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Generation of gluons from quark confinement

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Abstract. We study a model of quark confinement defined by the vanishing of colour currents. The model is shown to be equivalent to quantum chromodynamics and this equivalence is interpreted as due to the compositeness of the colour gluons. The Green's functions of the theory are found to contain nontrivial structure only for colour singlet composites which can be identified with hadrons.

Keywords. Nonabelian gauge theory; quantum chromodynamics; quark confinement; equivalence theorem; composite gluons.

1. Introduction

Ever since the idea of quarks as constituents of hadrons was proposed (Gell-Mann 1964, Zweig 1964), it has steadily gained strength as a working hypothesis. The recent discovery of asymptotically free nonabelian gauge theory (Gross and Wilczek 1973a, Politzer 1973, t'Hooft 1972) led to the construction of the field theory of quarks. Asymptotic freedom provided a natural explanation for the experimentally observed nearly free point-like behaviour of the constituents of the nucleon.

The nonabelian gauge group was taken to be the so-called colour SU(3) which acts on the colour index of the quark and the octet of gauge bosons are called gluons. This colour gauge theory of quarks and gluons named as quantum chromodynamics (QCD) has emerged as an attractive candidate for a field theory of strong interactions (Weinberg 1973, Fritzsch *et al* 1973).

But then, why are the quarks not observed**? This is the first problem faced by QCD. One possibility which is widely favoured is that quarks are permanently confined within the hadrons and can never be isolated. Considerable amount of theoretical effort has been already expended in aiming to prove confinement (see for instance Wilson 1974, Kogut and Susskind 1974, Cornwall and Tiktopoulos 1976 and Polyakov 1976) it is only fair to say that quark confinement still remains a hypothesis.

Our plan is to invert the problem and study the consequences. That is to say, instead of starting from QCD and trying to derive quark confinement which has proved very difficult, we accept quark confinement as a starting point and attempt to see how close we can approach QCD. It will be shown that QCD can in fact be derived from this starting point.

**LaRue *et al* (1977) have recently reported to have observed fractional charges. However, this is yet to be confirmed by indcpendent experiments.

This idea of starting with quark confinement is not a new one. Amati and Testa (1974) had proposed such an idea earlier, but there is a crucial difference between their approach and ours as will be indicated later.

The model of quark confinement is formulated in section 2 and its equivalence to QCD is shown in section 3. Section 4 discusses the same equivalence from the point of view of composite gluons. The structure of the Green's functions of the quark confinement model is analysed in section 5. In section 6, a comparison with other related works as well as a summary are presented.

2. The model of quark confinement

The model is defined by the Lagrangian*

$$\mathscr{L}_1=\bar{\psi}\,(i\,\gamma\cdot\partial-m)\,\psi \tag{1}$$

coupled with the constraint equations:

$$j_\mu{}^i\equiv\bar{\psi}\,\gamma_\mu\,\frac{\lambda^i}{2}\,\psi=0;\ i=1\ \dots\ 8. \tag{2}$$

Here, ψ denotes the quark field which is taken to be a triplet under colour SU(3) and $\lambda^i/2$ are the SU(3) generators in the triplet representation. The requirement of the vanishing of the colour octet currents (eq. 2) guarantees the absence of coloured states, and hence quark confinement. Since the Lagrangian $\mathscr{L}_1$ does not contain any explicit interaction terms, all the strong interactions of the hadrons are supposed to arise purely from the constraint eq. (2).

Under the infinitesimal gauge transformation

$$\psi\to\psi'=\left(1+ia^i(x)\frac{\lambda^i}{2}\right)\psi, \tag{3}$$

we have

$$\mathscr{L}_1\to\mathscr{L}_1'=\mathscr{L}_1-j_\mu{}^i\,\partial^\mu a^i. \tag{4}$$

However, in view of the constraint eq. (2), we see that

$$\mathscr{L}_1'=\mathscr{L}_1. \tag{5}$$

Thus, the model defined by (1) and (2) is gauge-invariant. Here we have the remarkable case of local gauge invariance being satisfied even without the presence of gauge fields.

