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Asymptotic bounds on pion form factor

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Abstract. A technique recently developed for inelastic electron proton scattering is applied for inelastic electron pion scattering. It is found that all the derivatives of off-shell form factor of pion near $s=m_\pi^2$ and for large Q^2 are bounded from above, provided that the dispersion relation for the form factor requires no more than one subtraction. The elastic pion form factor is bounded by $[\ln Q^2]^c/Q^2$, where c is any positive constant.

Keywords. Inelastic electron pion scattering; upper bounds.

1. Introduction

Asymptotic bound on electromagnetic form factors of hadrons is an important subject of current interest. Apart from expressing compositeness or elementarity of a hadron, its asymptotic upper bound puts constraints on the number of subtractions needed in the dispersion relation. Recently using a somewhat improved method over that existed earlier (Cooper and Pagels 1970), upper bounds on the derivatives of off-shell form factor of proton in asymptotic spacelike region have been derived (Broadhurst 1972) from considerations of electromagnetic vertex in inelastic electron proton scattering. It has been shown that the second and all the higher derivatives of the off-shell Dirac form factor of composite proton are bounded, provided the sideways dispersion relation requires no more than two subtractions. Further it has also been proved that the Drell-Yan-West (Drell and Yan 1970 and West 1970) relation is the extremum of an inequality imposed by unitarity and analyticity. In the present paper we observe that when a similar analysis is adopted for inelastic electron pion scattering, all the derivatives of the off-shell pion form factor are bounded, provided that the form factor satisfies a dispersion relation requiring no more than one subtraction. The pion form factor satisfies the upper bound

$$\lim_{Q^2\to\infty} |F_\pi(Q^2)| \leqslant B[\ln Q^2]^c/Q^2$$

where c is any positive constant and B is a constant related to the spectral functions, structure function and off-shell form factor.

2. Upper bound on the discontinuity of off-shell pion form factor

Consider the inelastic scattering or a space like virtual photon off pion target which could occur at the electromagnetic vertex of inelastic electron pion scattering and as shown in figure 1, pion (p) + space like photon $(q) \rightarrow$ off-shell pion (p'). Kinematics for this process is the same as that for inelastic electron-proton scattering:

$$p^2 = m_\pi^2$$

$$q^2 = -Q^2 < 0$$

$$p'^2 = W^2 = s$$

$$2pq = 2p_\mu q_\mu = 2m_\pi \nu = s - m_\pi^2 + Q^2 = \omega Q^2 \tag{1}$$

where m_π is the pion mass. We introduce photon-pion electromagnetic vertex function in terms of two off-shell form factors $F(W^2, Q^2)$ and $G(W^2, Q^2)$ as follows

$$\Gamma_\mu(p, q) = (p'+p)_\mu F(W^2, Q^2) + q_\mu G(W^2, Q^2) = \langle p' | j_\mu(o) | p \rangle \tag{2}$$

where $j_\mu(x)$ is the pion current. Application of LSZ reduction method then yields

$$\Gamma_\mu(p, q) = \frac{i}{(2\pi)^{3/2}} \int d^4x\, e^{ip'x} (\Box_x + m_\pi^2) \langle o | \theta(x_0) [\phi(x), j_\mu(o)] | p \rangle$$

$$= (p'+p)_\mu F(W^2, Q^2) + q_\mu G(W^2, Q^2). \tag{3}$$

Absorptive part A_μ of Γ_μ can be obtained if one replaces $i\theta(x_0)$ by $\frac{1}{2}$ and introduces physical intermediate states. Thus we can write

$$A_\mu = \frac{1}{2} \frac{(2\pi)^4}{(2\pi)^{3/2}} \sum_{p_n} \delta^4(p'-p_n)(-W^2+m_\pi^2) \langle o | \phi(o) | p_n \rangle \langle p_n | j_\mu(o) | p \rangle. \tag{4}$$

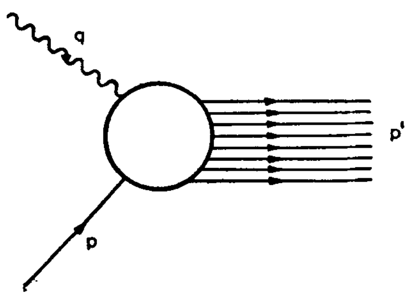

Figure 1. Off-shell electromagnetic vertex in inelastic electron pion scattering.

