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Static, c-number solutions of a two dimensional sine-Gordon-like field equation

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Abstract. A sine-Gordon type scalar field with a $\cos^4 [(\sqrt{\lambda}\,\phi)/m]$ potential in two dimensions is studied and a static, c-number, stable, finite energy solution obtained. The intersoliton potential is evaluated for this model and is shown to be long range, attractive and 'strong' in the weak coupling regime. A positronium-like spectrum of bound states of the soliton-antisoliton pair is derived using this potential. The coupling with a massless quark field is discussed and an exact, stable and analytic solution of the spinor field exhibiting confinement in one space dimension is obtained.

Keywords. Solitons; intersoliton potential; stability; quark confinement.

1. Introduction

A new chapter has been opened in field theory with the discovery of the existence of stable, localized finite energy solutions of certain classical field equations (see, for instance, Scott *et al* 1974). These solutions are often collectively known as 'solitons' (Jackiw 1977). When the classical solution is used as a spring-board for quantising the theory, the quantum version evinces a rich particle structure in the Hilbert space. Two dimensional sine-Gordon field theory is one such problem which has been investigated in meticulous detail (Coleman 1975; Gervais and Neveu 1975, Jackiw 1977; Neveu 1977; Rajaraman 1975; and references therein). The solitons in the quantum sine-Gordon theory are equivalent to fermions of the Thirring model (Coleman 1975). In this paper we study the exact static c-number solution of a non-linear scalar field equation in one space one time dimension derived from a 'sine-Gordon-type lagrangian'.

$$L(\phi) = \tfrac{1}{2}\,\partial_\mu\phi\;\partial^\mu\phi - \frac{m^4}{2\lambda}\cos^4\left(\frac{\sqrt{\lambda}}{m}\phi\right) \tag{1}$$

where m and λ are parameters and λ/m^2 is dimensionless. Since the field equation that follows from eq. (1) contains two sine terms, it may be called a double sine-Gordon (DSG) equation. We evaluate the classical energy of the static solution (soliton) and discuss its stability. As in the conventional SG system the solitons of the present theory are fermions. This is brought out by examining the symmetry of the form factor.

Recently Rajaraman (1977), developing the concept of a potential contained in the work of Perring and Skyrme (1962 a,b) has studied the intersoliton forces for

ϕ^4 and SG theories in two dimensions in the weak coupling approximation, and shown that the extended particles exert strong forces at large distances when the coupling of the underlying field theory is weak. Following his recipe of introducing discontinuities along the soliton profile, we calculate the intersoliton potential for solitons of our model. The potential is found to be attractive and of long range, and yields a positronium-like spectrum of bound states of the soliton-antisoliton configuration.

Considerable work (see for instance, Bardeen *et al* 1975, Chodos *et al* 1974, Vinciarelli 1975) has been done towards solving field equations which include fermions, with special emphasis on solutions which confine the fermion fields. We consider in the present work a massless fermi field coupled via Yukawa interaction to a DSG field in one spatial dimension and obtain an exact analytic, and stable c-number solution for the fermi field for the whole one dimensional space. This is shown to be a confined solution and, consequently, it seems to be of some importance in theories of quark confinement.

2. A double sine-Gordon equation

The potential term in eq. (1) is written as

$$V(\phi) = \frac{m^4}{2\lambda} \cos^4 \frac{(\sqrt{\lambda}\phi)}{m} = \frac{m^4}{16\lambda}\left[\cos \frac{(4\sqrt{\lambda}\phi)}{m} + 4 \cos \frac{(2\sqrt{\lambda}\phi)}{m} + 3\right]. \quad (2)$$

The Euler-Lagrange equation for the scalar field defined by eq. (1) is

$$\Box\phi = \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = - V'(\phi) = 2\frac{m^3}{\sqrt{\lambda}} \cos^3 \frac{(\sqrt{\lambda}\phi)}{m} \sin \frac{(\sqrt{\lambda}\phi)}{m}$$

where the prime denotes differentiation with respect to the argument. Equivalently, we have

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \frac{m^3}{4\sqrt{\lambda}}\left[\sin \frac{(4\sqrt{\lambda}\phi)}{m} + 2 \sin \frac{(2\sqrt{\lambda}\phi)}{m}\right]. \quad (3)$$

The right-hand side of this equation is a sum of two sines and this justifies it being called a double sine-Gordon (DSG) equation. On expanding the potential given by eq. (2), the lagrangian in eq. (1) is rewritten as a series

$$L[\phi] = \tfrac{1}{2}\, \partial_\mu\phi\, \partial^\mu\phi - \frac{m^4}{2\lambda} + m^2\phi^2 - \frac{5}{6}\, \lambda\phi^4 + \ldots \quad (4)$$

where the mass term appears with a perverse sign signalling spontaneous breakdown of symmetry.