It is important to note that such a model defined by the vanishing of the currents correspond to a nonabelian group symmetry. If we demand the vanishing of the

*Our metric, gamma matrices, etc. are the same as in Bjorken and Drell (1965).

abelian current $\bar{\psi}\gamma_\mu\psi$ this requires $\psi^\dagger\psi=0$ for which the only solution is $\psi=0$. The situation is quite different for nonabelian currents. Let us first consider the case of SU(2) for which the constraint equations can be written in the explicit form:

$$\sum_{i,j,\alpha} \psi_i^{\alpha *} \vec{\tau}_{ij} \psi_j^\alpha = 0,$$

$$\sum_{i,j,\alpha,\beta} \psi_i^{\alpha *} \vec{\tau}_{ij} (\gamma_0 \vec{\gamma})_{\alpha\beta} \psi_j^\beta = 0. \tag{6}$$

Here, $\vec{\tau}$ denotes the Pauli matrices acting on the SU(2) doublet ψ and the two indices i and α on ψ refer to SU(2) and Dirac space respectively. It can be verified that the solution:

$$\psi_1{}^1 = \psi_2{}^2 = \xi; \quad \psi_i{}^\alpha = 0 \text{ for others;} \tag{7}$$

satisfies the conditions (6) and thus provides a nonvanishing ψ. Clearly, such a nonvanishing ψ exists for the physically relevant colour SU(3) also; in particular, since SU(2) is a subgroup of SU(3), the solution (7) itself provides an example. Thus, for the very existence of the model, the nonabelian symmetry group is essential.

The model can also be defined by the generating functional obtained by functional integration:

$$W_1(\eta, \bar{\eta}) = \int \mathscr{D}\psi \mathscr{D}\bar{\psi}\, \delta(j_\mu{}^i) \exp i \int \{ \mathscr{L}_1 + \bar{\eta}\psi + \bar{\psi}\eta \} d^4x \tag{8}$$

where $\bar{\eta}$ and η are the source functions for ψ and $\bar{\psi}$. The δ-function takes care of the constraint (2) and it actually stands for

$$\prod_{i,\mu,x} \delta(j_\mu{}^i(x)).$$

3. Generation of gluons

In this section, we shall show that the model of quark confinement formulated in the last section is equivalent to the gauge-invariant theory of quarks interacting through nonabelian colour gauge gluons–namely quantum chromodynamics (QCD).

We first replace $\mathscr{L}_1$ and the constraint (2) by the equivalent Lagrangian $\mathscr{L}_2$.

$$\mathscr{L}_2 = \bar{\psi}(i\gamma.\partial - m)\psi - \bar{\psi}\gamma^\mu \frac{\lambda^i}{2}\psi A_\mu{}^i \tag{9}$$

where we have introduced the vector field $A_\mu{}^i$ belonging to the octet representation of colour SU(3). The Euler-Lagrange variational equation for $A_\mu{}^i$ is

$$0 = \frac{\partial \mathscr{L}_2}{\partial A_\mu{}^i} = -\bar{\psi}\gamma^\mu \frac{\lambda^i}{2}\psi \tag{10}$$

which is seen to be identical to the constraint equation (2) and when this is substituted into (9) we get back (1). An alternate way of seeing the same result is to consider the generating functional corresponding to $\mathscr{L}_2$:

$$W_2(\eta, \bar{\eta}) = \int \mathscr{D}\psi \, \mathscr{D}\bar{\psi} \, \mathscr{D} A_\mu{}^i \exp i \int \{\mathscr{L}_2 + \bar{\eta}\psi + \bar{\psi}\eta\} \, d^4x. \tag{11}$$

Since the exponential is linear in $A_\mu{}^i$, the integration over $A_\mu{}^i$ would produce the δ-function of the current and hence W_2 is the same as W_1 defined in (8). Thus the model of quark confinement can also be described by the Lagrangiar $\mathscr{L}_2$. However, the Lagrangian $\mathscr{L}_2$ does not yet correspond to that of QCD, since $A_\mu{}^i$ is not equipped with kinetic energy nor with the characteristic self-interaction terms.