The lowest non-vanishing production matrix element is for three pions, so that one can write dispersion relations for F and G which have only a right-hand cut in the s-plane extending from $s=(3m_\pi)^2$ to $s=\infty$, thus

$$F(s,\ Q^2)=\frac{1}{\pi}\int_{(3m_\pi)^2}^{\infty}\frac{\mathrm{Im}\ F(s',\ Q^2)}{(s'-s)}\,ds' \tag{5}$$

and similar relation for $G(s,\ Q^2)$ where we have not taken possible subtractions into account. Implication of subtractions has been discussed later.

Following Broadhurst (1972) we define projection operators

$$\epsilon_{L_\mu}=(p_\mu+m_\pi\nu\,q_\mu/Q^2)\,/\,[m_\pi(1+\nu^2/Q^2)^{1/2}] \tag{6}$$

and a unit space like vector orthogonal to both p and q such that

$$\epsilon_L q=\epsilon_T q=\epsilon_L\epsilon_T=0,\ \ \epsilon_L{}^2=-\ \epsilon_T{}^2=1,$$

$$\epsilon_L p=m_\pi(1+\nu^2/Q^2)^{1/2},\ \ \epsilon_T p=0. \tag{7}$$

From eqs (6) and (7) we obtain

$$\epsilon_{L_\mu}\,\Gamma_\mu=2m_\pi(1+\nu^2/Q^2)^{1/2}\ F(W^2,\ Q^2). \tag{8}$$

Equations (2) and (6) yield

$$\epsilon_{L_\mu}\,A_\mu=\frac{1}{2}\,\frac{(2\pi)^4}{(2\pi)^{3/2}}\sum_{p_n}\delta^4(p'-p_n)\,(-W^2+m_\pi{}^2)$$

$$\times\ \langle o\,|\,\phi(o)\,|\,p_n\rangle\ \langle p_n\,|\ \epsilon_{L_\mu}\ j_\mu(o)\,|\,p\rangle. \tag{9}$$

Using eqs (8) and (9) we obtain

$$2m_\pi(1+\nu^2/Q^2)^{1/2}\ \mathrm{Im}\ F(W^2,\ Q^2)=\frac{1}{2}\,\frac{(2\pi)^4}{(2\pi)^{3/2}}\sum_{p_n}\delta^4(p'-p_n)$$

$$\times\ (-W^2+m_\pi{}^2)\ \langle o\,|\,\phi(o)\,|\,p_n\rangle\ \langle p_n\,|\ \epsilon_{L_\mu}\ j_\mu(o)\,|\,p\rangle. \tag{10}$$

Application of Schwarz inequality to eq. (10) yields,

$$|\,\mathrm{Im}\ F(W^2,\ Q^2)\,|^2\leqslant\frac{(W^2-m_\pi{}^2)^2}{8m_\pi}\ \frac{\rho(W^2)}{(1+\nu^2/Q^2)}\ [(1+\nu^2/Q^2)\ W_2(W^2,\ Q^2)$$

$$-W_1(W^2,\ Q^2)]. \tag{11}$$

In deriving (11) we have used

$$(2\pi)^3\ \sum_{a}\ |\,\langle o\,|\,\phi(o)\,|\,p_n,\ a\rangle\,|^2=\rho(p_n{}^2),$$

where α sums the spin of the intermediate states, and

$$\frac{4\pi^2}{m_\pi} \sum_{p_n} \delta^4(p'-p_n) \langle p | j_\mu(o) | p_n \rangle \langle p_n | j_\nu(o) | p \rangle = (-g_{\mu\nu} + q_\mu q_\nu / Q^2)\, W_1$$

$$+ (p_\mu + pq q_\mu / Q^2)(p_\nu + pq q_\nu / Q^2)\, W_2 / m_\pi^2$$

Now repeating, with ϵ_{T_μ}, the steps that led to eqs (8) to (11) we obtain

$$W_1 \geqslant 0. \tag{12}$$

Combining inequalities (11) and (12) yields

$$|\,\mathrm{Im} F(W^2, Q^2)\,|^2 \leqslant \frac{(W^2 - m_\pi^2)^2}{8m_\pi}\, \rho(W^2)\, W_2(W^2, Q^2). \tag{13}$$

Inequalities (11) and (13) express upper bounds of discontinuity of $F(W^2, Q^2)$, however inequality (13) is stronger.