We consider a static c-number solution of eq. (3) designated as ϕ_c. In the static case eq. (3) becomes

$$\frac{d^2\phi_c}{dx^2} = - \frac{2m^3}{\sqrt{\lambda}} \cos^3 \frac{(\sqrt{\lambda}\phi_c)}{m} \sin \frac{(\sqrt{\lambda}\phi_c)}{m}. \quad (5)$$

This is solved by imposing the following boundary conditions

$$\phi_c(\pm\infty) = \pm\frac{\pi m}{2\sqrt{\lambda}};\quad \frac{d\phi_c}{dx}(x=\pm\infty)=0; \tag{6}$$

$$\phi_c(x = x_0) = 0.$$

A solution satisfying these conditions is

$$\phi_c(x) = \pm\frac{m}{\sqrt{\lambda}}\arctan m(x - x_0). \tag{7}$$

Here the constant x_0 refers to the origin of the co-ordinate system, and because of translational invariance of the theory, in the subsequent discussion we may set $x_0 = 0$ so that

$$\phi_c(x) = \pm\frac{m}{\sqrt{\lambda}}\arctan mx. \tag{8}$$

The solution with the positive sign is termed the soliton while that with the negative sign the anti-soliton (Coleman 1975). Figure 1 carries a display of the profiles of these solutions ϕ_c is a monotone function of x, the soliton solution monotonically increasing, and the anti-soliton solution monotonically decreasing.

The minima of the potential given by $V'(\phi) = 0$ are

$$\phi_{\min} = (2n+1)\frac{\pi m}{2\sqrt{\lambda}},\ n = 0, \pm 1, \pm 2, \ldots \tag{9}$$

These represent the constant solutions to the field eq. (3), and they all have the same energy zero, which is the lowest possible value of the energy integral

$$E = \int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)\right]dx \tag{10}$$

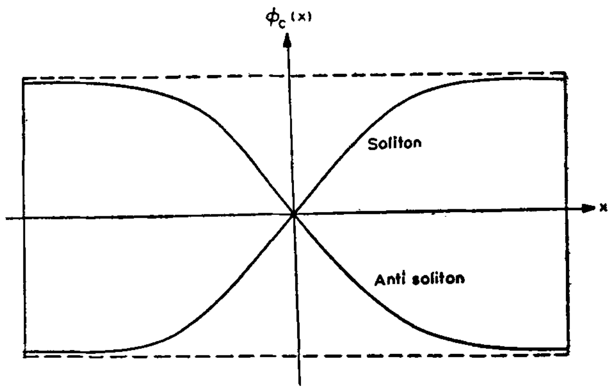

Figure 1. The plot of $\phi_c(x)$ versus x.

$\phi_{\min}$ thus represent the vacua of the DSG system with the soliton and anti-soliton configurations interpolating between them. In addition there exist boosted solutions for $|u| < 1$

$$\phi(t, x) = \pm \frac{m}{\sqrt{\lambda}} \text{ arc tan } m \left(\frac{x - ut}{\sqrt{1 - u^2}} \right). \tag{11}$$

The energy density of a time-independent solution is

$$\mathcal{H}(x) = \frac{1}{2} (\phi_c')^2 + V(\phi_c) = \frac{m^4}{\lambda} \frac{1}{(1 + m^2 x^2)^2}. \tag{12}$$

The energy density $\mathcal{H}(x)$, sketched in figure 2, is peaked around the origin, and vanishes asymptotically. The classical energy of the soliton, $E_c(\phi_c)$ is obtained by integrating $\mathcal{H}(x)$ over the whole configuration space:

$$E_c(\phi_c) = \frac{m^4}{\lambda} \int_{-\infty}^{\infty} \frac{dx}{(1 + m^2 x^2)^2} = \frac{\pi m^3}{2\lambda}. \tag{13}$$

This gives the mass M of the soliton (anti-soliton) in the classical approximation and is $O(1/\lambda)$. The soliton (anti-soliton) is a heavy object in the weak coupling limit $\lambda/m^2 \ll 1$. Here we remark that the introduction of a constant term, say $m^4/2\lambda$ into the lagrangian in eq. (1) would give a divergent result for the classical energy.

