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Green's functions for spin half field theory in Rindler space

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Abstract. The solutions of Dirac equation in different regions of the complete extension of Rindler space are obtained near the event horizons and in the asymptotic limits. Continuity of these solutions across the event horizons is established. The Green's functions are written down in the two causally disconnected regions, continued in the future (F) and past (P) regions using the techniques a la Boulware and a consistent scheme of Green's functions in all regions is exhibited.

Keywords. Rindler space; spin half field; Green's functions.

1. Introduction

The problem of generalization of quantum field theory to curved space-time has attracted a great deal of interest recently (see, for instance, DeWitt 1975). The sudden spurt in interest in this field has been caused by several remarkable developments beginning with the realization that quantum field theory in a strong gravitational background (such as that of a black hole) may give rise to a vacuum that is unstable against particle production. Hawking (1974, 1975) predicted on this basis the ' quantum evaporation ' of a black hole that is collapsing. More recently similar phenomenon has been predicted for the stationary gravitational field of a Kerr black hole (Ford 1975). For a recent review see Unruh (1976).

Of the many vexing problems of field theory in curved space, one that has merited particular attention is the construction of Feynman-Green's functions appropriate to the curved geometry in question. The Feynman-Green's functions satisfy the so-called positive frequency boundary conditions. In Minkowski space where there exists a universal time co-ordinate t, the definition of this boundary condition is unambiguous. In contrast, in the maximally extended Schwarzschild space, for instance, t is the time coordinate for $r>2M$ and r is the time co-ordinate for $r<2M$ ($2M$ is the Schwarzschild radius). In terms of Kruskal co-ordinates (u, v) on the other hand, v is globally the time co-ordinate. Evidently, the definition of Feynman-Green's function is in general not unique and one may adopt several different procedures. For instance, one may

(a) construct Green's functions region by region with the positive frequency referred to the time co-ordinate in that region, and continue them across the event horizons using appropriate boundary conditions. We shall call this the η prescription of Boulware (1975a) and its details will be given in section 3

(b) use the global time co-ordinate (if it exists) to define the positive frequency boundary condition in the entire space.

The motivation for investigating these prescriptions for the Rindler space is clear. Since the Rindler space is globally transformable to Minkowski space the second prescription above yields none other than the usual Feynman-Green's function. It is then a pertinent question whether the η prescription also determines a consistent set of Green's functions.

For the scalar field theory where the solutions in Rindler space are exactly known (Fulling 1973) (in contrast to the spin half case, where as we shall see, the solutions are known only in certain limits), such a consistent construction of Green's functions has been recently exhibited by Boulware (1975a) who has also shown its equivalence with the usual Minkowski-Green's functions. Equipped with this confidence, a natural question to ask is : Is a consistent scheme of Green's functions for spin half field theory in Rindler space possible using analogous techniques?

In this paper we take up the investigation of spin half field theory in Rindler space. In section 2 we obtain the solutions of Dirac equation in different regions of the extended Rindler space near the event horizons and in the asymptotic limits. The continuity of our solutions is then established by a proper re-alignment of frames. In section 3 we construct the Green's functions satisfying appropriate boundary conditions in different regions. The internal consistency of our scheme is next demonstrated and this supports the validity of Boulware's η prescription for the spin half case.

2. Spin half equation in Rindler space

2.1. *Rindler space*

The metric is given by

$$dS^2 = -\rho^2 d\tau^2 + d\rho^2 + dx^2 + dy^2; \; \rho \in [0, \infty), \tau \in (-\infty, \infty) \tag{1}$$

Rindler co-ordinates are co-moving co-ordinates for a uniformly accelerated observer. Since uniform acceleration can be globally transformed away, Rindler space is