We shall now show that the kinetic energy as well as the self-interaction terms can be generated by divergent radiative corrections. The technique is that used earlier (Kugo 1976, Kikkawa 1976, Eguchi 1976, Rajasekaran and Srinivasan 1977) in showing the equivalence of certain non-renormalizable interactions to renormalizable theories. We start with W_2 defined in (11) and perform the integration over ψ and $\bar{\psi}$. This can be done by a shift of variables:

$$\psi' = \psi + D^{-1}\eta \tag{12}$$

$$\bar{\psi}' = \bar{\psi} + \bar{\eta} D^{-1}$$

where

$$D^{-1} = \left(1 - S_F \gamma^\mu \frac{\lambda^i}{2} A_\mu{}^i\right)^{-1} S_F$$

$$S_F = (i\,\gamma.\partial - m)^{-1}.$$

Ignoring an inessential constant factor, the result can be written as

$$W_2 = \int \mathscr{D} A_\mu{}^i \det\left(1 - S_F \gamma^\mu \frac{\lambda^i}{2} A_\mu{}^i\right) \exp\left\{-i \int d^4x \, \bar{\eta} D^{-1} \eta\right\} \tag{13}$$

where the determinant occurs in the numerator rather than in the denominator because the integration variable was a Fermi field.

We expand the determinant in a series:

$$\det\left(1 - S_F \gamma^\mu \frac{\lambda^i}{2} A_\mu{}^i\right) = \exp\left\{\mathrm{Tr} \ln\left(1 - S_F \gamma^\mu \frac{\lambda^i}{2} A_\mu{}^i\right)\right\}$$

$$= \exp\left\{-\mathrm{Tr} \sum_n \frac{1}{n}\left(S_F \gamma^\mu \frac{\lambda^i}{2} A_\mu{}^i\right)^n\right\}. \tag{14}$$

The trace expression in (14) corresponds to the series depicted in figure 1, namely, diagrams with a single closed loop of ψ propagators and arbitrary number n of external $A_\mu{}^i$ lines. These are divergent up to $n=4$ and convergent for $n>4$.

Before we proceed to the actual calculation of the divergent parts, we should note that the Lagrangian $\mathscr{L}_2$ on which the calculation is based, is invariant under the infinitesimal gauge transformation:

$$\psi \to \psi' = \left\{ 1 + i\, a^i(x) \frac{\lambda^i}{2} \right\} \psi$$

$$A_\mu{}^i \to A_\mu{}^{i'} = A_\mu{}^i - f^{ijk} a^j(x) A_\mu{}^k - \partial_\mu a^i(x) \tag{15}$$

where f^{ijk} are the structure constants of SU(3). Hence, one should choose a gauge-invariant regularization such as dimensional regularization (t'Hooft and Veltman 1972) in computing the divergent integrals. When this is done, one can verify that the apparent quadratic divergence of the $n=2$ term represented by the first diagram in figure 1 does not materialize. Or, stated in the language of dimensional regularization, the pole at dimension $l=2$ cancels. One can further verify that the divergent parts of the $n=2$, 3 and 4 terms have to appear in the following gauge-invariant combination:

$$-i \frac{I_0}{3} \int d^4x\, G^i_{\mu\nu}\, G_i^{\mu\nu} \tag{16}$$

where

$$G^i{}_{\mu\nu} = \partial_\mu A_\nu{}^i - \partial_\nu A_\mu{}^i - f^{ijk} A_\mu{}^j A_\nu{}^k \tag{17}$$

and the constant I_0 is logarithmically divergent (or pole at $l \geq 4$). Isolating the pole at dimension $l=4$, we get

$$I_0 = \frac{1}{(2\pi)^2} \frac{1}{l-4}. \tag{18}$$

Combining (13), (14), and (16) we may write

$$W_2 = \int \mathscr{D}A^i{}_\mu \exp i \int d^4x \left\{ -\frac{I_0}{3} G^i_{\mu\nu}\, G_i^{\mu\nu} - \bar{\eta}\left(1 - S_F \gamma^\mu \frac{\lambda^i}{2} A_\mu{}^i\right)^{-1} S_F \eta + F(A_\mu{}^i) \right\}. \tag{19}$$

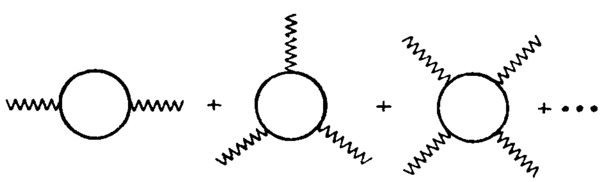

Figure 1. Closed loop diagrams corresponding to the series in eq. (14). Wavy lines and ordinary lines denote the gluons and quarks respectively.

