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Extremum uncertainty product and sum states

C L MEHTA and SUNIL KUMAR
Department of Physics, Indian Institute of Technology, New Delhi 110 029

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Abstract. We consider the states with extremum products and sums of the uncertainties in non-commuting observables. These are illustrated by two specific examples of harmonic oscillator and the angular momentum states. It shows that the coherent states of the harmonic oscillator are characterized by the minimum uncertainty sum $\langle(\triangle\hat{q})^2\rangle + \langle(\triangle\hat{p})^2\rangle$. The extremum values of the sums and products of the uncertainties of the components of the angular momentum are also obtained.

Keywords. Minimum uncertainty states.

1. Introduction

It is well known that two non-commuting observables cannot simultaneously have sharply defined values. In fact, if

$$[\hat{A}, \hat{B}] = i\hat{C}, \tag{1}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ are Hermition operators, then the uncertainties $\langle(\triangle\hat{A})^2\rangle$ and $\langle(\triangle\hat{B})^2\rangle$ satisfy the inequality

$$\langle(\triangle\hat{A})^2\rangle \, \langle(\triangle\hat{B})^2\rangle \geqslant \tfrac{1}{4} \langle\hat{C}\rangle^2. \tag{2}$$

Further, if the observables representing $\hat{A}$ and $\hat{B}$ have the same dimensions, we find from the relation

$$\{\langle(\triangle\hat{A})^2\rangle^{\frac{1}{2}} - \langle(\triangle\hat{B})^2\rangle^{\frac{1}{2}}\}^2 \geqslant 0$$

that

$$\langle(\triangle\hat{A})^2\rangle + \langle(\triangle\hat{B})^2\rangle \geqslant 2\langle(\triangle\hat{A})^2\rangle^{\frac{1}{2}} \, \langle(\triangle\hat{B})^2\rangle^{\frac{1}{2}}. \tag{3}$$

From (2) and (3), we obtain

$$\langle(\triangle\hat{A})^2\rangle + \langle(\triangle\hat{B})^2\rangle \geqslant |\langle\hat{C}\rangle|. \tag{4}$$

In case when (4) is an equality, the relations (2) and (3) must also necessarily reduce to equalities and

$$\langle(\triangle\hat{A})^2\rangle = \langle(\triangle\hat{B})^2\rangle = \tfrac{1}{2}|\langle\hat{C}\rangle|. \tag{5}$$

Let us briefly consider the proof of the uncertainty relation (2). If λ is real, we observe that for an arbitrary state, the inequality

$$\langle (\hat{A} - i\,\lambda\,\hat{B})\;(\hat{A} + i\,\lambda\,\hat{B})\rangle \geqslant 0 \tag{6}$$

holds. The left hand side is minimum when

$$\lambda = \tfrac{1}{2}\frac{\langle \hat{C}\rangle}{\langle \hat{B}^2\rangle}. \tag{7}$$

Setting this value of λ in (6), we find that

$$\langle \hat{A}^2\rangle\;\langle \hat{B}^2\rangle \geqslant \tfrac{1}{4}\,\langle \hat{C}\rangle^2. \tag{8}$$

The equality sign in (8) will hold, if and only if it holds in (6). This implies that (8) is an equality only for those states (or a statistical mixture of such states) which are the eigenstates of the operator $(\hat{A} + i\,\lambda\,\hat{B})$ belonging to the eigenvalue zero:

$$(\hat{A} + i\,\lambda\,\hat{B})\,|\,\psi\rangle = 0. \tag{9}$$

Now, since the operators $\triangle\,\hat{A} = \hat{A} - \langle \hat{A}\rangle$ and $\triangle\,\hat{B} = \hat{B} - \langle \hat{B}\rangle$ also satisfy the commutation relation

$$[\triangle\hat{A},\;\triangle\hat{B}] = i\,\hat{C}, \tag{10}$$

we may replace $\hat{A}$ and $\hat{B}$ in (8) by $\triangle\hat{A}$ and $\triangle\hat{B}$ respectively

$$\langle(\triangle\hat{A})^2\rangle\;\langle(\triangle\hat{B})^2\rangle \geqslant \tfrac{1}{4}\,\langle \hat{C}\rangle^2. \tag{11}$$

