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# Upper bounds on the wave function renormalization constant of pion

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**Abstract.** Using unitarity, analyticity and the hypothesis of Bjorken scaling inequalities have been derived for the upper bounds on the wave function renormalization constant of pion.

**Keywords.** Upperbounds; wave function; renormalization constant of pion.

## 1. Introduction

It has been suggested (Tirapegui 1971) that the vanishing of wave function renormalization constant for a particle should be taken as a criterion for identifying as it composite, although arguments have been put forward that such a criterion is not sufficient. By now there is sufficient evidence from theoretical and experimental investigations that proton is a composite particle. West (1971) has demonstrated that if

$$\text{(a)}\quad \lim_{Q^2\to\infty} Fi(Q^2, W)$$

are asymptotically bounded in both variables,

$$\text{(b)}\quad G_E(Q^2, M) + 2M\,\partial\, G_E(Q^2, M)/\partial W \neq 1$$

and

$$\text{(c)}\quad Q^2\sigma_L = 0$$

in both scaling and Regge limits, then the wave function renormalization constant of proton vanishes. Using unitarity, Schwarz inequality, the hypothesis of Bjorken scaling and the experimental data on the structure functions of inelastic electron proton scattering Broadhurst (1972) has been able to estimate upper bounds on $Z_2$ for proton with certain assumptions on the subtraction constants and the cross section ratio $R$. A stronger result has also been obtained (Hirayama and Ishida 1972) for the vanishing of $Z_2$. It has also been remarked that all these investigations are very weak applications of unitarity and numerical upper bounds using better methods have also been obtained (Baluni and Broadhurst 1973, 1977).

Fractional value of the renormalization constant has been interpreted as the manifestation of degree of compositeness of proton (Baluni and Broadhurst 1973, 1977).

Pion is the least massive of hadrons taking part in strong interactions. If its composite nature is established, other hadrons may be treated as composite in a naive sense. Nevertheless it is important to investigate in the conditions leading to the vanishing of its wave function renormalization constant, $Z_3$, and establish possible upper bounds on it. Recently using unitarity, analyticity, and Bjorken scaling, upper bounds on the discontinuity of the off-shell form factor of pion have been obtained by the present authors (Parida and Giri 1977, hereinafter referred to as paper I). It has been shown in paper I that all the derivatives of the off-shell form factor of pion near $s=m_\pi^2$ are bounded, provided that the dispersion relation for the form factor requires not more than one subtraction. Out of two inequalities derived for the off-shell form factor only one was used to establish asymptotic upper bound on the elastic form factor of the pion. In this paper the usefulness of both the inequalities has been demonstrated to obtain upper bounds on $Z_3$. Since the off-shell form factor is an analytic function with only the right-hand cut starting from $s=qm_\pi^2$ to $\infty$, all its derivatives must be bounded near $s=m_\pi^2$. Out of the two alternatives for the dispersion relation with one or no subtraction as concluded in paper I, using a once subtracted dispersion relation for the off-shell form factor and inequalities of paper I we obtain inequalities which relate $Z_3$ to the experimentally measurable quantities like $F_2(\omega)$, the pion structure function and $R$, the ratio of the longitudinal to transverse photo-absorption cross section of pion. Unlike deep inelastic lepton-nucleon scattering, experimental data on electron-pion scattering have not been available yet due to the instability of a pion target. For lack of relevant information from experiment, the inequalities could not be numerically computed in the present work, but they can be readily evaluated if data can be obtained in future from inelastic electron-pion scattering or from indirect experiments (Schierholz and Schmidt 1976). Even then our inequalities supply information on the conditions under which pion is composite. Our principal results in this work are

$$\frac{Z_3}{Z_3-1} \leqslant \int_1^\infty \frac{F_2(\omega)}{\omega} \frac{R(\omega)}{R(\omega)+1} d\omega \tag{1}$$

and

$$\frac{Z_3}{Z_3-1} \leqslant \int_1^\infty \frac{F_2(\omega)}{\omega} d\omega. \tag{2}$$

This paper is divided into four short sections. In section 2 we clarify normalization and restate our inequalities derived earlier. In section 3 we derive upper bounds on $Z_3$. In section 4 we discuss our results.

