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Modulational instability and envelope solutions of nonlinear dispersive wave equations

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Abstract. The nonlinear Schrödinger equation describing the evolution of the plane wave solutions of the Hirota equation and of the Boussinesq equation are obtained. The conditions for modulational instability and the localised stationary solutions are derived.

Keywords. Modulational instability; Nonlinear Schrödinger equations; localised stationary solutions; envelope solutions.

1. Introduction

The nonlinear Schrödinger (NS) equation governs a variety of phenomena, e.g., the self-focussing and self-modulation of plane waves, the propagation of heat pulses in solids, the propagation of a number of plasma waves (Langmuir, ion-acoustic and magnetosonic waves), etc. (Scott *et al* 1973). The modified Korteweg-de Vries (KdV) equation on the other hand arises in the study of acoustic waves in anharmonic lattices, Alfven waves in a collisionless plasma, etc. (Jaffrey and Kakutani 1972). Both these equations can be unified into the Hirota equation:

$$i\frac{\partial \chi}{\partial t}+i3\alpha|\chi|^2\frac{\partial \chi}{\partial x}+\beta\frac{\partial^2 \chi}{\partial x^2}+i\gamma\frac{\partial^3 \chi}{\partial x^3}+\delta|\chi|^2\chi=0, \tag{1}$$

where α, β, γ and δ are real constants (Hirota 1973a). This equation reduces to a NS equation for $\alpha=\gamma=0$, and to a modified KdV equation for $\beta=\delta=0$. For the case $\alpha\beta=\gamma\delta$, the exact envelope soliton solutions of eq. (1) were obtained by Hirota (1973a).

The propagation of waves in shallow water under gravity propagating in both directions are described by the nonlinear wave equation

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-6\frac{\partial^2\phi^2}{\partial x^2}-\frac{\partial^4\phi}{\partial x^4}=0, \tag{2}$$

where ϕ, x and t are normalized to the quantities appropriate to the particular problem. This is the well known Boussinesq equation. The exact-N-soliton solution of this equation was obtained by Hirota (1973b).

Both eqs (1) and (2) are nonlinear and their plane wave solutions are dispersive, and hence they describe nonlinear dispersive media. It is well known that such media can give rise to modulational instability, i.e., the plane wave solutions of these equations can be unstable against long wavelength modulations. If the plane wave solutions are modulationally unstable we get envelope soliton solutions and if stable we get envelope hole solutions (Hasegawa 1975). Here we study the modulational instability of eqs (1) and (2) and their consequent stationary solutions. We derive the NS equation governing the evolution of the amplitudes of the plane wave solutions of eqs (1) and (2) in the next section. In section 3 we study the conditions for modulational instability and obtain the corresponding localized stationary solutions. We conclude with a discussion of the results in the last section.

2. Nonlinear Schrödinger equation

Let us first consider eq. (1). To obtain the NS equation describing the systems governed by this equation we use the Krylov-Bogoliubov-Mitropolsky method (Kakutani and Sugimoto 1974; Sharma and Buti 1976). The solution to eq. (1) can then be written as

$$\chi = \epsilon\,\chi_1(a, \bar{a}, \psi) + \epsilon^2\chi_2(a, \bar{a}, \psi) + \ldots, \tag{3}$$

where χ_1 is chosen to be the monochromatic plane wave given by

$$\chi_1 = a\exp(i\psi) + \bar{a}\exp(-i\psi). \tag{4}$$

Here a is the complex amplitude, $\psi = kx - \omega t$ is the phase factor and $\bar{a}$ is the complex conjugate of a. In eq. (3), χ_1, χ_2, ..., are functions of x and t only through a, $\bar{a}$ and ψ. The complex amplitude a is a slowly varying function of x and t:

$$\frac{\partial a}{\partial t} = \epsilon A_1(a, \bar{a}) + \epsilon^2 A_2(a, \bar{a}) + \ldots \tag{5}$$

and

$$\frac{\partial a}{\partial x} = \epsilon B_1(a, \bar{a}) + \epsilon^2 B_2(a, \bar{a}) + \ldots$$

The unknown functions A_1, B_1, B_2, A_2,... are determined by using the requirement that the perturbation solution (3) is secular free to all orders.

On substituting eq. (3) into eq. (1), we get equations of different orders in ϵ. The equation to order ϵ gives the linear dispersion relation

$$D(k, \omega) \equiv \omega - \beta k^2 + \gamma k^3. \tag{6}$$

To order ϵ^2, eq. (1) contains terms proportional to exp $(\pm i\psi)$ which give rise to resonant secularity. The condition for the removal of this secularity is

$$A_1 + V_g B_1 = 0, \tag{7}$$

where $V_g = 2\beta k - 3\gamma k^2$, is the group velocity of the plane waves. The secular free solution can then be written as

$$\chi_2 = b(a, \bar{a})\, e^{i\psi} + c.c. + \chi_{20}(a, \bar{a}), \tag{8}$$

where $b(a, \bar{a})$ and $\chi_{20}(a, \bar{a})$ are constants with respect to ψ.