Here $F(A_\mu^i)$ is the finite part of the trace in (14) and denotes the finite contribution to the effective Lagrangian of the vector fields arising from closed fermion loops. We have thus generated the kinetic cum self-interaction term of the gauge field: $G^i_{\mu\nu}\, G^{\mu\nu}_i$ in the effective Lagrangian.

For comparison, let us now consider the Lagrangian of QCD:

$$\mathscr{L}_3 = \bar{\psi}(i\gamma.\partial - m)\psi - \frac{1}{4} G^i_{\mu\nu} G^{\mu\nu}_i - g\bar{\psi}\gamma^\mu \frac{\lambda^i}{2}\psi A^i_\mu \tag{20}$$

where

$$G^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g f^{ijk} A^j_\mu A^k_\nu \tag{21}$$

and perform the same one loop integrations as before. The result would be

$$\begin{aligned} W_3(\eta, \bar{\eta}) &\equiv \int \mathscr{D}\psi \mathscr{D}\bar{\psi} \mathscr{D}A^i_\mu \exp i \int \{\mathscr{L}_3 + \bar{\eta}\psi + \bar{\psi}\eta\} d^4x \\ &= \int \mathscr{D}A^i_\mu \exp i \int d^4x \left\{ -\frac{1}{4}\left(1 + \frac{4}{3} g^2 I_0\right) G^i_{\mu\nu} G^{\mu\nu}_i \right. \\ &\qquad \left. - \bar{\eta}\left(1 - S_F\, g\, \gamma^\mu \frac{\lambda^i}{2} A^i_\mu\right)^{-1} S_F\, \eta + F\left(g\, A^i_\mu\right) \right\} \end{aligned} \tag{22}$$

where $F(g\, A^i_\mu)$ is the same finite functional as in (19).

The generating functional W_2 and W_3 are seen to be identical in form and the main difference is in the infinite coefficients of the gauge field term $G^i_{\mu\nu} G^{\mu\nu}_i$. In the case of W_3 this infinite coefficient is absorbed by renormalization:

$$\begin{aligned} &(1 + \tfrac{4}{3} g^2 I_0)^{1/2} A_\mu{}^i = \tilde{A}_\mu{}^i \\ &g(1 + \tfrac{4}{3} g^2 I_0)^{-1/2} = \tilde{g} \end{aligned} \tag{23}$$

and so a similar procedure can be adopted for W_2.

$$\begin{aligned} &(\tfrac{4}{3} I_0)^{1/2} A_\mu{}^i = \tilde{A}_\mu{}^i \\ &(\tfrac{4}{3} I_0)^{-1/2} = \tilde{g}. \end{aligned} \tag{24}$$

As a result of these scaling conditions (23) and (24), W_2 and W_3 become completely identical. Thus, $\mathscr{L}_2$ is equivalent to $\mathscr{L}_3$ and hence we have shown the equivalence between QCD and the quark confinement model of section 2.

In contrast to QCD which involves the fields, ψ, $\bar{\psi}$ and $A_\mu{}^i$, the model of quark confinement was formulated in terms of ψ and $\bar{\psi}$ alone. So, the above equivalence suggests that $A_\mu{}^i$ is a composite of ψ and $\bar{\psi}$ in which case $A_\mu{}^i$ could be eliminated from the theory. In the next section, we shall discuss this point of view.

4. Gluons as composite bosons

Field theoretically, compositeness is connected with the vanishing of the renormalization constants (see for instance Lurie 1968). However, renormalization constants are generally dependent on the ultraviolet cut-off Λ and so much of the earlier work on the compositeness conditions was plagued by our ignorance of this cut-off dependence.

This difficulty may be overcome by using the technique of renormalization group. The wavefunction renormalization constant Z of any field satisfies a renormalization group equation of the type:

$$\left(\Lambda \frac{\partial}{\partial \Lambda} - \beta(g)\frac{\partial}{\partial g} + 2\gamma(g)\right)Z = 0 \tag{25}$$

where $\beta(g)$ is the Callan–Symanzik function and $\gamma(g)$ is the anomalous dimension of the field. The solution of this equation is

$$Z(t, g) = Z(0, \bar{g}(g, t)) \exp\left\{-2\int_0^t \gamma(\bar{g}(g, t'))dt'\right\} \tag{26}$$

where $t = \ln(\Lambda/M)$, M is the renormalization point and $\bar{g}(g, t)$ is the effective coupling constant defined by

$$\frac{\partial \bar{g}}{\partial t}(g, t) = \beta(\bar{g}(g, t)); \quad \bar{g}(g, 0) = g. \tag{27}$$

In a theory with positive metric (which is not guaranteed for gauge theory) the positivity of the spectral density function of the propagator leads to positivity of $\gamma(g)$. This implies that the limit of Z as $\Lambda \to \infty$ is either finite or zero, depending on whether the integral in (26) is finite or infinite.