## 2. Bounds on discontinuity of off-shell form factor

In this section we clarify some normalizations adopted in paper I and rewrite inequalities. Throughout this paper we adopt the notations of paper I, in which we obtained

$$|\,\mathrm{Im}\, F(s,\, Q^2)\,|^2 \leqslant C\,(s - m_\pi^{\,2})^2\, \rho\,(s) / (1 + \nu^2/Q^2)$$

$$\times\, [(1 + \nu^2/Q^2)\, W_2\,(s,\, Q^2) - W_1\,(s,\, Q^2)] \tag{3}$$

and

$$|\,\mathrm{Im}\, F(s,\, Q^2)\,|^2 \leqslant C\,(s - m_\pi^{\,2})^2\, \rho\,(s)\, W_2\,(s,\, Q^2) \tag{4}$$

where $C = 1/8m_\pi$. However, if we use a normalization for the pion similar to that for the nucleon (Broadhurst 1972) i.e.,

$$\Gamma_\mu\,(p,\, q) = (p' + p)_\mu\, F(s,\, Q^2) + q_\mu\, G\,(s,\, Q^2) \tag{5}$$

$$= i\,(2\,p_0)^{1/2} \int dx\, \exp\,(ip'\, x)\, (\Box_x + m_\pi^{\,2})\, \langle\, O\,|\, \theta\,(x_0)\, [\phi(x),\, j_\mu\,(0)]\,|\, p\, \rangle \tag{6}$$

$$W_{\mu\nu} = \frac{(2\pi)^3}{\sqrt{2}m_\pi} \sum\nolimits_{p_n,\, a} (2p_0)\, \langle\, p\,|\, j_\mu(0)\,|\, p_n,\, a\, \rangle$$

$$\langle\, p_n,\, a\,|\, j_\nu\,(0)\,|\, p\, \rangle\, \delta^4\,(p' - p_n)$$

$$= (-g_{\mu\nu} + q_\mu\, q_\nu/Q^2)\, W_1 + (p_\mu + pqq_\mu/Q^2)\,(p_\nu + pqq_\nu/Q^2) W_2/m_\pi^{\,2} \tag{7}$$

and repeat all the steps with projection operators of paper I, we obtain (3) and (4) with $C = \pi^2/2m_\pi$. Now defining

$$R\,(s,\, Q^2) = \frac{\sigma_L}{\sigma_T} = (1 + \nu^2/Q^2)\, \frac{W_2}{W_1} - 1 \tag{8}$$

where $\sigma_L$ and $\sigma_T$ are photoabsorption cross sections of pion for longitudinal and transverse photons, respectively, in (3) and (4) we obtain

$$|\,\mathrm{Im}\, F\,(s,\, Q^2)\,|^2 \leqslant \frac{\pi^2}{2m_\pi}\, (s - m_\pi^{\,2})^2\, \rho\,(s)\, W_2\,(s,\, Q^2)\, \frac{R\,(s,\, Q^2)}{R(s,\, Q^2) + 1} \tag{9}$$

$$|\,\mathrm{Im}\, F\,(s,\, Q^2)\,|^2 \leqslant \frac{\pi^2}{2m_\pi}\, (s - m^2_{\ \pi})^2\, \rho\,(s)\, W_2\,(s,\, Q^2). \tag{10}$$

In the next section we will use inequalities (9) and (10) to derive bounds on $Z_3$. It may be observed that with the present normalization the constant $A$ in paper I is to be redefined by mutliplying by a factor $4\pi^2$ on the right-hand side of eq. (21b).

