Pramāṇa, Vol. 10, No. 3, March 1978, pp. 267–272, © printed in India.

Compressibility and collisional effects on thermal instability of a partially ionized medium

R C SHARMA and K C SHARMA
Department of Mathematics, Himachal Pradesh University, Simla 171 005

MS received 23 June 1977

Abstract. The thermal instability of a finitely conducting hydromagnetic composite and compressible medium is studied to include the frictional effects with neutrals. The effect of compressibility is found to be stabilizing. In contrast to the nonoscillatory modes for $(C_p/g)\beta > 1$ in the absence of a magnetic field; C_p, β and g being specific heat at constant pressure, uniform adverse temperature gradient and acceleration due to gravity respectively, the presence of magnetic field introduces oscillatory modes in the system. The overstable case is also discussed. The magnetic field is found to have a stabilizing effect on the system for $(C_p/g)\beta > 1$.

Keywords. Compressibility; collisions; thermal instability.

1. Introduction

The problem of thermal instability of fluids, both in hydrodynamics and hydromagnetics, has been treated in detail by Chandrasekhar (1961). The Boussinesq approximation has been used throughout.

The equations governing the system become quite complicated when the fluids are compressible. Spiegel and Veronis (1960) made the following assumptions:

(i) the fluctuations in density, pressure and temperature introduced due to motion, do not exceed their total static variations and

(ii) the depth of fluid layer is much less than the scale height as defined by them.

Using the above assumptions, Spiegel and Veronis (1960) have found the flow equations to be the same as for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic.

Following Hans (1968) the medium has been idealized as a composite mixture of a hydromagnetic (ionized) component and a neutral component, the two interacting through mutual collisional (frictional) effects. As it is quite frequent that the medium is not fully ionized and may be permeated with neutral atoms, the above idealization of the medium as a composite mixture was called for. Hans (1968) also found that these collisions have stabilizing effect on the Rayleigh-Taylor instability. Sharma (1976) studied the thermal hydromagnetic instability of a partially ionized medium.

The object of the present paper is to study the compressibility and collisional effects on thermal instability of a composite medium.

2. Formulation of the problem and dispersion relation

Consider an infinite horizontal compressible and composite layer of thickness d consisting of a finitely conducting hydromagnetic fluid of density ρ, permeated with neutrals of density ρ_d, acted on by a uniform vertical magnetic field $\mathbf{H}(0, 0, H)$ and gravity force $\mathbf{g}(0, 0, -g)$. This layer is heated from below such that a steady adverse temperature gradient $\beta(=|dT/dz|)$ is maintained. Both the ionized fluid and neutral atoms are assumed to behave like continuum fluids.

Following Spiegel and Veronis (1960), Sharma (1976), the basic linearized equations governing the motion of compressible and composite medium are:

$$\rho \frac{\partial \mathbf{q}}{\partial t} = -\nabla\, \delta p + \mathbf{g}\, \delta\rho + \rho\nu \nabla^2 \mathbf{q} + \frac{1}{4\pi} (\nabla \times \mathbf{h}) \times \mathbf{H} + \rho_d \nu_c (\mathbf{q}_d - \mathbf{q}), \tag{1}$$

$$\frac{\partial \mathbf{q}_d}{\partial t} = -\nu_c (\mathbf{q}_d - \mathbf{q}), \tag{2}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{q} \times \mathbf{H}) + \eta \nabla^2 \mathbf{h}, \tag{3}$$

$$\nabla \cdot \mathbf{q} = 0, \ \nabla \cdot \mathbf{h} = 0, \tag{4}$$

$$\frac{\partial \theta}{\partial t} = \left(\beta - \frac{g}{C_p}\right) w + \kappa \triangle^2 \theta, \tag{5}$$

where $\mathbf{q}(u, v, w)$, $\mathbf{h}(h_x, h_y, h_z)$, θ, $\delta\rho$ and δp denote respectively the perturbations in velocity, magnetic field $\mathbf{H}$, temperature T, density ρ and pressure p; g/C_p, μ, $\nu\,(=\mu/\rho_m)$, κ', $\kappa\,(=\kappa'/\rho_m C_p)$, ν_c, $\mathbf{q}_d$ and η stand for the adiabatic gradient, the viscosity, the kinematic viscosity, the thermal conductivity, the thermal diffusivity, the collisional frequency between the two components of the composite medium, the velocity of the neutral component and the resistivity respectively. $\alpha_m(=\alpha$, say) is the coefficient of thermal expansion and $\rho_m(=\rho$, say) is the density of the ionized medium. In writing eq. (2), it is assumed that the effects on the neutral component resulting from the fields of gravity and pressure are neglected and that the neutral particles are non-conducting.