It is easily verified that the system possesses the discrete symmetries corresponding to

$$\phi \to \pm \phi + \frac{n\pi m}{\sqrt{\lambda}}, \; n = 0, \pm 1, \pm 2, \ldots \tag{14}$$

but the minima of the potential are given by eq. (9). If we take $\phi_{\min} = \pi m/(2\sqrt{\lambda})$ as the vacuum state, and build a quantum theory on this minimum, the non-vanishing vacuum expectation value of the quantum field Φ is

$$\langle 0 | \Phi(t, x) | 0 \rangle = \frac{\pi m}{2\sqrt{\lambda}}. \tag{15}$$

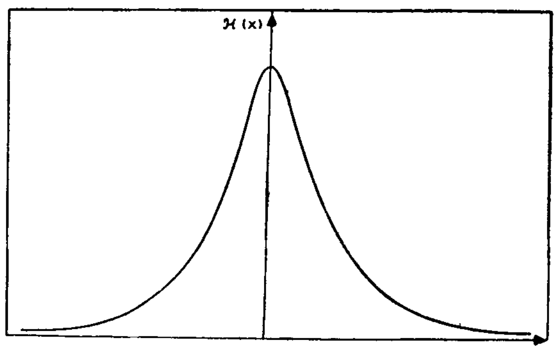

Figure 2. The variation of $\mathcal{H}(x)$ against x.

This constitutes a direct proof of spontaneous symmetry breakdown.

In the usual way, we suppose that there exist 'meson' states which are energy-momentum eigenstates. The lowest order mass of the meson in the present theory, defined by the curvature of $V(\phi)$ at $\phi_{\min}$ turns out to be zero.

3. Stability of the classical solution

Since the existence of a classical soliton implies that of a quantum soliton, the stability of the former would ensure the stability of the latter.

The classical stability of the soliton is assured when the eigenvalues ω_n^2 of the Schrödinger-like equation for the quantum fluctuation amplitude ψ_n

$$\left[-\frac{d^2}{dx^2}+V''(\phi_c)\right]\psi_n=\omega_n^2\,\psi_n \tag{16}$$

are non-negative. It is easy to see that $\psi_0(x)=d\phi_c/dx$ is an eigenfunction of this equation, for ϕ_c satisfies.

$$\phi''_c(x)=V'(\phi_c) \tag{17}$$

and the corresponding eigenvalue is zero.

Since $\phi_c\,(x)$ given by eq. (8) is a monotone function of x, ϕ'_c has no nodes, and for a one-dimensional Schrödinger equation with an arbitrary potential, the eigenfunction with no nodes is the eigenfunction of lowest energy (Morse and Feshbach 1953). Hence we conclude that $\omega_n^2\geqslant 0$.

The stability of the soliton, despite its large mass, can also be understood as a result of a topological conservation law. The charge associated with the trivially conserved current

$$J^{\mu}=\epsilon^{\mu\nu}\,\partial_\nu\phi \tag{18}$$

where $\quad\epsilon^{\mu\nu}=-\,\epsilon^{\nu\mu}$ and $\epsilon^{01}=1$, is

$$N=\int dx\;J^0=\frac{\pi m}{\sqrt{\lambda}} \tag{19}$$

$N\neq 0$ in the one soliton sector, and its conservation takes care of the stability of the soliton. This conservation is founded on the special topology of the field which assumes two different values at $x=\pm\infty$.