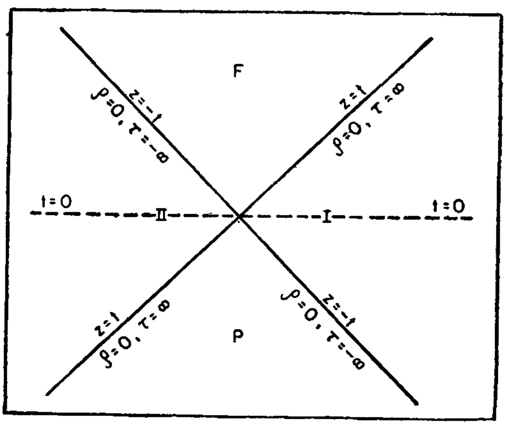

Figure 1. Complete analytic extension of Rindler space (Minkowski space). The thick lines represent the event horizons, $z = \pm t$.

equivalent to the usual Minkowski (flat) space-time. This is easily seen by the transformations

$$z = \rho \cosh \tau, \quad t = \rho \sinh \tau; \quad z, t \in I \tag{2a}$$

which gives

$$dS^2 = -dt^2 + dz^2 + dx^2 + dy^2. \tag{3}$$

However, Rindler co-ordinates span only the region I of the Minkowski space (figure 1). Thus the co-ordinates z, t determine the extension of Rindler space. In the remaining regions the Rindler co-ordinates are given by

$$z = -\rho \cosh \tau, \quad t = -\rho \sinh \tau; \quad z, t \in II \tag{2b}$$

$$z = \rho \sinh \tau, \quad t = \rho \cosh \tau; \quad z, t \in F \tag{2c}$$

$$z = -\rho \sinh \tau, \quad t = -\rho \cosh \tau; \quad z, t \in P. \tag{2d}$$

The metric in Rindler co-ordinates is given by

$$dS^2 = \pm(-\rho^2 \, d\tau^2 + d\rho^2) + dx^2 + dy^2 \tag{4}$$

Where the $+$ ($-$) refers to I and II (F and P). The metric in terms of the continuous co-ordinates z and t, is, of course, same everywhere, given by eq. (3).

From eq. (4), τ is the time co-ordinate in I and II and ρ is the time co-ordinate in F and P. Further from the transformation equations above it is seen that in I (II) the direction of increasing (decreasing) τ is the direction of increasing proper time t and in F (P) the direction of increasing (decreasing) ρ is the direction of increasing proper time. We may also note that the metric has a co-ordinate singularity at $\rho = 0$. The two event horizons $\rho = 0$, $\tau = \pm\infty$ correspond to $z = \pm t$ in figure 1.

2.2. *Dirac equation in Rindler space*

The general formulation of Dirac equation in curved space is considerably involved mainly because the transformation property of the spinor ψ is given only with respect to Lorentz transformations. To tackle this situation, one introduces in the curved space local Lorentzian frames (this is always possible in a Riemannian space) and demands that the Dirac equation be covariant with respect to transformations among these local frames. The connection between the local axes and the general co-ordinate axes is given by the so called 'Vierbein Components'. A good review of this formalism can be found in Boulware (1975b) and for other relevant details see Iyer and Kumar (1977). We shall quote the results here.

Dirac equation in a general curved space is

$$\left(\frac{\gamma^a}{i} \nabla_a + m\right) \psi = 0 \tag{5}$$

where $\nabla_a = e_a^{\;\mu}\left(\partial_\mu + \frac{i}{2} S^{ab}\,\omega_{ab\mu}\right)$

and $S^{ab} = \frac{i}{4}[\gamma^a, \gamma^b].$

The ' Vierbein components' $e_a^{\;\mu}$ and ' Spinor affinity ' $\omega_{ab\mu}$ are defined as

$$e_a^{\;\mu} = \eta_{ab}\, g^{\mu\nu} \frac{\partial \xi^b}{\partial x^\nu}$$

$$\omega_{ab\nu} = e_a^{\;\mu}\, e_{b\mu;\nu}$$

$$e_{b\mu;\nu} = \partial_\nu\, e_{b\mu} - \Gamma^\lambda{}_{\mu\nu}\, e_{b\lambda}$$

ξ^b are the locally inertial co-ordinates. (Note that the latin indices a, b run over 0, 1, 2, 3 of the local Lorentzian axes and the greek indices μ, ν run over the general co-ordinates).