To go further, one has to assume that the theory has an ultraviolet-stable fixed point at $g = g_\infty$; i.e.,

$$\beta(g_\infty) = 0; \quad \left.\frac{d\beta}{dg}\right|_{g_\infty} < 0. \tag{28}$$

It then follows that

$$\left.\begin{aligned} &\bar{g}(g, t) \underset{t\to\infty}{\longrightarrow} g_\infty \\ \text{and} \quad &\gamma(\bar{g}(g, t)) \underset{t\to\infty}{\longrightarrow} \gamma(g_\infty). \end{aligned}\right\} \tag{29}$$

It is clear from (26) that the behaviour of Z on Λ for $\Lambda \to \infty$ is now controlled by this fixed point. In particular,

$$\text{if } \gamma(g_\infty) \neq 0, \text{ then } Z \underset{\Lambda\to\infty}{\longrightarrow} 0, \tag{30}$$

whereas if $\gamma(\bar{g}(g, t))$ vanishes sufficiently rapidly for $t\rightarrow\infty$, then Z tends to a non-vanishing limit. However, for most of the field theoreis, the existence of an ultraviolet-stable fixed point is a pure hypothesis and so the high-Λ behaviour of Z does not yet become computable.*

In contrast, for a nonabelian gauge theory, the origin $g=0$ is known to be an ultraviolet—stable fixed point (i.e. the theory is asymptotically free) as long as the number of fermion triplets is less than 16 (Gross and Wilczek 1973a, Politzer 1973). Further, $\gamma(\bar{g}(g, t))$ for large t can be computed by perturbation theory and hence the high-Λ behaviour of Z in nonabelian gauge theory can be worked out. This has been done by Ng Wing-Chiu and Young (1974) and let us now recapitulate their results (see also Cheng *et al* 1974).

For the Lagrangian of QCD given by

$$\mathscr{L}_3=\bar{\psi}(i\gamma.\partial-m)\psi-\tfrac{1}{4}(\partial_\mu A_\nu{}^i-\partial_\nu A_\mu{}^i-gf^{ijk}A_\mu{}^jA_\nu{}^k)^2-g\bar{\psi}\gamma^\mu\frac{\lambda^i}{2}\psi A_\mu{}^i \quad (31)$$

the renormalization constances can be introduced through the following equations (ignoring the ghosts):

$$\psi=Z_2^{\frac{1}{2}}\psi_R,$$
$$A^i_\mu=Z_3^{\frac{1}{2}}A^i_{\mu R}, \quad (32)$$
$$g=Z_1Z_3^{-\frac{3}{2}}g_R.$$

Here, the suffix R refers to the renormalized quantity, Z_1 is the vertex renormalization constant and Z_2 and Z_3 are the wave-function renormalization constants of the quark and gluon fields respectively. Actually, for gauge theory, there is an additional term in the renormalization group eq. (25) which describes the gauge dependence of the Z's. However, Ng Wing-Chiu and Young (1974) have shown that the zero of the gauge parameter also is an ultraviolet stable fixed point, provided the number of fermion triplets is greater than 10. In this case the renormalization constants become asymptotically gauge-independent. When the perturbatively computed formulae for $\gamma(\bar{g}(g\ t))$ are then substituted into (26), the results turn out to be

$$Z_i \xrightarrow[\Lambda\rightarrow\infty]{} \left(\ln\frac{\Lambda}{M}\right)^{-\delta_i}; \quad i=1, 2, 3 \quad (33)$$

where

$$\delta_1>0; \quad \delta_3>0; \quad \delta_2=0. \quad (34)$$

In other words, in the limit of infinite cut-off,

$$Z_1=Z_3=0 \quad (35)$$

*Eguchi (1977) conjectures an ultraviolet-stable fixed point $g_\infty\neq 0$ for a Yukawa type theory and then argues that since $\gamma(g_\infty)$ is not expected to be zero, the renormalization constants Z_3 and Z_4 vanish, thus making the scalar bosons composite. Actually, the logical conclusion of this argument is that all the renormalization constants vanish. This follows from (30). In other words, we have the theorem valid for Yukawa and ϕ^4 theories: either there is no ultraviolet-stable fixed point, or all the renormalization constants vanish.