4. Solitons of the double sine-Gordon system are fermions

The matrix element of the quantum field $\Phi\,(t,x)$ between one soliton states, evaluated at the origin, $\langle P'\,|\,\Phi\,(0)\,|\,P\,\rangle$ can be shown to satisfy the following equation.

$$\langle P'\,|\,\Phi\,(0)\,|P\,\rangle=\int dx\,\exp\,[i\,(P'-P)x]\,\frac{m}{\sqrt{\lambda}}\,\text{arc tan}\,mx$$
$$+\text{ higher powers in }\lambda \tag{20}$$

(Jackiw 1977). Here $|P\rangle$ and $|P'\rangle$ are one-soliton quantum states which are momentum and energy eigenstates. $\langle P'|\Phi(0)|P\rangle = f(P', P)$ is called the field form-factor. To leading order in $\sqrt{\lambda}, f(P', P)$ depends only on the difference $P'-P$ so that, in this order,

$$f(P'-P) = \int dx \exp\left[i(P'-P)x\right] \frac{m}{\sqrt{\lambda}} \arctan mx. \tag{21}$$

Since arc tan mx is an odd function of x, $f(P'-P)$ would undergo a sign change when P and P' are interchanged. The antisymmetry of the field form-factoe proclaims that the solitons of the present model are fermions, just as in ϕ_2^4 theory (Jackiw and Goldstone 1975, Jackiw 1977).

5. Intersoliton potential

This section is devoted to a calculation of the intersoliton forces exerted by the extended particles of this DSG system. We follow the recipe of Rajaraman (1977) for defining and evaluating this potential $V(R)$, when the soliton and anti-soliton are separated by a distance R much larger than the size of the soliton.

We first rewrite the two dimensional scalar field lagrangian in eq. (1) in terms of rescaled coordinates and fields:

$$X_\mu \to m x_\mu$$

$$\eta \to \frac{\sqrt{\lambda}}{m}\phi \tag{22}$$

so that

$$L = \frac{m^4}{2\lambda}\left[(\partial_\mu \eta)^2 - \cos^4 \eta\right] \tag{23}$$

where $\partial_\mu \eta = \partial\eta / \partial X_\mu$. This transformation is introduced purely for computational case.

A static field η satisfies the equation

$$d^2\eta/dX^2 = -2\cos^3\eta \sin\eta. \tag{24}$$

As in section 2 we obtain the soliton or anti-soliton configuration by integrating (24) subject to the conditions,

$$\eta = \pi/2, \frac{d\eta}{dX} = 0 \text{ at } X = \infty$$

$$\eta = 0 \qquad \text{at } X = X_0$$

and hence,

$$\eta = \pm \arctan(X - X_0). \tag{25}$$

The static soliton-antisoliton profile held a distance R apart is obtained by adopting the following procedure. We set $X_0 = -R/2$ and integrate (24) from $X = -\infty$ to $X = -R/2$ to get half of a soliton. Next we introduce a discontinuity in the slope $d\eta/dX$ at $X = -R/2$. Specifically we demand $\eta(-R/2) = 0$ such that

$$\lim_{\delta\to 0} \frac{d\eta}{dX}\left(-\frac{R}{2}+\delta\right) = \beta \tag{26}$$

and

$$\lim_{\delta\to 0} \frac{d\eta}{dx}\left(-\frac{R}{2}-\delta\right) = 1 \ (\beta \neq 1). \tag{27}$$

The constant β will be explicitly evaluated shortly. Integrating eq. (24) between $X = -R/2$ and $X = 0$ subject to the above conditions, we find

$$\frac{d\eta}{dX} = (\cos^4\eta + \beta^2 - 1)^{1/2} \tag{28}$$

and therefore,

$$X + \frac{R}{2} = \int_0^{\eta(X)} \frac{d\eta}{(\cos^4\eta + \beta^2 - 1)^{1/2}} \text{ for } -R/2 < X < 0. \tag{29}$$

To fix the constant β we demand space-inversion symmetry for η and also continuity at $X = 0$. These imply

$$\left(\frac{d\eta}{dX}\right)_{X=0} = 0 \tag{30}$$

or equivalently, from eq. (28)

$$\eta(0) = \cos^{-1}(1-\beta^2)^{1/4} \tag{31}$$

Putting $X = 0$, in eq. (29) and using eq. (31)

$$\frac{R}{2} = \int_0^{\cos^{-1}\alpha} \frac{d\eta}{(\cos^4\eta - \alpha^4)^{1/2}} \tag{32}$$

where

$$\alpha = (1-\beta^2)^{1/4}$$

This integral is evaluated in terms of elliptic integrals (Byrd and Friedman 1971) yielding

$$\frac{R}{2} = \frac{1}{\sqrt{2}\,\alpha} K\left[(1-\alpha^2/2)^{1/2}\right] \tag{33}$$

where K is the complete elliptic integral of the first kind of modulus $[(1-\alpha^2)/2]^{1/2}$. This fixes α, and, consequently, β as a function of R.