Analyzing in terms of normal modes, we seek solutions of the above equations, whose dependence on space coordinates x, y, z and time t is of the form

$$[w, \theta, h_z] = [W(z), \Theta(z), K(z)] \exp(ik_x x + ik_y y + nt), \tag{6}$$

where k_x, k_y are horizontal wave numbers of the harmonic disturbance, $k^2 = k_x^2 + k_y^2$ and n is the frequency. Eliminating $\mathbf{q}_d$ between eqs (1) and (2) and using expression (6), eqs (1)–(5) give

$$(D^2 - a^2)\left(D^2 - a^2 - \sigma\, \overline{1 + \frac{\alpha_0 \nu_c\, d^2/\nu}{\sigma + \nu_c d^2/\nu}}\right) W + \frac{H}{4\pi\rho\nu} (D^2 - a^2) DK -$$

$$\frac{g\alpha d^2}{\nu} a^2\, \Theta = 0, \tag{7}$$

$$(D^2-a^2-p_2\sigma)K = -\left(\frac{Hd}{\eta}\right)DW, \tag{8}$$

$$(D^2-a^2-p_1\sigma)\,\Theta = -\frac{d^2}{\kappa}\left(\beta-\frac{g}{C_p}\right)W, \tag{9}$$

where $a=kd$, $\sigma=nd^2/\nu$, $p_1=\nu/\kappa$, $p_2=\nu/\eta$, $\alpha_0=\rho_d/\rho$, $D=d/dz$ and x, y, z stand for the coordinates in the new unit of length d. Use has also been made of the Boussinesq equation of state $\delta\rho=-\alpha\rho\theta$.
Eliminating Θ and K between eqs (7)–(9),
we get

$$(D^2-a^2)(D^2-a^2-p_1\sigma)$$

$$\left[(D^2-a^2-p_2\,\sigma)\left(D^2-a^2-\sigma\,\overline{1+\frac{\alpha_0\nu_c d^2/\nu}{\sigma+\nu_c d^2/\nu}}\right)-QD^2\right]W$$

$$= -R\left(\frac{G-1}{G}\right)(D^2-a^2-p_2\sigma)a^2W, \tag{10}$$

where $R=g\alpha\beta d^4/\nu\kappa$ is the Rayleigh number, $Q=H^2d^2/4\pi\rho\nu\eta$ is the Chandrasekhar number and $G=C_p\beta/g$.

Consider the case in which both the boundaries are free and the medium adjoining the fluid is nonconducting. The boundary conditions appropriate for the problem are (Chandrasekhar 1961):

$$\left.\begin{array}{l} W=D^2W=0,\ \Theta=0 \\ X=0 \text{ and } \mathbf{h} \text{ is continuous} \end{array}\right\} \text{at } z=0 \text{ and } 1. \tag{11}$$

In the absence of any surface current, the tangential components of magnetic field are continuous. Hence the boundary conditions in addition to (11) are

$$DK=0, \tag{12}$$

on the boundaries.

Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for $z=0$ and 1 and hence the proper solution of W characterizing the lowest mode is

$$W = W_0 \sin \pi z, \tag{13}$$

where W_0 is a constant.
Substituting (13) in eq. (10), we obtain the dispersion relation

$$R_1 = \left(\frac{G}{G-1}\right)\left[(1+b)\left(1+b+p_1\frac{\sigma}{\pi^2}\right)\frac{\left\{\left(1+b+p_2\frac{\sigma}{\pi^2}\right)\left(1+b+\frac{\sigma}{\pi^2}\,\overline{1+\frac{\alpha_0\,\nu_c\,d^2/\nu}{\sigma+\nu_c\,d^2/\nu}}\right)+Q_1\right\}}{b\left(1+b+p_2\frac{\sigma}{\pi^2}\right)}\right], \tag{14}$$

where

$$Q_1 = \frac{Q}{\pi^2},\ R_1 = \frac{R}{\pi^4} \text{ and } b = \frac{a^2}{\pi^2}.$$

3. The stationary convection

When instability sets in as ordinary convection, the marginal state will be characterized by $\sigma=0$ and eq. (14) reduces to

$$R_1 = \left(\frac{G}{G-1}\right)\left[\frac{1+b}{b}\{(1+b)^2 + Q_1\}\right]. \tag{15}$$

For the fixed value of Q_1, let the nondimensional number G accounting for the compressibility effects be also kept as fixed, then we find that

$$\overline{R}_c = \left(\frac{G}{G-1}\right) R_c, \tag{16}$$

where R_c and $\overline{R}_c$ denote respectively the critical Rayleigh number in the absence and presence of compressibility. The effect of compressibility is, thus, to postpone the onset of thermal instability. Hence we obtain a stabilizing effect of compressibility. The cases $G<1$ and $G=1$ correspond to negative and infinite values of critical Rayleigh numbers in the presence of compressibility which are not relevant for the present problem.

4. Stability of the system and non-oscillatory modes

Multiplying eq. (7) by W^*, the complex conjugate of W, and using eqs (8) and (9), we get

$$I_1 + \sigma\left(1 + \frac{a_0 \nu_c d^2/\nu}{\sigma + \nu_c d^2/\nu}\right) I_2 + \frac{\eta}{4\pi \rho\nu}(I_3 + p_2 \sigma^* I_4)$$

$$+ \frac{C_p \alpha \kappa a^2}{\nu(1-G)}(I_5 + p_1 \sigma^* I_6) = 0, \tag{17}$$

where

$$I_1 = \int_0^1 (|D^2 W|^2 + 2a^2 |DW|^2 + a^4 |W|^2)\, dz,$$

$$I_2 = \int_0^1 (|DW|^2 + a^2 |W|^2)\, dz,$$

$$I_3 = \int_0^1 (|D^2 K|^2 + 2a^2 |DK|^2 + a^4 |K|^2)\, dz,$$

$$I_4 = \int_0^1 (|DK|^2 + a^2 |K|^2)\, dz,$$

$$I_5 = \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2)\, dz,$$

$$I_6 = \int_0^1 (|\Theta|^2)\, dz, \tag{18}$$

which are all positive definite and σ^* is the complex conjugate of σ.

Putting $\sigma = \sigma_r + i\,\sigma_i$ and then equating the real and imaginary parts of eq. (17), we obtain

$$I_1 + \frac{(\sigma_r + \nu_c d^2/\nu)\,\{\sigma_r^2 + \sigma_i^2 + \sigma_r (\nu_c d^2/\nu)(1+a_0)\}}{(\sigma_r + \nu_c d^2/\nu)^2 + \sigma_i^2}\, I_2 + \frac{\eta}{4\pi\rho\nu}(I_3 + p_2 \sigma_r I_4)$$

$$+ \frac{C_p\, a\, \kappa a^2}{\nu(1-G)}(I_5 + p_1 \sigma_r I_6) = 0, \tag{19}$$

and

$$i\sigma_i \left[\frac{\sigma_r^2 + \sigma_r (\nu_c d^2/\nu)(1+a_0) + (\nu_c d^2/\nu)^2 (1+a_0) + \sigma_i^2}{(\sigma_r + \nu_c d^2/\nu)^2 + \sigma_i^2}\, I_2\right.$$

$$\left. - \frac{\eta}{4\pi\rho\nu}\, p_2 I_4 + \frac{C_p\, a\, \kappa a^2}{\nu(G-1)}\, p_1 I_6\right] = 0. \tag{20}$$

It follows from eq. (20) that if $G>1$ and if the magnetic field is absent, $\sigma_i = 0$, which means that the oscillatory modes are not allowed for $G>1$ and in the absence of magnetic field. The presence of magnetic field, in contrast to nonoscillatory modes for $G>1$ in the absence of magnetic field, introduces oscillatory modes in the system.