which are just the compositeness conditions for the gluons. Thus, the compositeness conditions for the gluons are shown to be satisfied at least for a class of gauge models —namely for those with the number of quark triplets lying between 10 and 16.

Since $A_\mu{}^i$ becomes a composite of ψ and $\bar{\psi}$, it can be eliminated and the theory can be formulated equivalently in terms of ψ and $\bar{\psi}$ as we have done in section 2.

It should be pointed out here that although $A_\mu{}^i$ is a composite field, it does not manifest itself as a particle in the spectrum of asymptotic states, for as we shall see in the next section, along with the quark, the gluon also will be confined.

5. Green's functions

Having discussed the equivalence of QCD to the quark-confinement model, we now return to this original model. We shall study the general structure of the Green's functions of the quark-confinement model. In particular, we shall exhibit the form of the quark-propagator. One expects quark-confinement to be manifested as the absence of a pole in the quark-propagator and we shall verify this expectation.

The Green's functions of the quark-confinement model can be defined in terms of functional integration:

$$G(x-y) \equiv \langle\, \psi(x)\, \bar{\psi}(y)\, \rangle\rangle$$

$$= N^{-1} \int \mathscr{D}\psi \mathscr{D}\bar{\psi}\, \delta(j_\mu{}^i)\, \psi(x)\bar{\psi}(y) \exp i \int d^4z\, \{\bar{\psi}(i\gamma.\partial - m)\psi\} \tag{36}$$

where

$$N = \int \mathscr{D}\psi \mathscr{D}\bar{\psi}\, \delta(j_\mu{}^i) \exp i \int d^4z\, \{\bar{\psi}(i\gamma.\partial - m)\psi\}. \tag{37}$$

As already noted in section 2, this theory is invariant under local gauge transformations. We shall first derive a Ward–Takahashi identity following from this invariance. For this purpose, let us make the following change of variable in the functional integration:

$$\psi(x) \to \psi'(x) = R(x)\psi(x) \tag{38}$$

where $R(x)$ is a local gauge transformation:

$$R(x) = \exp\left(\frac{i\lambda^i}{2}\, \alpha^i(x)\right). \tag{39}$$

Under this change of variable, the right side of (36) becomes

$$N^{-1} \int \mathscr{D}\psi \mathscr{D}\bar{\psi}\, \delta(j_\mu{}^i) R(x)\psi(x)\bar{\psi}(y) R^{-1}(y) \exp i \int d^4z\, \{\bar{\psi}(i\gamma.\partial - m)\psi\}.$$

Hence we get the Ward–Takahashi identity:

$$G(x-y) = R(x)\, G(x-y)\, R^{-1}(y). \tag{40}$$

The solution of this equation is

$$G^{\alpha\beta}_{ab}(x-y) = \delta_{ab}\, A^{\alpha\beta}\, \delta^4(x-y) \tag{41}$$

where a, b refer to colour SU(3) indices and α, β refer to Dirac indices. The constant $A^{\alpha\beta}$, is not determined by the Ward–Takahashi identity.

The fourier transform of (41) is a constant and so it does not have the quark pole. It is remarkable that the constraint of quark-confinement has not only removed the pole in the quark-momentum variable, but banished any dependence on this variable. Stated in terms of space-time variable one sees from (41) that the quark simply does not propagate!