The equality sign in (11) will hold only for those states which satisfy [cf. eq. (9) and also Jackiw (1968)]

$$(\triangle\hat{A} + i\lambda\triangle\hat{B})\,|\psi\rangle = 0,$$

or

$$(\hat{A} + i\lambda\hat{B})\,|\psi\,\rangle = [\,\langle \hat{A}\rangle + i\lambda\,\langle \hat{B}\rangle\,]\,|\psi\rangle, \tag{12}$$

where λ is an arbitrary real number. The uncertainties $\langle(\triangle\hat{A})^2\rangle$ and $\langle(\triangle\hat{B})^2\rangle$ for these states are in fact given by

$$\langle(\triangle\hat{A})^2\rangle = \tfrac{1}{2}\,\lambda\,\langle \hat{C}\rangle \tag{13}$$

and

$$\langle(\triangle\hat{B})^2\rangle = \frac{1}{2\lambda}\,\langle \hat{C}\rangle. \tag{14}$$

Equations (13) and (14) are readily derived by (12) on setting $\triangle\hat{A} + i\lambda\, \triangle\hat{B} = \hat{K}$ so that $\triangle\hat{A} = \frac{1}{2}(\hat{K} + \hat{K}^\dagger)$, $\triangle\hat{B} = \frac{1}{2}(i\lambda)^{-1}(\hat{K} - \hat{K}^\dagger)$ and observing that

$$\langle \hat{K}^2 \rangle = \langle \hat{K}^{\dagger 2} \rangle = \langle \hat{K}^\dagger \hat{K} \rangle = 0,$$

$$\langle \hat{K}\hat{K}^\dagger \rangle = \langle [\hat{K}, \hat{K}^\dagger] + \hat{K}^\dagger \hat{K} \rangle = 2\lambda \langle \hat{C} \rangle.$$

The states for which the inequality (4) is an equality are now readily obtained by requiring [cf. eq. (5)] $\langle (\triangle\hat{A})^2 \rangle = \langle (\triangle\hat{B})^2 \rangle$. This gives us $\lambda = \pm 1$ depending on whether $(\hat{C})$ is positive or negative.

We also observe an important conclusion from this result. It is readily seen from eq. (13) that if $\hat{C}$ is a positive definite operator, λ is necessarily positive. This implies that *in the case when* $[\hat{A}, \hat{B}] = i\hat{C}$ *with* $\hat{C}$ *positive definite there are no eigenstates of the operator* $\hat{A} + i\lambda\hat{B}$ *with* $\lambda < 0$.

When the commutator $[\hat{A}, \hat{B}]$ is a C-number ($= i\,C$) we find that $\langle C \rangle$ is, simply a constant and does not depend on the state. In such cases, we refer to the states, for which equality sign in (2) holds as the minimum uncertainty *product* states and to those for which equality sign in (4) holds as the minimum uncertainty *sum* states. Obviously, the minimum uncertainty sum states are also necessarily the minimum uncertainty product states, and, for such states

$$\langle (\triangle\hat{A})^2 \rangle = \langle (\triangle\hat{B})^2 \rangle = \tfrac{1}{2}\, C. \tag{15}$$

The minimum uncertainty product states are the eigenstates of $\hat{A} + i\lambda\hat{B}$ *with* λ *real and having the same sign as C. The minimum uncertainty sum states are the eigenstates of* $\hat{A} + i\hat{B}$ *if* $C > 0$ *and of* $\hat{A} - i\hat{B}$ if $C < 0$. It may further be observed that the eigenstates of $\hat{A} + i\lambda\hat{B}$ with λ having a sign opposite to that of C do not exist.

When the commutator $[\hat{A}, \hat{B}]$ is a q-number $= i\hat{C}$, we find that $\langle \hat{C} \rangle$ will in general depend on the given state. The nomenclature (cf. Ruschin and Ben Aryeh 1976), of calling the states which satisfy (2) as an equality the " minimum uncertainty product states " is therefore misleading. For such states, only the quantity $\langle (\triangle\hat{A})^2 \rangle \langle (\triangle\hat{B})^2 \rangle / |\langle \hat{C} \rangle|$ is minimum which is not the product of uncertainties since $|\langle \hat{C} \rangle|$ is neither an uncertainty nor a normalization constant. These states, may, however, be interpreted as " minimum uncertainty product states " in a restrictive or relative sense: Among those states for which $|\langle \hat{C} \rangle|$ is fixed, the uncertainty product $\langle (\triangle\hat{A})^2 \rangle \langle (\triangle\hat{B})^2 \rangle$ is minimum for such states and these are then the eigenstates of the operator $\hat{A} + i\lambda\hat{B}$. Similar remarks also hold for states which satisfy (4) as an equality.