The equation to order ϵ^3, which is obtained by using eqs (4) and (8) in eq. (1), has two sources of secularities: the resonant secularity arising from exp $(+i\psi)$ terms, and the second one due to ψ independent terms which become proportional to ψ on integration. The condition for the removal of the latter secularity determines χ_{20} to be an absolute constant. The resonant secularity however is removed by the condition

$$i(A_2 + V_g B_2) + \frac{1}{2}\frac{dV_g}{dk}\left(B_1 \frac{\partial B_1}{\partial a} + \bar{B}_1 \frac{\partial B_1}{\partial \bar{a}}\right) - 3(\alpha k - \delta)|a|^2 a = 0.$$

Defining

$$P_1 = \frac{1}{2}\frac{dV_g}{dk} = \beta - 3\gamma k$$

and

$$Q_1 = -3(\alpha k - \delta) \tag{9}$$

and using eq. (5), the above equation can be written as

$$i\left(\frac{\partial a}{\partial t_2} + V_g \frac{\partial a}{\partial x_2}\right) + P_1 \frac{\partial^2 a}{\partial x_1^2} + Q_1 |a|^2 a = 0.$$

This, on introducing the new variables, namely $\tau = t_2 = \epsilon t_1 = \epsilon^2 t$ and $\xi = (x_2 - V_g t_2)/\epsilon = x_1 - V_g t_1 = \epsilon(x - V_g t)$, reduces to the familiar NS equation

$$i\frac{\partial a}{\partial \tau} + P_1 \frac{\partial^2 a}{\partial \xi^2} + Q_1 |a|^2 a = 0. \tag{10}$$

Exactly similar analysis is carried out for eq. (2). The linear dispersion relation for the plane wave solutions in this case is given by

$$D(k, \omega) \equiv -\omega^2 + k^2 - k^4.$$

To order ϵ^2, secular free solution of eq. (2) is

$$\phi_2 = \frac{2a^2}{k^2}\exp(2i\psi) + c(a, \bar{a})\exp(i\psi) + c.c. + \phi_{20}(a, \bar{a}).$$

From the secularity removal condition for eq. (2) (to order ϵ^4) $\phi_{20}(a, \bar{a})$ is found to be

$$\phi_{20} = \frac{12}{V_g^2 - 1}\,\bar{a}\, a + \mu,$$

where μ is an absolute constant. As before the condition for the removal of the resonant secularity in eq. (2) to order ϵ^3 yields the NS equation

$$i\frac{\partial a}{\partial \tau}+P_2\frac{\partial^2 a}{\partial \xi^2}+Q_2\,|\,a\,|^2 a+Ra=0, \tag{11}$$

with

$$P_2=\frac{1}{2}\frac{dV_g}{dk}=-\frac{k^2(3-2k^2)}{2\omega(1-k^2)}, \tag{12}$$

$$Q_2=\frac{3(12-17k^2+12k^4)}{\omega(3-4k^2)},$$

and

$$R=-6\mu k^2$$

3. Modulational instability and envelope solutions

Equations (10) and (11) describe how the amplitudes of the plane wave solutions of eqs (1) and (2) respectively will evolve according to their dispersion, determined by P_i and nonlinearity, determined by Q_i ($i=1$ and 2). If $P_iQ_i>0$, perturbations with $K<K_c=(2Q_i\,\rho_0/P_i)^{1/2}$ are unstable and grows with the maximum growth rate $\Gamma=Q_i\,\rho_0$ for $K=(Q_i\,\rho_0/P_i)^{1/2}$ where $\rho_0=|\,Q_0\,|^2$ is the initial intensity. However, if $P_i\,Q_i<0$, perturbations of all wavelengths are stable (Hasegawa 1975). From eq. (9), we find that

$$P_1Q_1=-3(\beta-3\gamma k)(\alpha k-\delta).$$

So that modulational instability of eq. (1) is decided by the values of α, β, γ and δ. For $\alpha=\gamma=0$, when eq. (1) becomes a NS equation itself, $P_1Q_1=3\beta\delta$; which in confirmation with earlier results is unstable for $\beta\delta>0$. For $\beta=\delta=0$, eq. (1) reduces to the modified KdV equation with $P_1Q_1=9\alpha\gamma k^2{}_j$ this unlike KdV equation is unstable provided $\gamma\,\alpha>0$. For nonvanishing α, β, γ, δ, however, modulational instability can arise only if $k>k^*$ where

$$k^*=[(\alpha\beta+3\gamma\delta)\pm\{(\alpha\beta+3\gamma\delta)^2-12\alpha\beta\gamma\delta\}^{1/2}]/6\alpha\gamma.$$

For k^* to be real we must have either $\alpha\beta\gamma\delta<0$ or $\alpha\beta\gamma\delta>0$ with $(\alpha\beta+3\gamma\delta)^2\geqslant 12\alpha\beta\gamma\delta$.