## 3. Upper bounds on the wave function renormalization constant of pion

In this section while relating $Z_3$ to the experimentally measurable quantities like $R$ and $F_2\,(\omega)$, we will demonstrate the usefulness of both the inequalities (3), and (4), whereas in paper I only (4) was used. Some of the results of paper I can be briefly summarized in the following manner: If $\rho(s)$ and $F(s, Q^2)$ are bounded like $s^a$ and $s^b$, respectively, as $s \to \infty$, all the derivatives of the off-shell pion form factor starting from the $n$th one, where $n > b$ and $n \geqslant 1 + (p+a)/2$, are bounded, provided that the off-shell form factor requires atmost $n$ subtractions in the dispersion relation. Here $p$ is the exponent determining threshold behaviour of pion structure function i.e., $F_2(\omega) \sim (\omega - 1)^p$, near $\omega = 1$. From the experimental information on the threshold behaviour (Schierholz and Schmidt 1977) that $p=1$ and the value $a=-1$, corresponding to a composite pion, we have $n \geqslant 1$. This conclusion is also valid for an elementary pion corresponding to $Z_3 = 1$ or pion with some degree of compositeness corresponding to $0 \leqslant Z_3 < 1$, where

$$\frac{1}{Z_3} - 1 = \int_{qm_\pi^2}^{\infty} \rho(s) ds \tag{11}$$

because in such cases one of the conditions is that the integral (11) at the upper limit at least, must vanish. This condition restricts $a < -1$. For example with $a = -3$, we have $n \geqslant 0$, which means that the off-shell form factor itself and all its derivatives near $s = m_\pi^2$ are bounded and no subtractions are needed in the dispersion relation. Even though no subtraction is needed in this case, one can use subtracted dispersion relation to make the dispersion integral more convergent, provided one knows the subtraction constants.

Following Bincer (1961) it is easy to verify the expression for the imaginary part of $\Gamma_\mu$ as given in paper I with the present normalization, i.e.,

$$A_\mu = \frac{(2\pi)^4 (2p_0)^{1/2}}{2} \sum_{p_n} \delta^4(p' - p_n)(-s + m_\pi^2) \langle O | \phi(0) | p_n \rangle$$

$$\times \langle p_n | j_\mu(0) | p \rangle$$

and

$$\operatorname{Im} F(s, Q^2) = \frac{(2\pi)^4 (2p_0)^{1/2}}{4m_\pi (1 + \nu^2/Q^2)^{1/2}} \sum_{p_n} \delta^4(p' - p_n)(-s + m_\pi^2)$$

$$\langle O | \phi(0) | p_n \rangle \langle p_n | \epsilon L_\mu j_\mu(0) | p \rangle .$$

From these two expressions we see that the lowest non-vanishing production matrix element is for three points and $\operatorname{Im} F(s, Q^2)$ is nonzero for $s \geqslant qm_\pi^2$. Then we know that $F(s, Q^2)$ is an analytic function in the cut-$s$ plane with only the right-hand cut starting from $s = qm_\pi^2$ to $\infty$. As an anayltic function, $F(s, Q^2)$ and all its derivatives near $s = m_\pi^2$ must be bounded. In view of this and discussions in the preceding paragraph we are led to the following conclusions: *Unitarity, analyticity, and Bjorken scaling relate the subtracted dispersion relation for the off-shell form factor with the compositeness or elementarity of a pion. Similar analysis can be carried out and conclusions derived for any hadron.* To obtain more concrete results we start with a once subtracted dispersion relation for the off-shell form factor,

$$F(s, Q^2) = F_\pi(Q^2) + \frac{(s - m_\pi^2)}{\pi} \int_{qm_\pi^2}^{\infty} \frac{\operatorname{Im} F(s', Q^2) ds'}{(s' - s)(s' - m_\pi^2)} . \tag{12}$$

The Ward-Takahashi identity in this case reads

$$q_\mu \, \Gamma_\mu (p, q) = s - m_\pi^2. \tag{13}$$

From eqs (5) and (13) we get

$$\frac{(s-m_\pi^2)}{Q^2} [F(s, Q^2)-1] = G(s, Q^2). \tag{14}$$

Thus, if $G(s, Q^2)$ is bounded in both the variables like $s^\alpha$ and $(Q^2)^\beta$ where $\alpha<1$, and $\beta<-1$, $\lim\limits_{s\to\infty} F(s, Q^2)=1$ or $\lim\limits_{Q^2\to\infty} F(s, Q^2)=1$.