We illustrate our observations by considering two specific examples, viz., the harmonic oscillator and the angular momentum states.

2. Harmonic oscillator states

Let $[\hbar/(m\omega)]^{\frac{1}{2}}\,\hat{q}$ and $(m\omega\hbar)^{\frac{1}{2}}\,\hat{p}$ denote the position and momentum operators respectively of a harmonic oscillator (so that $\hat{q}$ and $\hat{p}$ are both dimensionless). $\hat{q}$ and $\hat{p}$ then satisfy the commutation relation

$$[\hat{q}, \hat{p}] = i. \tag{15}$$

We identify $\hat{A}$, $\hat{B}$ and C by $\hat{q}, \hat{p}$ and 1 respectively. We thus find that the states with the minimum product of the uncertainties in position and momentum variables are the eigenstates of $(\hat{q} + i\lambda\hat{p})$ where λ is an arbitrary positive number. (Since $C = 1$, is positive definite, there are no eigenstates with λ negative). If $\hat{a}$ and $\hat{a}^\dagger$ denote the annihilation and creation operators defined as

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}), \tag{16}$$

then the minimum uncertainty product states are the eigenstates of the operator

$$\hat{A}_\lambda = \frac{1+\lambda}{\sqrt{2}}\hat{a} + \frac{1-\lambda}{\sqrt{2}}\hat{a}^\dagger, \quad \lambda \geqslant 0. \tag{17}$$

The extreme values of λ, viz. 0 and ∞ correspond respectively to the operators proportional to $\hat{q}$ and $\hat{p}$.

The eigenstates $|\psi_0\rangle$ of $\hat{A}_\lambda$ with eigenvalue zero may readily be expressed in terms of the number states $|n\rangle$. Thus writing

$$|\psi_0\rangle = \sum_n C_n |n\rangle,$$

where $\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle$ and requiring $\hat{A}_\lambda|\psi_0\rangle = 0$ we obtain

$$|\psi_0\rangle = C_0 \sum_{m=0}^{\infty} \left(-\frac{1}{2}\frac{1-\lambda}{1+\lambda}\right)^m \frac{\{(2\,m)!\}^{\frac{1}{2}}}{m!}\,|2m\rangle. \tag{18}$$

The general eigenstate $|\psi_\alpha\rangle$ with eigenvalue α of the operator $\hat{A}_\lambda$ is obtained from $|\psi_0\rangle$ in the following way:

$$|\psi_\alpha\rangle = \exp\left(\frac{\alpha\hat{A}_\lambda{}^\dagger - \alpha^*\hat{A}_\lambda}{2\lambda}\right)|\psi_0\rangle. \tag{19}$$

The state $|\psi_\alpha\rangle$ is the most general state [Schrödinger 1926; cf. also Schiff 1955] with the minimum product of the uncertainties in $\hat{q}$ and $\hat{p}$, $\{\langle(\Delta\hat{q})^2\rangle = \frac{1}{2}\lambda;$ $\langle(\Delta\hat{p})^2\rangle = \frac{1}{2}\lambda;$ $\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle = \frac{1}{4}\}$.

The states with the minimum sum of the uncertainties in $\hat{q}$ and $\hat{p}$ are obtained by setting $\lambda = 1$. In this case $\langle(\Delta\hat{q})^2\rangle = \langle(\Delta\hat{p})^2\rangle = \frac{1}{2}$ and $\langle(\Delta\hat{q})^2\rangle + \langle(\Delta\hat{p})^2\rangle = 1$. These states are the eigenstates of the annihilation operator $\hat{a}$, which are the usual coherent states (Glauber 1963). We further observe that there are in fact no eigenstates of the operator $\hat{a}^\dagger$.