It is straightforward to calculate the $\Gamma^\alpha{}_{\beta\gamma}$, $e_a^{\;\mu}$ and $\omega_{ab\mu}$. The non-vanishing $\Gamma^\alpha{}_{\beta\gamma}$ in all regions are given by

$$\Gamma^\tau{}_{\rho\tau} = \Gamma^\tau{}_{\tau\rho} = \frac{1}{\rho}; \quad \Gamma^\rho{}_{\tau\tau} = \rho. \tag{6}$$

The non-vanishing $\omega_{ab\mu}$ in *all regions* are found to be

$$\omega_{30\tau} = -\omega_{03\tau} = 1 \tag{7}$$

Table 1. Vierbein components $e_a^{\;\mu}$ for the complete Rindler space. Here a stands for the Lorentz index and μ for the co-ordinate index. The local frames are oriented so that the axes are ' parallel ' to the co-ordinate axes. The upper signs refer to regions *I*, *F* while the lower signs refer to regions *II*, *P*.

Lorentz index a \ Rindler co-ordinate index μ	τ	x	y	ρ
0	$\pm\frac{1}{\rho}$ ($\rho \in$ I, II)	0	0	± 1 ($\rho \in F, P$)
1	0	1	0	0
2	0	0	1	0
3	$\pm\frac{1}{\rho}$ ($\rho \in F, P$)	0	0	± 1 ($\rho \in$ I, II)

The 'Vierbein components' $e_a^{\,\mu}$ in different regions of the Rindler space are given in table 1. Using these explicit values Dirac equation in various regions is obtained: In I and II

$$\frac{1}{i}\left\{\gamma^1\partial_x+\gamma^2\partial_y+\epsilon\left(\frac{\gamma^0}{\rho}\,\partial_\tau+\gamma^3\partial_\rho+\frac{\gamma^3}{2\rho}\right)\right\}\psi+m\psi=0;$$

$$\epsilon=+1\,(-1)\text{ in }I\,(II) \tag{8a}$$

In F and P

$$\frac{1}{i}\left\{\gamma^1\partial_x+\gamma^2\partial_y+\epsilon\left(\frac{\gamma^3}{\rho}\,\partial_\tau+\gamma^0\partial_\rho+\frac{\gamma^0}{2\rho}\right)\right\}\psi+m\psi=0$$

$$\epsilon=+1\,(-1)\text{ in }F\,(P). \tag{8b}$$

In each region the equation is translationally invariant with respect to x, y and τ. We write

$$\psi=\int\frac{d\omega\;dk_1\;dk_2}{(2\pi)^{3/2}}\,e^{i(k_1x+k_2y-\omega\tau)}\;\frac{\psi\,(\rho,\,\omega,\,k_1,\,k_2)}{\sqrt{\rho}} \tag{9}$$

where $\psi\,(\rho,\,\omega,\,k_1,\,k_2)$ satisfies

$$\left[m+k_1\gamma^1+k_2\gamma^2+\epsilon\left(-\frac{\omega}{\rho}\gamma^0+\frac{\gamma^3}{i}\,\partial_\rho\right)\right]\psi\,(\rho,\omega,k_1,k_2)=0;$$

$$\rho\in I\,(II) \tag{10a}$$

$$\left[m+k_1\gamma^1+k_2\gamma^2+\epsilon\left(-\frac{\omega}{\rho}\gamma^3+\frac{\gamma^0}{i}\,\partial_\rho\right)\right]\psi\,(\rho,\omega,k_1,k_2)=0;$$