We define the function

$$\mathscr{F}(Q^2) = \Big| \lim_{s\to\infty} F(s, Q^2) - F_\pi(Q^2) \Big|^2$$
$$= | 1 - F_\pi(Q^2) |^2. \tag{15}$$

Taking the limit $s\to\infty$ in (12) we obtain

$$1 = F_\pi(Q^2) - \frac{1}{\pi} \int_{qm_\pi^2}^{\infty} \frac{\text{Im}\, F(s', Q^2)}{(s'-m^2)} ds'. \tag{16a}$$

From (15) and (16a) we easily obtain the following inequality

$$\mathscr{F}(Q^2) \leqslant \frac{1}{\pi^2} \int_{qm_\pi^2}^{\infty} \frac{|\text{Im}\, F(s', Q^2)|^2}{(s'-m_\pi^2)^2} ds'. \tag{16b}$$

Now using the inequality (9) in the integrand of (16b) we obtain

$$\mathscr{F}(Q^2) \leqslant \frac{1}{2m_\pi} \int_{qm_\pi^2}^{\infty} \rho(s') W_2(s', Q^2) \frac{R(s', Q^2)}{R(s', Q^2)+1} ds'. \tag{17}$$

Using Schwarz inequality for integration over $s'$ in (17) and the definition (11) we have

$$\mathscr{F}(Q^2) \leqslant \frac{1}{2m_\pi} \int_{qm_\pi^2}^{\infty} \rho(s') ds' \int_{qm_\pi^2}^{\infty} ds' W_2(s', Q^2) \frac{R(s', Q^2)}{R(s', Q^2)+1} ds'$$
$$= \left(\frac{1}{Z_3} - 1\right) \frac{1}{2m_\pi} \int_{qm_\pi^2}^{\infty} ds' W_2(s', Q^2) \frac{R(s', Q^2)}{R(s', Q^2)+1}. \tag{18}$$

The integration over $s'$ can now be converted to the one over $\omega$ using the relation $\omega = 2m_\pi \nu / Q^2 = 1 + (s' - m_\pi^2)/Q^2$ which yields

$$\mathscr{F}(Q^2) \leqslant (Z_3^{-1} - 1) \int_{1+8m_\pi^2/Q^2}^{\infty} \frac{d\omega}{\omega} \nu W_2 \frac{R}{R+1}. \tag{19}$$

Taking the limit $Q^2 \to \infty$ and then using the Bjorken scaling hypothesis

$$\lim_{Bj} \nu W_2(s, Q^2) = F_2(\omega) \tag{20}$$

and

$$\lim_{Bj} R(s, Q^2) = R(\omega) \tag{21}$$

in (19) yields

$$\lim_{Q^2 \to \infty} \mathcal{F}(Q^2) \leqslant (Z_3^{-1} - 1) \int_1^{\infty} \frac{F_2(\omega)}{\omega} \frac{R(\omega)}{R(\omega)+1}. \tag{22}$$

According to the result of paper I the pion form factor is asymptotically bounded by

$[\ln Q^2]^c / Q^2$. Thus $\lim_{Q^2 \to \infty} \mathcal{F}(Q^2) = 1$.

Using this relation in (22) we get inequality (1).

Instead of using inequality (9) if we use inequality (10) in (16b) and repeat all the steps those led to inequality (1) we obtain inequality (2). Inequalities (1) and (2) express upper bounds on $Z_3$. Thus we can formulate a theorem on the compositeness of pion.

**Theorem:** If $G(s, Q^2)$ is bounded in both the variables like $s^{\alpha}$ and $(Q^2)^{\beta}$ where $\alpha < 1$ and $\beta < -1$, and $R(\omega) \to 0$ in the Bjorken limit for all $\omega$, $Z_3 = 0$ or the pion is composite.