Let us next consider the four-point function

$$\langle \psi_a{}^\alpha(x_1)\, \psi_b{}^\beta(x_2)\, \bar{\psi}_l{}^\lambda(y_1)\, \bar{\psi}_m{}^\mu(y_2)\rangle$$

given by an expression similar to (36). By the same procedure as before, we can derive the Ward-Takahashi identity:

$$\langle \psi_a{}^\alpha(x_1)\, \psi_b{}^\beta(x_2)\, \bar{\psi}_l{}^\lambda(y_1)\, \bar{\psi}_m{}^\mu(y_2)\rangle$$

$$= R_{ac}(x_1) R_{bd}(x_2) R^{-1}_{ln}(y_1) R^{-1}_{mp}(y_2)\, \langle \psi_c{}^\alpha(x_1)\psi_d{}^\beta(x_2)\bar{\psi}_n{}^\lambda(y_1)\bar{\psi}_p{}^\mu(y_2)\rangle. \tag{42}$$

The general solution of this is

$$\langle \psi_a{}^\alpha(x_1)\, \psi_b{}^\beta(x_2)\, \bar{\psi}_l{}^\lambda(y_1)\, \bar{\psi}_m{}^\mu(y_2)\rangle$$

$$= \delta_{al}\, \delta_{bm}\, \delta^4(x_1-y_1)\delta^4(x_2-y_2) f^{\alpha\beta\lambda\mu}(x_1-x_2) + \text{exchange term}. \tag{43}$$

We have used translational invariance in writing this solution and the exchange term is obtained by permuting the fermion labels. Although the single quark-propagation is again restricted to a δ-function, a non-trivial propagation for the colour singlet composite field operator $\sum_a \psi_a(x_1)\bar{\psi}_a(y_1)\,|_{x_1=y_1}$ is allowed by this solution. In particular this allows colour singlet bound states (mesons) manifesting as poles in the Fourier transform of the function $f(x_1-x_2)$. (However note that the colour octet combination has no non-trivial propagation and so the composite gluon also is confined).

The structure of the Ward–Takahashi identity for any higher Green's function is clear from (42). As a final exmaple, let us write down the solution for the six-point function:

$$\langle \psi_a{}^\alpha(x_1)\psi_b{}^\beta(x_2)\psi_c{}^\gamma(x_3)\bar{\psi}_l{}^\lambda(y_1)\bar{\psi}_m{}^\mu(y_2)\bar{\psi}_n{}^\nu(y_3)\rangle$$

$$= \{\delta_{al}\,\delta_{bm}\,\delta_{cn}\,\delta^4(x_1-y_1)\delta^4(x_2-y_2)\delta^4(x_3-y_3) g^{\alpha\beta\gamma\lambda\mu\nu}(x_1-x_2;\, x_2-x_3)$$

$$+ \text{exchange terms}\}$$

$$+ \epsilon_{abc}\,\epsilon_{lmn}\,\delta^4(x_1-x_2)\delta^4(x_2-x_3)\delta^4(y_1-y_2)\delta^4(y_2-y_3) h^{\alpha\beta\gamma\lambda\mu\nu}(x_1-y_1) \tag{44}$$

where $h^{\alpha\beta\gamma\lambda\mu\nu}$ is symmetric in the Dirac indices $\alpha\beta\gamma$ and $\lambda\mu\nu$. The last term containing $h^{\alpha\beta\gamma\lambda\mu\nu}(x_1 - y_1)$ is allowed by the Ward–Takahashi identity for the six-point function because of the invariance of the antisymmetric tensor ϵ_{abc}:

$$R_{ai}(x)\, R_{bj}(x)\, R_{ck}(x)\, \epsilon_{ijk} = \epsilon_{abc}. \tag{45}$$

It is clear that $g(x_1 - x_2;\ x_2 - x_3)$ describes the non-trivial three-point function for colour singlet mesons while $h(x_1 - y_1)$ describes the non-trivial propagator for the colour singlet baryons.

Thus, the general structure of the Green's functions allows non-trivial propagation as well as non-trivial interaction among colour singlet composite field operators retaining δ-function propagation for coloured objects.

As a result of colour confinement, there is a considerable simplification in the space and colour dependence of the Green's functions. For instance, instead of the usual dependence on the three spatial coordinate—differences expected of a four-point function, we need consider only a function of a single coordinate difference $f(x_1 - x_2)$ defined in (43). It may be possible to exploit the simplification achieved here in further investigations of this problem.