5. The overstable case

In this section we consider the possibility of whether instability may occur as an overstability. Put $\sigma/\pi^2 = i\sigma_1$, it being remembered that σ may be complex. Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (14) will admit of solution with σ_1 real. Equation (14) becomes

$$R_1 = \left(\frac{G}{G-1}\right)\left[(1+b)(1+b+i\,p_1\sigma_1)\right.$$

$$\left.\frac{\left\{(1+b+i\,p_2\sigma_1)\left(1+b+i\,\sigma_1\,\overline{1+\frac{a_0\,\nu_c\,d/^2\nu}{\sigma+\nu_c\,d^2/\nu}}\right)+Q_1\right\}}{b\,(1+b+i\,p_2\sigma_1)}\right]. \tag{21}$$

Equating real and imaginary parts of eq. (21) and eliminating R_1 between them, we obtain

$$(p_2^2\pi^2\nu/d^2)\,[p_1\alpha_0\nu_c + (1+b)(1+p_1)(\pi^2\nu/d^2)]\,\sigma_1^4$$
$$+ (1+b)\,[p_2^2\nu_c^2(1+p_1+\alpha_0) + p_1(1+b)(\pi^2\nu/d^2)\,\alpha_0\nu_c$$
$$+ (1+b)^2(1+p_1)(\pi^4\nu^2/d^4) + Q_1(p_1-p_2)(\pi^4\nu^2/d^4)]\sigma_1^2$$
$$+ (1+b)\,[(1+b)^2\nu_c^2(1+p_1+\alpha_0) + Q_1(p_1-p_2)\nu_c^2] = 0. \tag{22}$$

Equation (22) is quadratic in σ_1^2 and does not allow any of its roots to be such that $\mathrm{Re}(\sigma_1)$ is positive so that σ_1 is imaginary when $p_1 \geqslant p_2$. Hence $p_1 \geqslant p_2$ i.e.

$$\kappa \leqslant \eta, \tag{23}$$

is a sufficient condition for the nonexistence of overstability. The condition (23) is the same as in the absence of compressibility as well as frictional (collisional) effects with neutrals on thermal instability (Chandrasekhar 1961).

Thus $\kappa \leqslant \eta$ is a sufficient condition for the nonexistence of overstability which holds both in the presence or absence (Chandrasekhar 1961) of compressibility and collisional effects with neutrals on thermal instability of a composite and compressible medium. To study the effect of magnetic field on thermal instability of a compressible and composite medium, we examine the nature of dR_1/dQ_1. It follows from eq. (14) that

$$\frac{dR_1}{dQ_1} = \left(\frac{G}{G-1}\right)\left(\frac{1+b}{b}\right)\left[\frac{(1+b)^2+p_1p_2\sigma_1^2+i\sigma_1(1+b)(p_1-p_2)}{(1+b)^2+p_2\sigma_1^2}\right]. \tag{24}$$

The imaginary part of eq. (24) equated to zero gives

$$p_1 = p_2. \tag{25}$$

Equating real parts of eq. (24) and substituting (25) in it, we obtain

$$\frac{dR_1}{dQ_1} = \left(\frac{G}{G-1}\right)\left(\frac{1+b}{b}\right). \tag{26}$$

It follows from eq. (26) that dR_1/dQ_1 is positive if $G>1$. Hence for $G>1$, the Rayleigh number increases as the magnetic field increases, showing the stabilizing effect of magnetic field. The cases $G<1$ and $G=1$ are not allowed here as those would mean negative and infinite Rayleigh numbers respectively which are not relevant in the present discussion.

References

Chandrasekhar S 1961 *Hydrodynamic and Hydromagnetic Stability* (Oxford: Clarendon Press) Chap. 4
Hans H K 1968 *Nucl. Fusion* **8** 89
Sharma R C 1976 *Physica* **C81** 199
Spiegel E A and Veronis G 1960 *Astrophys. J.* **131** 442