If $\alpha=\delta=0$, eq. (1) is linear and dispersive and if $\beta=\gamma=0$, i.e.(1) is nonlinear but dispersionless. Evidently in these cases we cannot obtain the NS equation.

In case of eq. (2), eq. (12) gives

$$P_2Q_2=-\frac{3}{2}\frac{(3-2k^2)(12-17k^2+12k^4)}{(1-k^2)^2(3-4k^2)}=-\frac{3}{2}\frac{f(k)}{(1-k^2)^2(3-4k^2)^2}$$

where $f(k)=96k^8-352k^6+510k^4-369k^2+108$. Thus eq. (2) is modulationally unstable for $f(k)<0$ i.e., for $k>k_0=0{\cdot}866$.

Having settled the question of modulational stability of eqs (1) and (2), the corresponding envelope solutions are obtained immediately. Hasegawa (1975) has shown that if $P_t Q_t > 0$, i.e., unstable case, the localized stationary solutions are the envelope solitons:

$$\rho = \rho_s \operatorname{sech}^2 \{(Q_t \rho_s / 2P_t)^{1/2} \xi\},$$

where ρ_s is the amplitude of the envelope soliton. And if $P_t Q_t < 0\,(P_t Q_t = -|P_t Q_t|)$, i.e., stable case, the localized solutions are the envelope holes representing a depletion in the intensity of the plane waves. The stationary solution in the latter case being

$$\rho = \rho_1 [1 - \tilde{a}^2 \operatorname{sech}^2 \{(|P_t Q_t| \rho_1 / 2P_t^2)^{1/2} \tilde{a} \xi\}],$$

where ρ_1 is the level with respect to which the depletion region (hole) is formed, and a is the depth of the depletion (modulation).

Now the localized stationary solutions of eqs (1) and (2) may be easily obtained from these considerations. For example, in eq. (1) the cases (a) $\alpha = \gamma = 0$, β and δ both positive, and (b) $\beta = \delta = 0$, α and γ both positive, admit envelope soliton solutions. In case of eq. (2), the stationary solution is an envelope solution for $k > 0.866$ and an envelope hole otherwise.

4. Conclusions and discussion

We have shown that the amplitudes of the plane wave solutions of eq. (1) are governed by the nonlinear Schrödinger equation and that these plane waves can be modulationally unstable. When $\beta = \delta = 0$ eq. (1) reduces to the modified KdV equation, which has soliton solutions. And eq. (10) with $\beta = \delta = 0$ describes the evolution of the amplitudes of the plane wave solutions of the modified KdV equation. The plane waves are modulationally unstable for $\alpha\gamma > 0$ and stable otherwise, and give rise to envelope soliton or envelope hole solutions respectively. It may be noted that in the KdV equation the sign of the nonlinear term is not important because it can be changed by the transformation $X \to (-X)$. However in the modified KdV equation the sign of the nonlinear term α cannot be changed by such a transformation and hence the sign of α is important.

In the case of the Boussinesq equation, plane waves with $k > 0{\cdot}886$ are unstable against perturbations with $K < (2Q_2\rho_0/P_2)^{1/2}$, and consequently give rise to envelope solitons. When one of these two conditions is violated, it is stable and hence has envelope hole solutions.

Equations (1) and (2) describe nonlinear dispersive media. In such media we can study two types of phenomena. One is the dynamics of the wave form itself and the other that of the wave envelope. Here we have studied the relation between these two phenomena.

The KdV equation or the modified KdV equation describes the dynamics of the wave form itself. These equations admit stationary solutions called solitons. The KdV equation can be obtained only for a specific type of nonlinear dispersive medium and hence solitons are properties of such restricted media only.

On the other hand one can always obtain a plane wave solution of the system of equation describing a nonlinear dispersive medium. The slow variations of the amplitudes of the plane waves due to the nonlinearity and dispersion are described by the NS equation, provided the lowest nonlinearity is cubic or less. If the medium is modulationally unstable it has envelope soliton solutions and if stable it has envelope hole solutions. Equations (1) and (2), whose corresponding NS equations are eqs (10) and (11) respectively, are examples of this fact.

The NS equation, if modulationally stable, can be converted into the KdV equation (Taniuti and Yajima 1969). In the present analysis, however, we have shown that the modified KdV equation can also lead to the NS equation which may be modulationally stable or unstable depending on the constants α and γ.

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