For $X > 0$ we fix $\eta(X)$ by inversion symmetry,

$$\eta(X) = \eta(-X). \tag{34}$$

The static soliton-antisoliton configuration is sandwiched between the half of a free soliton extending from $X = -\infty$ to $X = -R/2$ and the half of a free anti-soliton extending from $X = +R/2$ to $X = +\infty$.

The classical energy $E(R)$ of the soliton-antisoliton configuration may be evaluated as a function of R. Accordingly

$$E(R) = \frac{m^3}{\lambda}\int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{d\eta}{dX}\right)^2 + \frac{1}{2}\cos^4\eta\right] dX. \tag{35}$$

Using eq. (28), and also the fact that outside the 'distortion region', $-R/2<X<R/2$, the energy is equal to the mass of one soliton, namely $\pi m^3/2\lambda$ we obtain

$$E(R) = \frac{\pi m^3}{2\lambda} + \frac{m^3\alpha^4 R}{2\lambda} + \frac{2m^3}{\sqrt{2}\lambda}\left[\alpha E - \alpha(1+\alpha^2)K + \left(\frac{\alpha}{2} + \frac{1}{2\lambda}\right)\Pi\right] \tag{36}$$

where K and E are complete elliptic integrals of the first and second kind respectively of modulus $[(1-\alpha^2)/2]^{1/2}$, and

$$\Pi = \Pi(\rho^2, k)$$

$$= \Pi\left(\frac{\alpha^2-1}{2\alpha^2}, \sqrt{\frac{1-\alpha^2}{2}}\right)$$

is a complete elliptic integral of the third kind (Byrd and Friedman 1971). From eq. (33) it is clear that as $R \to \infty$, $\alpha \to 0$ and this implies $\beta \to 1$ which is the natural slope at the centre of a free soliton. For large but finite R let $\alpha = \epsilon$ be a small parameter. Using standard expansions of elliptic integrals

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \dots\right] \tag{37}$$

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{k^6}{5} + \dots\right] \tag{38}$$

$$\Pi(\rho^2, k) = \frac{\pi}{2\sqrt{(1-\rho^2)}} + \frac{\pi}{4\rho^2}\left(\frac{1}{\sqrt{(1-\rho^2)}} - 1\right)k^2$$

$$+ \frac{3\pi}{32\rho^2}\left(\frac{2}{\sqrt{(1-\rho^2)}} - 2 - \rho^2\right)k^4 + \dots \tag{39}$$

where $k = [(1 - \epsilon^2)/2]^{1/2}$ and $\rho^2 = [-(1 - \epsilon^2)/2\epsilon^2]$ with the restrictions $k^2 < 1$, $-\rho^2 > 1$. For large R we have the approximation

$$\epsilon \approx \frac{2{\cdot}616}{mR}. \tag{40}$$

This enables us to expand $\mathrm{E}(R)$ for large R or small ϵ, and retaining terms up to first order in ϵ,

$$\mathrm{E}\,(R) = 2 \times \frac{\pi m^3}{2\lambda} - 0{\cdot}1108 \times \sqrt{2}\frac{m^3}{\lambda}\,\epsilon. \tag{41}$$

The effective soliton-antisoliton potential (Rajaraman 1977) $V(R)$ defined as

$$V(R) = \mathrm{E}(R) - \mathrm{E}(\infty) \tag{42}$$

is therefore calculated to be

$$V(R) = -\,0{\cdot}4095\,\frac{m^2}{\lambda}\,\frac{1}{R} \tag{43}$$

which is attractive and becomes strong in the weak coupling limit. The long range nature of this potential confirms the result obtained in section 2 that the mass of the underlying mesons is zero.