6. Discussion and summary

We shall first compare our results with those of other connected investigations. As already mentioned, Amati and Testa (1973) also started with the model of quark confinement, but argued that this model is the strong coupling limit ($g \to \infty$) of QCD. If this were true, this would have the disadvantage that asymptotic freedom of the theory may be lost. For, asymptotic freedom exists only if the gauge coupling constant g lies in the domain of attraction of the ultraviolet stable fixed point $g=0$. Since the behaviour of $\beta(g)$ for large g is unknown, we do not know whether the strong coupling limit $g \to \infty$ lies in this domain. In contrast, our claim is that the model of quark confinement is equivalent to QCD at a finite g so that asymptotic freedom and quark confinement can be simultaneously preserved.

Next let us consider the work of Kikkawa (1976), Eguchi (1976), Terazawa *et al* (1976) and Saito and Shigemoto (1976). These authors start with the four-fermion coupling

$$G\left(\bar{\psi}\,\gamma_\mu \frac{\lambda^i}{2}\,\psi\right)^2$$

which is rewritten in the equivalent form

$$\bar{\psi}\,\gamma_\mu \frac{\lambda^i}{2}\,\psi\, A_\mu{}^i - \frac{1}{4G}\,(A_\mu{}^i)^2$$

in the Lagrangian. Divergent radiative corrections involving closed fermion loops are then used to generate the kinetic and self-interaction terms for the gauge field. Thus, the four-fermion coupling is argued to be equivalent to the nonabelian gauge theory. However, this procedure suffers from the criticism that it is not gauge invariant (Shizuya 1977). In order to cancel the bare mass term $(4G)^{-1}$ $(A_\mu{}^i)^2$ of the gauge field, the closed fermion loop has to generate a quadratically divergent mass term for the gauge field and this requires the employment of a gauge-noninvariant regularization. On the other hand, we have seen in section 3 that a completely gauge invariant procedure leads to the equivalence of the nonabelian gauge theory to the delta-function model of quark confinement rather than to the fermi coupling. Since the troublesome term $(4G)^{-1}$ $(A_\mu{}^i)^2$ vanishes for $G\to\infty$, one can say that the correct gauge-invariant model is obtained only in this limit; in fact,

$$\operatorname*{lt}_{G\to\infty} \exp\left\{iG\int d^4x\left(\bar\psi\gamma_\mu\frac{\lambda^i}{2}\psi\right)^2\right\} \sim \delta\left(\bar\psi\gamma_\mu\frac{\lambda^i}{2}\psi\right).$$

It was conjectured (Rajasekaran 1971) that the violent infrared disease of the nonabelian gauge theory might prevent the materialization of the massless gauge quantum as a physical particle. The same infrared problem was later invoked in the context of quark-gluon theory to prevent the materialization of quarks as well (Weinberg 1973, Gross and Wilczek 1973b). As a result of the recent work by Appelquist *et al* (1976) as well as Yao (1976), it seems that in perturbation theory the infrared behaviour is well controlled for colour-averaged objects and hence cannot be invoked for quark confinement. However, nonperturbative calculations such as those by Cornwall and Tiktopoulos (1976) do suggest quark confinement. Pagels (1976) has examined the consistency of quark confinement in a nonperturbative treatment of QCD using the infrared singularity of the gluon propagator. In the present approach, we bypass the infrared problem by eliminating the gauge field altogether and showing that the resulting theory leads to perfect quark confinement.

To sum up, we have formulated a model for strong interactions which has all the following features:

(a) It corresponds to a nonabelian colour gauge theory which is asymptotically free.
(b) The colour gluons are composites of quarks with $Z_1 = 0$ and $Z_3 = 0$ so that they can be eliminated from the theory.
(c) Quark confinement is simply expressed by the constraint $j_\mu{}^i = 0$.
(d) The Green's functions of the theory contain nontrivial structure only for colour singlet composites which can be interpreted as hadrons.

We should also add a few remarks concerning questions which need further investigation. Our argument for the equivalence of QCD with the quark-confinement model which is based on functional methods is rather formal and it is worthwhile to check the results by explicit calculations. This can perhaps be done along the lines of the recent work by Bender *et al* (1977) on the equivalence between ϕ^4 and $\sigma\phi^2$ theories. Such a calculation may also clarify whether the restriction placed on the number of triplets to lie between 10 and 16 is essential. Further, although we argued that the composite gluon field can be eliminated, the precise relationship between the composite gluon field and the elementary fermion field is yet to be determined. The relation between the Green's function given in (41) and the free-field behaviour expected for $(x-y)^2\to 0$ in asymptotic free theory also has to be elucidated.

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