3. Upper bounds on elastic form factor

We have proved that $F(s, Q^2)$ is analytic in the out s-plane with a right-hand cut in the region $(3m_\pi)^2 \leqslant s \leqslant \infty$. $F(s, Q^2)$ reduces to elastic pion form factor $F_\pi(Q^2)$ at $s = m_\pi^2$, i.e. $F_\pi(Q^2) = F(m_\pi^2, Q^2)$. The imaginary part of $F(s, Q^2)$ is restricted by the inequalities (11) and (13). We can write the inequality (13), for convenience, in the form

$$|\,\mathrm{Im} F(s, Q^2)\,| \leqslant \frac{1}{2}\, \frac{1}{\sqrt{2m_\pi}}\, (s - m_\pi^2)\, \sqrt{\rho(s)\, W_2\,(s, Q^2)}. \tag{14}$$

Let us assume that the off-shell pion form factor $F(s, Q^2)$ and the pion spectral function $\rho(s)$ are bound by s^b and s^a respectively. Then for $n > b$ we can write an n times subtracted dispersion relation:

$$F(s, Q^2) = \sum_{m=0}^{n} a_m(Q^2)\, s^m + \frac{(s - m_\pi^2)^n}{\pi} \int_{9m_\pi^2}^{\infty} \frac{\mathrm{Im}\, F(s', Q^2)}{(s'-s)\,(s'-m_\pi^2)^n}\, ds' \tag{15}$$

where we have taken the subtraction points at m_π^2. Now differentiating both sides of (15) n times with respect to s we obtain

$$\left| \frac{d^n F}{ds^n}\, (m_\pi^2, Q^2) \right|^2 \leqslant \frac{(n!)^2}{4\pi^2} \left| \int_{9m\pi^2}^{\infty} \frac{ds'}{(s'-m_\pi^2)^n} \left[\frac{W_2(s', Q^2) \rho(s')}{2m_\pi} \right]^{\frac{1}{2}} \right|^2. \tag{16}$$

It is to be noted that the inequality (16) can also be obtained from the unsubtracted dispersion relation (5), but then b has to be negative. Obviously unsubtracted dispersion relation is too restrictive a condition on $F(s,Q^2)$, but subtracted dispersion relation is more general. Now using Schwarz inequality for integration over s' in (16) we obtain

$$\left|\frac{d^nF}{ds^n}(m_\pi^2,Q^2)\right|^2 \leqslant \frac{(n!)^2}{4\pi^2}\int_{9m_\pi^2}^{\infty} ds'\rho(s')(s'-m_\pi^2)^{p+1-\epsilon-2n}$$

$$\times \int_{9m_\pi^2}^{\infty} ds'\,\frac{W_2(s',Q^2)}{2m_\pi}(s'-m_\pi^2)^{\epsilon-p-1} \tag{17}$$

where ϵ is any arbitrary positive number and p is a real number which decides existence of the two integrals in (17). We will see that p is related to the threshold behaviour of structure function νW_2. Now changing from variable s' to ω in the integral over the structure function and taking the limit $Q^2\to\infty$ yields,

$$\lim_{Q^2\to\infty}\left|Q^{p+1-\epsilon}\frac{d^nF}{ds^n}(m_\pi^2,Q^2)\right|^2 \leqslant \frac{(n!)^2}{4\pi^2}\int_{9m_\pi^2}^{\infty} ds'\rho(s')(s'-m_\pi^2)^{p+1-\epsilon-2n}$$

$$\times \int_1^{\infty}\frac{d\omega}{\omega}\,\frac{\nu W_2}{(\omega-1)^{p+1-\epsilon}}. \tag{18}$$

