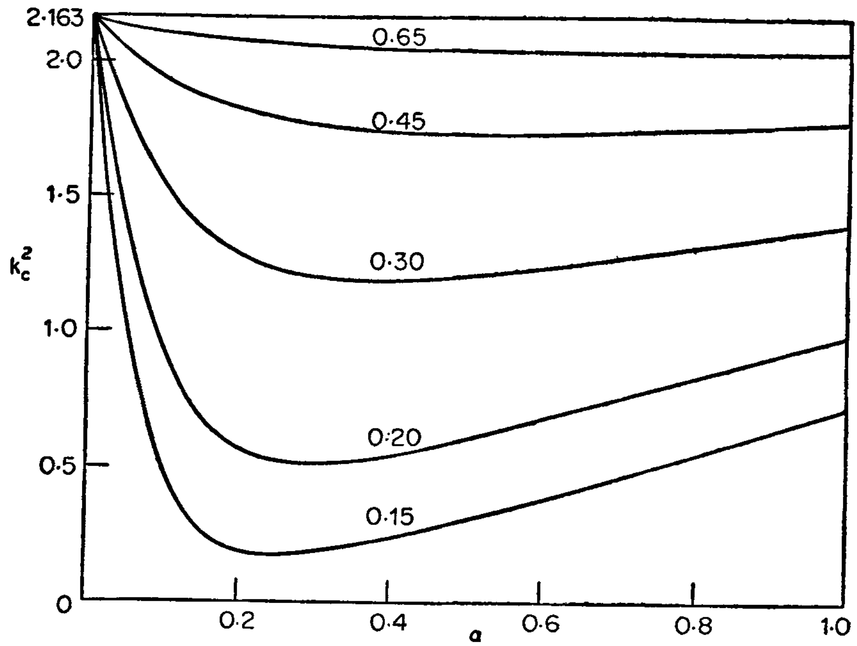

Figure 1. Variation of critical wavenumber $k_c{}^2$ with the ratio α of the densities of the cold and hot electron components for β=0·15, 0·2, 0·3, 0·45 and 0·65.

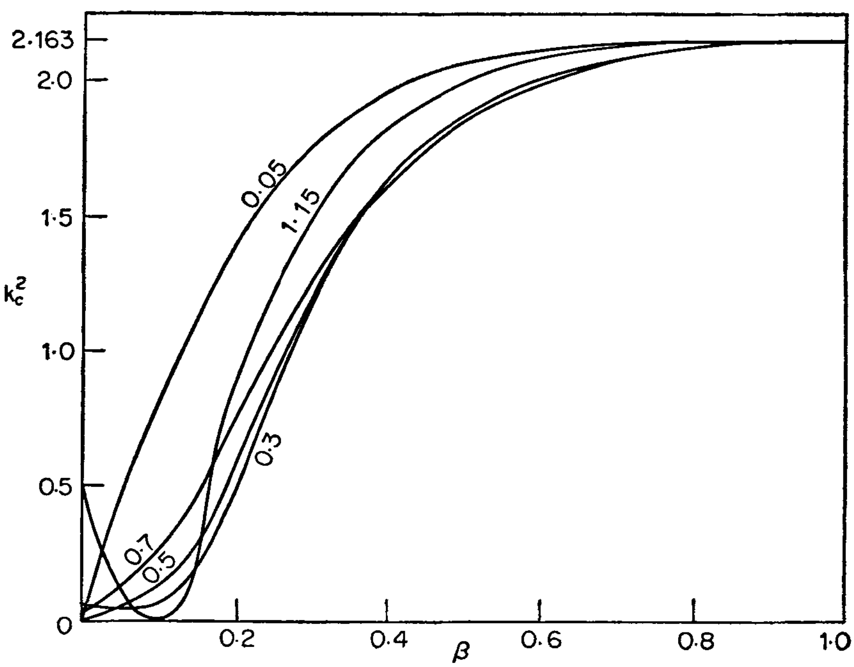

Figure 2. Variation of critical wavenumber $k_c{}^2$ with the ratio β of the temperatures of the cold and hot electron components for α=0·05, 0·15, 0·3, 0·5 and 0·7.

References

Bogoliubov N N and Mitropolsky Y A 1961 *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Delhi: Hindustan)
Buti B 1977 *Phys. Rev. Lett.* **38** 498
Goswami B N and Buti B 1976 *Phys. Lett.* **A57** 149
Hasegawa A 1975 *Plasma Instabilities and Nonlinear Effects* (Berlin, Heidelberg: Springer-Verlag)
Jones W D, Lee A, Gleman S M and Doucet H J 1975 *Phys. Rev. Lett.* **35** 1349
Kakutani T and Sugimoto N 1974 *Phys. Fluids* **17** 1617
Krall N A and Trivelpiece A W 1973 *Principles of Plasma Physics* (New York: McGraw Hill)
Morales G J and Lee Y C 1974 *Phys. Rev. Lett.* **33** 1534
Oleson N L and Found C G 1949 *J. Appl. Phys.* **20** 416
Sharma A S and Buti S 1976 *J. Phys.* **A9** 1823
Sudan R N 1973 *Proc. VI European Conf. on Controlled Fusion and Plasma Phys.*, Moscow Vol. II