$$\rho\in F\,(P). \tag{10b}$$

The behaviour of solutions near the event horizons ($\rho\to 0$) and in the asymptotic regions ($\rho\to\infty$) can now be obtained from the above equations. Consider the equation in region I. Writing $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ and using the standard representation of matrices, we obtain in the limit $\rho\to 0$ two coupled equations

$$\frac{\sigma^3}{i}\,\partial_\rho\psi_2-\frac{\omega}{\rho}\psi_1=0\;;\qquad -\frac{\sigma^3}{i}\,\partial_\rho\psi_1+\frac{\omega}{\rho}\psi_2=0.$$

These equations are easily solved giving two types of solutions :

$$\begin{pmatrix}\chi\\ -\sigma^3\chi\end{pmatrix}\rho^{-i\omega}\quad\text{and}$$

$$\begin{pmatrix}\chi\\ \sigma^3\chi\end{pmatrix}\rho^{i\omega}.$$

Each type of solution has, as usual, ' spin up ' (χ_u), ' spin down ' (χ_d) components. Following a similar procedure the solutions in the asymptotic limits are also obtained.

We now define two different complete sets of solutions in region I

(a) $\psi^{(+)}_{I\,(u,d)}$ and $\psi^{(-)}_{I\,(u,d)}$.

These are characterized by their behaviour near the event horizon ($\rho\to 0$) as given above

(b) $\psi^{a}_{I\,(u,d)}$ and $\psi^{b}_{I\,(u,d)}$.

These are characterized by their asymptotic behaviour ($\rho\to\infty$).

Both the sets are needed to implement the boundary conditions on Green's functions discussed in section 3. Analogous sets may be defined in other regions of the Rindler space. We list these below:

Regions I and II

$$\psi^{a}_{I\,(II)} \underset{\rho\to\infty}{\sim} \begin{pmatrix} \chi \\ \dfrac{k_1\sigma^1 + k_2\sigma^2 + \epsilon\, iq\sigma^3}{m}\,\chi \end{pmatrix} e^{-q\rho} \tag{11a}$$

$$\psi^{b}_{I\,(II)} \underset{\rho\to\infty}{\sim} \begin{pmatrix} \chi \\ \dfrac{k_1\sigma^1 + k_2\sigma^2 - \epsilon\, iq\sigma^3}{m}\,\chi \end{pmatrix} e^{q\rho} \tag{11b}$$

$$\psi^{(+)}_{I\,(II)} \underset{\rho\to 0}{\sim} \begin{pmatrix} \chi \\ -\sigma^3\chi \end{pmatrix} \rho^{-i\omega} \tag{11c}$$

$$\psi^{(-)}_{I\,(II)} \underset{\rho\to 0}{\sim} \begin{pmatrix} \chi \\ \sigma^3\chi \end{pmatrix} \rho^{i\omega}. \tag{11d}$$

The two sets are related by

$$\psi^{a}_{I} = A^{s}_{aI}\ \psi^{(-)}_{Is} + B^{s}_{aI}\ \psi^{(+)}_{Is}, \text{ etc.} \tag{12}$$

The spin summation over s is implied.

Regions F *and* P

$$\psi^{a}_{F(P)} \underset{\rho\to\infty}{\sim} \begin{pmatrix} \chi \\ (\epsilon q - m)\,\dfrac{(k_1\sigma^1 + k_2\sigma^2)}{k_1^2 + k_2^2}\,\chi \end{pmatrix} e^{-iq\rho} \tag{13a}$$

$$\psi^{b}_{F(P)} \underset{\rho\to\infty}{\sim} \begin{pmatrix} \chi \\ -\,\epsilon(q+m)\,\dfrac{(k_1\sigma^1 + {}_2k\sigma^2)}{k_1^2 + k_2^2}\,\chi \end{pmatrix} e^{iq\rho} \tag{13b}$$