3. Angular momentum states

Let $\hbar\hat{J}_x, \hbar\hat{J}_y, \hbar\hat{J}_z$ be the Cartesian components of the angular momentum operator. Since $[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$ we may identify $\hat{A}, \hat{B}, \hat{C}$ with the operators $\hat{J}_x, \hat{J}_y, \hat{J}_z$ respectively. We thus find that the states for which [cf. relation (12)]

$$\langle(\Delta\hat{J}_x)^2\rangle\ \langle(\Delta\hat{J}_y)^2\rangle = \tfrac{1}{4}\langle\hat{J}_z\rangle^2, \tag{20}$$

are the eigenstates of $\hat{J}_x + i\lambda\hat{J}_y$ where λ is an arbitrary real number. Examples of these states for $-1 \leqslant \lambda \leqslant 1$ are the recently introduced atomic coherent states [Arecchi *et al* 1972] $|\theta, \phi\rangle$ with $\phi = 0, \pm\frac{1}{2}\pi$ or π. The state $|\theta, \phi\rangle$ is obtained by the rotation of lowest $\hat{J}_z$-eigenstate $|-j\rangle$ in the angular momentum space and is given by

$$|\theta, \phi\rangle = \exp\{-i\theta(\hat{J}_x \sin\phi - \hat{J}_y \cos\phi)\}\ |-j\rangle. \tag{21}$$

For these states one may verify that

$$\langle(\Delta\hat{J}_x)^2\rangle = \tfrac{1}{2}j(1 - \sin^2\theta\cos^2\phi), \tag{22}$$

$$\langle(\Delta\hat{J}_y)^2\rangle = \tfrac{1}{2}j(1 - \sin^2\theta\sin^2\phi), \tag{23}$$

$$\langle\hat{J}_z\rangle \quad = -j\cos\theta. \tag{24}$$

Therefore, we observe that* $\langle(\Delta\hat{J}_x)^2\rangle\ \langle(\Delta\hat{J}_y)^2\rangle = \frac{1}{4}\langle\hat{J}_z\rangle^2$ for $\phi = 0, \pm\frac{1}{2}\pi$ or π. The states for which

$$\langle(\Delta\hat{J}_x)^2\rangle + \langle(\Delta\hat{J}_y)^2\rangle = |\langle\hat{J}_z\rangle| \tag{25}$$

are obtained by setting $\lambda = \pm 1$. These are then the eigenstates of $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. There is only one eigenstate of $\hat{J}_+$ ($\hat{J}_-$) viz., the state $|j\rangle$ ($|-j\rangle$) with maximum

* It has been stated by Arecchi *et al* (1972) that the atomic coherent states $|\theta, \phi)\rangle$ satisfy the "minimum uncertainty relation" $\langle(\Delta\hat{J}_x')^2\rangle\ \langle(\Delta\hat{J}_y')^2\rangle = \frac{1}{4}\langle\hat{J}_z'\rangle^2$, where primes denote the rotated angular momentum components $\hat{J}_x' = \hat{R}\hat{J}_x\hat{R}^{-1}$, etc., and the averages are taken in the state $|\theta, \phi\rangle = \hat{R}|-j\rangle$. The statement is however trivial since what it amounts to, is that $\langle(\Delta\hat{J}_x^2)\rangle\ \langle(\Delta\hat{J}_y^2)\rangle = \frac{1}{4}\langle\hat{J}_z^2\rangle$ where the average are taken in the state $|-j\rangle$ and thus the atomic coherent states as such have nothing to do with the 'minimum uncertainty product' in that context.

(minimum) m-value and the corresponding eigenvalue is zero. The uncertainty sum in this case is given by

$$\langle(\triangle\hat{J}_x)^2\rangle + \langle(\triangle\hat{J}_y)^2\rangle = j.$$

It is however obvious that the states satisfying eq. (20) are not the minimum uncertainty product states in the absolute sense—nor those satisfying eq. (25) are the minimum uncertainty sum states. It may readily be seen that the absolute minimum value of the product $\langle(\triangle\hat{J}_x)^2\rangle\ \langle(\triangle\hat{J}_y)^2\rangle$ is in fact zero and it occurs only for the states for which either $\langle(\triangle\hat{J}_x)^2\rangle$ or $\langle(\triangle\hat{J}_y)^2\rangle$ is zero, i.e. for the eigenstates of either $\hat{J}_x$ or of $\hat{J}_y$. We have not been able to obtain the absolute minimum value of the sum $\langle(\triangle\hat{J}_x)^2\rangle + \langle(\triangle\hat{J}_y)^2\rangle$ but it is certainly not j. For the state $|\hat{J}_x = j\rangle$ we find that $\langle(\triangle\hat{J}_x)^2\rangle = 0$, $\langle(\triangle\hat{J}_y)^2\rangle = \frac{1}{2}j$ so that the above uncertainty sum has the value $\frac{1}{2}j$. It appears that this is the lower bound.