The bound states of the soliton-antisoliton pair can be interpreted as resonances. We calculate these states by plugging $V(R)$ into the non-relativistic two-body Schrödinger equation. The non-relativistic approach adopted here is justified by the fact that the soliton is very heavy in the weak coupling domain. The Schrödinger equation in the relative coordinate R is

$$\left[-\frac{1}{M}\frac{d^2}{dR^2} + V(R)\right]\psi_N\,(R) = E_N\,\psi_N\,(R) \tag{44}$$

where $V(R)$ is given by eq. (43) and M by eq. (13). Solutions are obtained subject to the condition $\psi_N\,(R) \to 0$ as $R \to \infty$. The eigenvalues are

$$E_N = -\frac{\pi m^7}{8\lambda^3 N^2} \times (0{\cdot}4095)^2;\; N = 1, 2, 3, \ldots. \tag{45}$$

Clearly the spectrum resembles that of positronium, as it should, because of the $1/R$ dependance of $V(R)$. Up to a normalization constant the eigenfunctions are found to be

$$\psi_N\,(R) = e^{-\kappa R}\sum_s \frac{(-2\kappa)^s\,(N-1)!\,R^{s+1}}{(N-1-s)!\,s!\,(s+1)!} \tag{46}$$

where

$$\kappa = \sqrt{M\,|E_N|}.$$

6. Double sine-Gordon field for quark confinement

In this section we couple a massless quarkfield ψ to the DSG field defined in section 2 through a Yukawa-type interaction in two dimensions. The corresponding lagrangian is

$$L = \tfrac{1}{2}\, \partial_\mu \phi\, \partial^\mu \phi - \frac{m^4}{2\lambda} \cos^4\left(\frac{\sqrt{\lambda}}{m}\,\phi\right)$$

$$+ \tfrac{1}{2}\, [\bar{\psi}\gamma^\mu\, (\partial_\mu\, \psi) - (\partial_\mu\, \bar{\psi})\, \gamma^\mu\, \psi] - i\, g\, \bar{\psi}\, \gamma^5\, \psi\, \phi \tag{47}$$

where g is a positive coupling constant with mass dimensionality. The field equations are

$$\gamma^\mu\, \partial_\mu\, \psi = g\, \gamma^5\, \psi\, \phi \tag{48}$$

$$\partial^\mu\, \partial_\mu\, \phi = \frac{2m^3}{\sqrt{\lambda}} \cos^3\left(\frac{\sqrt{\lambda}}{m}\,\phi\right) \sin\left(\frac{\sqrt{\lambda}}{m}\,\phi\right) - i\, g\, \bar{\psi}\, \gamma^5 \psi. \tag{49}$$

We postulate that

$$\bar{\psi}\, \gamma^5\, \psi = 0. \tag{50}$$

This simplifies eqs (48) and (49), and in the static case, they assume the form

$$\frac{d^2\phi}{dx^2} = - \frac{2m^3}{\sqrt{\lambda}} \cos^3\left(\frac{\sqrt{\lambda}}{m}\,\phi\right) \sin\left(\frac{\sqrt{\lambda}}{m}\,\phi\right) \tag{51}$$

$$\gamma^1\, \frac{d\psi}{dx} = -\, g\, \gamma^5\, \psi\, \phi. \tag{52}$$

We take the following representation for the two dimensional Dirac matrices:

$$\gamma^0 = \sigma^1, \quad \gamma^1 = i\, \sigma^3, \quad \gamma^5 = \gamma^0, \quad \gamma^1 = \sigma^2 \tag{53}$$

where σ^1, σ^2 and σ^3 are Pauli matrices, so that $\bar{\psi} = \psi^\dagger\, \gamma^0 = \psi^\dagger\, \sigma^1$. Writing

$$\psi = \begin{pmatrix} u\,(x) \\ v\,(x) \end{pmatrix} \tag{54}$$

eq. (50) reads

$$u^*\, u = v^*\, v. \tag{55}$$

The solutions of this equation are

$$v = \pm\, u, \quad \pm\, i\, u. \tag{56}$$

Let us pick the solution $v = -u$

$$\text{or } \psi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(x). \tag{57}$$

Equation (51) possesses the exact soliton solution

$$\phi(x) = \pm \frac{m}{\sqrt{\lambda}} \arctan m(x - x_0). \tag{58}$$

Equation (52) is separated into two identical equations so that

$$\frac{du}{dx} = -g\phi u. \tag{59}$$

This is directly integrated to give

$$u(x) = A\,(1 + m^2x^2)^{g/2\sqrt{\lambda}} \exp\left(\frac{-gm}{\sqrt{\lambda}} x \arctan mx\right) \tag{60}$$

where we have set $x_0 = 0$ for the sake of convenience and A is a normalisation constant. It is easy to see that because of the presence of the factor. x arc tan mx in the exponential $u(x)$ will be a good, localised function. The variation of $u(x)$ is sketched in figure (3) alongside with that of the soliton solution.