$$\psi^{(+)}_{F(P)} \underset{\rho\to 0}{\sim} \begin{pmatrix} \chi \\ -\sigma^3\chi \end{pmatrix} \rho^{-i\omega} \tag{13c}$$

$$\psi^{(-)}_{F(P)} \underset{\rho\to 0}{\sim} \begin{pmatrix} \chi \\ \sigma^3\chi \end{pmatrix} \rho^{i\omega} \tag{13d}$$

$$\psi^{a}_{F} \;=\; A^{s}_{aF} \;\; \psi^{(-)}_{Fs} \;+\; B^{s}_{aF} \;\; \psi^{(+)}_{Fs}, \text{ etc.} \tag{14}$$

In all the equations above, $q = +(k_1{}^2+k_2{}^2+m^2)^{1/2}$

2.3. *Continuity across the event horizons*

The fourier transform $\psi\ (\rho,\ \omega)$ contains information about both the boundaries ($\rho{=}0,\ \tau{=}\pm\infty$). To obtain continuity across each event horizon, it is necessary to construct wave-packets such that they are localized at one of the event horizons and vanish at the other. Consider the wave-packet

$$\phi^{(\pm)}_{I}\ (\rho,\ \tau) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp\ (-i\omega\tau) \tilde{f}\ (\omega)\ \frac{\psi^{(\pm)}_{I}\ (\rho,\ \omega)}{\sqrt{\rho}} \tag{15}$$

where

$$\int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp\ (-i\omega\tau) \tilde{f}\ (\omega) = f(\tau)\ e^{-i\omega_0\tau} \tag{16}$$

We choose $\tilde{f}\ (\omega)$ such that $f(\tau)$ is peaked at the zero value of its argument. Near the event horizon, the packets become

$$\phi^{(\pm)}_{I}\ (\rho,\ \tau) \underset{\rho\to 0}{\longrightarrow} \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp\ (-i\omega\tau)\ \tilde{f}\ (\omega) \begin{pmatrix} \chi \\ \mp\sigma^3\chi \end{pmatrix} \rho^{\mp i\omega-1/2}$$

$$= \frac{1}{\sqrt{\rho}} \begin{pmatrix} \chi \\ \mp\sigma^3\chi \end{pmatrix} f(\tau \pm \ln\rho) \exp\ (-i\omega_0\ (\tau\pm\ln\rho)). \tag{17}$$

Thus ϕ^{+}_{I} is localized at the $\rho = 0,\ \tau = +\infty$ event horizon and vanishes at the $\rho = 0,\ \tau = -\infty$ event horizon, while $\phi^{(-)}_{I}$ is localized at $\rho = 0,\ \tau = -\infty$ event horizon and vanishes at the other horizon. Similar wave-packets can be constructed in all the other regions.

Although the explicit forms of the wave-packets above look same in all regions, it is wrong to conclude that they are continuous across the event horizons. In fact in terms of continuous co-ordinates z and t, the wave packets are *not* yet continuous.

This happens because the local frames (with respect to which the spinor solutions are given) on either side of the event horizons are oriented differently. What is needed therefore is a proper re-alignment of local frames. To this end, we calculate the Vierbein components with respect to the continuous co-ordinates:

$$e_a^{\ z} = e_a^{\ \rho} \frac{\partial z}{\partial \rho} + e_a^{\ \tau} \frac{\partial z}{\partial \tau} \tag{18}$$

$$e_a^{\ t} = e_a^{\ \rho} \frac{\partial t}{\partial \rho} + e_a^{\ \tau} \frac{\partial t}{\partial \tau}.$$

We then get in all regions

$$e_a^{\ \mu} = \begin{matrix} \\ 0 \\ 3 \end{matrix} \overset{\begin{matrix} t & \quad z \end{matrix}}{\begin{pmatrix} \cosh \tau & \sinh \tau \\ \sinh \tau & \cosh \tau \end{pmatrix}}. \tag{19}$$

It is easily seen that the Lorentz transformation

$$\begin{pmatrix} \cosh \tau & -\sinh \tau \\ -\sinh \tau & \cosh \tau \end{pmatrix} \tag{20}$$

transforms the $e_a^{\ \mu}$ (eq. 19) to the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in all regions. It should be noted that although the form of Lorentz transformation required to realign the frames looks the same in all the regions, it in fact corresponds to different transformations in different regions because τ, for instance is along 0-axis in I and along the 3-axis in F, (see table 1).