We note that in establishing inequalities (1) and (2) we did not have complications due to subtraction constant unlike the nucleon case. This fact has been pointed out by West (1971). Further we need not assume boundedness of both the form factors. Ward-Takahashi identity along with the boundedness assumption upon $G(s, Q^2)$ guarantees boundedness of $F(s, Q^2)$. Inequality (2) is weaker than (1) and it agrees with the inequality (29d) of Broadhurst (1972) for proton. However, inequality (1) is different from all other inequalities for proton derived by Broadhurst and the inequalities of West (1971) and Hirayama and Ishida (1972). Unlike the inequality of West and inequality (29a) of Broadhurst (1972), inequalities (1) or (2) can never give rise to divergence at the lower limit of the integral which could have occurred in their integrals if it were found that $F_2(\omega) \sim$ constant, near $\omega = 1$. But if $F_2(\omega) \sim (\omega - 1)^p$ both for the case of proton and pion, which is experimentally observed our integrals are less convergent than some of the integrals of Broadhurst (1972). If $\lim_{\omega \to \infty} F_2(\omega) \to$ constant inequality (2) gives a logarithmic divergence and inequality (1) yields the same type of divergence if in addition $\lim_{\omega \to \infty} R(\omega) \to$ const. These cases correspond to the usual result

$$Z_3 = 1.$$

However, if it is found that $\lim_{\omega \to \infty} F_2(\omega) \to 0$ and $R(\omega)$ does not grow with $\omega$ both the inequalities give $Z_3 < 1$. On the other hand if it is found that $\lim_{\omega \to \infty} F_2(\omega) \to$ const. but $\lim_{\omega \to \infty} R(\omega) \to O(\omega^{-\epsilon})$ where $\epsilon > 0$, integral in the inequality (1) is convergent and will yield $Z_3 < 1$. At present we have not been able to obtain the bound numerically due to want of experimental data on $F_2(\omega)$ and $R(\omega)$. However, if experimental data are available from inelastic electron-pion scattering or from indirect experiments (Schierholz and Schmidt 1976) in future, these inequalities can be used to evaluate the bound numerically.

## 4. Results and discussion

Using unitarity, analyticity, and the hypothesis of Bjorken scaling we have expressed upper bound on the wave function renormalization constant of pion, $Z_3$, as integrals containing experimentally measurable structure functions $F_2(\omega)$ and/or $R(\omega)$. Our inequalities are different from that of West (1971), Hirayama and Ishida (1972), and some of the inequalities of Broadhurst (1972) for proton. We conclude that pion is composite if the inelastic form factor $G(s, Q^2)$ is bounded in both the variables like $s^{\alpha}$ and $(Q^2)^{\beta}$, where $\alpha < 1$ and $\beta < -1$, and $R(\omega) \to 0$ for all $\omega$. If $R(\omega) \neq 0$ for all $\omega$ then also it is possible that $Z_3 < 1$. This corresponds to a case where pion is not elementary but definitely possessing certain degree of compositeness as in the case of nucleon (Baluni and Broadhurst 1973, 1977). For lack of experimental data the upper bounds could not be evaluated numerically.

In our derivation of the upper bound we have used once subtracted dispersion relation (12) for the off-shell form factor according to one of the possibilities pointed out in paper I. The same type of analysis cannot be carried out with unsubtracted dispersion, the remaining possibility pointed out in paper I.

## Note

After completion of this work, the work of Goldberger and Blankenbecler (1977) was brought to our notice. Section 2 of this paper has been devoted to deriving inequality (3) with $C = \pi^2/4m_\pi$ instead of $\pi^2/2m_\pi$ in the present paper. As pointed out in this paper this inequality has already been derived by us earlier than these authors in our paper I with $C = 1/8m_\pi$. In the derivation of Goldberger and Blankenbecler (1977) the inequality differs by a factor of 2 from the inequality (9) due to a different normalization for $W_{\mu\nu}$ adopted by them. In their definition the factor $1/\sqrt{2}$ in (7) has not been taken. Omission of this factor would imply different definitions for $W_{\mu\nu}$ for proton (Broadhurst 1972) and pion, and introduces a spurious factor of $\frac{1}{2}$ in the right-hand sides of (1) and (2).

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