Now using Bjorken scaling behaviour of structure functions for inelastic electron pion scattering:

$$\lim_{Q^2\to\infty}\nu W_2=F_2(\omega) \tag{19}$$

in inequality (18) we obtain

$$\lim_{Q^2\to\infty}\left|Q^{p+1-\epsilon}\frac{d^nF}{ds^n}(m_\pi^2,\,Q^2)\right|^2 \leqslant \frac{(n!)^2}{4\pi^2}\int_{9m_\pi^2}^{\infty} ds'\rho(s')\,(s'-m_\pi^2)^{p+1-\epsilon-2n}$$

$$\times \int_1^{\infty}\frac{d\omega}{\omega}\,\frac{F_2(\omega)}{(\omega-1)^{p+1-\epsilon}}. \tag{20}$$

We observe that for the case of meson also the inequality (20) retains the same form as that for proton (Broadhurst 1972) but differs by a constant $(n!/2\pi)^2$. The first integral over the spectral function on the right hand side of (20) is finite for $n\geqslant(p+1+a)/2$. If the structure function behaves like $(\omega-1)^p$ near threshold $(\omega=1)$ and does not grow faster than $\omega^{p'}$ as $\omega\to\infty$, where $p'=p+1-2\epsilon$, the second integral over the structure function is also finite for any arbitrary small positive ϵ. Thus we obtain for $n>b$ and $n\geqslant(p+2+a)/2$

$$\lim_{Q^2\to\infty}\left|\frac{d^nF}{ds^n}(m_\pi^2, Q^2)\right|\leqslant A[\ln Q^2]^c/Q^{p+1} \tag{21a}$$

where

$$A=\left(\frac{n!}{2\pi}\right)^2\int_{9m_\pi^2}^{\infty} ds\rho(s)(s-m_\pi^2)^{p+1-\epsilon-2n}\int_1^{\infty}\frac{F_2(\omega)\,d\omega}{\omega(\omega-1)^{p+1-\epsilon}}. \tag{21b}$$

Now we make factorizability assumption on the function $F(s, Q^2)$ for $Q^2\to\infty$ and near $s=m_\pi^2$ upon which our derivation of the asymptotic bound on elastic pion form factor depends crucially. Such an assumption is justified if near $s=m_\pi^2$ and for $Q^2\to\infty$, $F(s, Q^2)$ is a smooth but sufficiently varying function of s. Thus making factorizability assumption on the function $F(s, Q^2)$ i.e.

$\lim_{Q^2\to\infty} F(s, Q^2)=f(s)F_\pi(Q^2)$ near $s=m_\pi^2$, we obtain from (21a)

$$F_\pi(Q^2)\leqslant B[\ln Q^2]^c/Q^{p+1} \tag{22}$$

where

$$B=\frac{A}{[d^n f(m_\pi^2)]/(ds^n)}.$$

Parton model predicts (Drell *et al* 1970) the value of p to be even (odd) integer if, nucleon (pion) current dominates. There are many other analyses (Hama *et al* 1972 and Niegawa 1971, 1972) which suggest scaling behaviour of structure functions for inelastic electron meson scattering. More important than these theoretical evidences is the information from experiment (Schierholz and Schmidt 1976) suggesting that $p=1$. Thus taking the experimental value $p=1$ and $a=-1$, which corresponds to a composite pion, our results hold for $n\geqslant1$ provided that dispersion relation requires no more than one subtraction. The pion form factor for large Q^2 is thus bounded by $[\ln Q^2]^c/Q^2$. Unlike the proton case where the first derivative could not be bounded all the derivatives in the case of pion are bounded. It may be mentioned that, recently, asymptotic behaviour of the type $F_\pi(Q^2)\sim Q^{-2}$ has been derived in the Bethe-Salpeter model (Goldberger *et al* 1976) by considering pion as quark-antiquark bound state. This result saturates our bound with $C=0$.

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