It is of interest to note extremum values, for a given j, of some related sums and products of the uncertainties in $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$:

$$0 \leqslant \langle(\triangle\hat{J}_a)^2\rangle \leqslant j^2; \quad (a = x, y, z), \tag{26}$$

$$\langle(\triangle\hat{J}_x)^2\rangle + \langle(\triangle\hat{J}_y)^2\rangle \leqslant j(j+1) - \epsilon_j^2, \tag{27}$$

$$0 \leqslant \langle(\triangle\hat{J}_x)^2\rangle\ \langle(\triangle\hat{J}_y)^2)\rangle \leqslant \tfrac{1}{4}\,[j(j+1) - \epsilon_j^2]^2, \tag{28}$$

$$j \leqslant \langle(\triangle\hat{J}_x)^2\rangle + \langle(\triangle\hat{J}_y)^2\rangle + \langle(\triangle\hat{J}_z)^2\rangle \leqslant j(j+1), \tag{29}$$

$$0 \leqslant \langle(\triangle\hat{J}_x)^2\rangle\ \langle(\triangle\hat{J}_y)^2\rangle\ \langle(\triangle\hat{J}_z)^2\rangle \leqslant \frac{1}{27}[j(j+1)]^2, \tag{30}$$

where ϵ_j is the lowest non-negative eigenvalue of $\hat{J}_z$:

$$\begin{aligned} \epsilon_j &= 0 \text{ when } j \text{ is an integer,} \\ &= \tfrac{1}{2} \text{ when } j \text{ is half odd integer.} \end{aligned} \tag{31}$$

Relation (26) is obvious since $j^2 \geqslant \langle\hat{J}_a^2\rangle \geqslant \langle(\triangle\hat{J}_a)^2\rangle \geqslant 0$, and that $\langle(\triangle\hat{J}a)^2\rangle$ is in fact zero for any of the $(2j+1)$ eigenstates of $\hat{J}_a$ and is equal to j^2 for the two states $2^{-\frac{1}{2}}\,[|\hat{J}_a = j\rangle \pm |\hat{J}_a = -j\rangle]$. There may of course be several other mixed states for which $\langle(\triangle\hat{J}_a)^2\rangle = j^2$. Relation (27) is obtained by observing that

$$\langle(\triangle\hat{J}_x)^2\rangle + \langle(\triangle\hat{J}_y)^2\rangle = \langle\hat{J}^2\rangle - [\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2 + \langle\hat{J}_z^2\rangle], \tag{32}$$

and that the minimum value of the quantity in the square bracket is ϵ_j^2 and this occurs for example for the state $|\hat{J}_z = \epsilon_j\rangle$. For this state $\langle(\Delta\hat{J}_x)^2\rangle\,\langle(\Delta\hat{J}_y)^2\rangle$ is also maximum since in this case $\langle(\Delta\hat{J}_x)^2\rangle = \langle(\Delta\hat{J}_y)^2\rangle$. This establishes the relation (28).

Relation (29) is obtained by noting that

$$\langle(\Delta\hat{J}_x)^2\rangle + \langle(\Delta\hat{J}_y)^2\rangle + \langle(\Delta\hat{J}_z)^2\rangle = \langle\hat{J}^2\rangle - |\langle\hat{J}\rangle|^2, \tag{33}$$

and that $0 \leqslant |\langle\hat{J}\rangle| \leqslant j$. Thus for example $|\langle J\rangle| = j$ for the state $|\hat{J} = j\rangle$ whereas $|\langle\hat{J}\rangle| = 0$ for the mixed state described by the density operator

$$\hat{\rho} = (2j+1)^{-1}\hat{I} \tag{34}$$

where $\hat{I}$ is the identity operator in $(2j+1)$ dimensions. In fact, for the state described by the above density operator we find that

$$\langle(\Delta\hat{J}_x)^2\rangle = \langle(\Delta\hat{J}_y)^2\rangle = \langle(\Delta\hat{J}_z)^2\rangle = \tfrac{1}{3}j(j+1), \tag{35}$$

from which relation (30) follows immediately. There may of course be several other states for which (35) holds.

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