As $g/\sqrt{\lambda}$ increases the single-colour quark field $\psi(x)$ becomes confined to a narrower and narrower region around the centre of the soliton. The quark contribution to the bound state mass given by the hamiltonian,

$$H = \tfrac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{m^4}{2\lambda}\cos^4\left(\frac{\sqrt{\lambda}}{m}\phi\right)$$

$$+ \tfrac{1}{2}\left[\bar{\psi}\gamma'\frac{d\psi}{dx} - \frac{d\bar{\psi}}{dx}\gamma'\psi\right] + ig\bar{\psi}\gamma^5\psi\phi \tag{61}$$

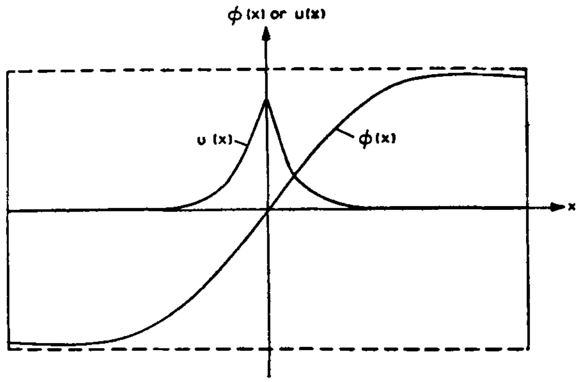

Figure 3. The plot of $\phi(x)$ and $u(x)$ against x.

is obviously zero on account of the vanishing of the interaction term implied by eq. (50), as well as the vanishing of the kinetic term for the static solution considered. The bound-state mass is due entirely to the scalar field $\phi(x)$ and is given by eq. (13). Because of the trapping of the quark field $\phi(x)$ the DSG field $\phi(x)$ can be regarded as a 'container' more or less similar to the SLAC bag (Bardeen *et al* 1975) in which confinement is achieved by a scalar field with quartic coupling or the two dimensional bag based on the supersymmetric SG model of Hruby (1977). In the present case the coupling cunstants g and λ can be suitably chosen so as to limit the quark field almost to the centre of the scalar field. The stability of the solution in eq. (60) trivially follows from an argument due to Uchiyama (1976).

7. Conclusion

We have introduced a doube sine-Gordon scalar field with a $\cos^4[(\sqrt{\lambda}/m)\phi]$ potential and obtained a static c-number, stable soliton solution, ϕ_c of the corresponding field equation in two dimensions. It exhibits several interesting features similar to the solitons familiar in the literature, especially those associated with ϕ_2^4 and SG systems. By examining the symmetry of the form factor, ϕ_c has been shown to represent a fermion. The 'mesons' of the DSG field system are massless. Since there is spontaneous breakdown of symmetry, built into the DSG lagrangian through a mass term with the wrong sign, we may speculate that these 'mesons' are Goldstone bosons. It is gratifying to note that the intersoliton potential for the soliton-antisoliton configuration for large separations, calculated by Rajaraman's method (Rajaraman 1977) is long range while being attractive, so that the exchanged 'mesons' are massless. This shows the consistency of the procedure followed in evaluating this potential.

A positronium-like spectrum with $1/N^2$ — dependence of bound states of the soliton-antisoliton pair in the DSG model has been derived using the $1/R$ potential referred to above. Even though no quantization of the DSG field theory has been attempted we may suggest that these bound states correspond to resonances. We have also discussed a coupling of the massless fermi field to the DSG field and obtained an exact, stable, and analytic solution for the spinor field exhibiting confinement in one space dimension. The scalar field acts like a vessel for the trapped fermi-field. This model shares the gross features of bag-models such as the SLAC model (Bardeen *et al* 1975) and the supersymmetric SG model of Hruby (Hruby 1977).

The results herein obtained seem to show that solitons exist not only in the conventional SG model but in theories related to it such as the DSG model. Quantization of the DSG theory is being studied by us with a view to gaining additional information regarding the structure and properties of the quantal Hilbert space of particles.

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