The Lorentz transformation eq. (20) is a boost by $-\tau$ in the τ-ρ plane. Under this transformation the spinor transforms by $\exp\,[i\,(-\tau)\,\sigma^{03}/2] = \exp\left(\frac{\tau}{2}\,\alpha^3\right)$. With the local frames so aligned, we now establish the continuity of our solutions across the event horizons. Consider

$$\exp\,(-i\omega\tau)\exp\left(\frac{\tau}{2}\,\alpha^3\right)\frac{\psi^{(-)}\,(\rho,\,\omega)}{\sqrt{\rho}} \xrightarrow[\rho\to 0]{} \frac{1}{\sqrt{\rho}}$$

$$\exp\,[i\omega\,(\ln\rho-\tau)]\,\exp\left(\frac{\tau}{2}\,\alpha^3\right)\begin{pmatrix} \chi \\ \sigma^3\chi \end{pmatrix}$$

$$= \frac{\exp\,[i\omega(\ln\rho-\tau)]}{\exp\,[\frac{1}{2}(\ln\rho-\tau)]}\begin{pmatrix} \chi \\ \sigma^3\,\chi \end{pmatrix} \tag{21}$$

where we have used

$$\exp\left(\frac{\tau}{2}\,\alpha^3\right)\begin{pmatrix} \chi \\ \sigma^3\,\chi \end{pmatrix} = \exp\left(\frac{\tau}{2}\right)\begin{pmatrix} \chi \\ \sigma^3\,\chi \end{pmatrix}.$$

In terms of continuous co-ordinates z and t we then get:

$$\exp(-i\omega\tau)\exp\left(\frac{\tau}{2}\alpha^3\right)\frac{\psi^{(-)}(\rho,\omega)}{\sqrt{\rho}} \underset{\rho\to 0}{\longrightarrow} [\pm(z-t)]^{i\omega-1/2}\begin{pmatrix}\chi\\ \sigma^3\chi\end{pmatrix} \tag{22}$$

where the upper sign refers to I and P and the lower sign refers to II and F. We thus see that the ' rotated ' solution $\psi^{(-)}$ is continuous across the $z=-t$ ($\rho=0$, $\tau=-\infty$) event horizon. At the other event horizon ($z=t$) it is not well defined but as already shown the wave-packet constructed out of $\psi^{(-)}$ vanishes there. Similarly

$$\exp(-i\omega\tau)\exp\left(\frac{\tau}{2}\alpha^3\right)\frac{\psi^{(+)}(\rho,\omega)}{\sqrt{\rho}} \underset{\rho\to 0}{\longrightarrow} [\pm(z+t)]^{-i\omega-1/2}\begin{pmatrix}\chi\\ -\sigma^3\chi\end{pmatrix} \tag{23}$$

where the $+(-)$ sign refers to I and F (II and P). This time, the ' rotated ' solution $\psi^{(+)}$ is continuous across the $z=t$ ($\rho=0$, $\tau=+\infty$) event horizon and not well-defined at the other horizon but again its wave-packet vanishes there.

In the above results it is to be found the key to the technique of construction of continued solutions and Green's functions in different regions of Rindler space, to be done in section 3. Suppose continuation from region I to F is desired. This involves going across the $\rho=0$, $\tau=+\infty$ horizon. Across this horizon the solution $\psi_I^{(+)}$ goes over to $\psi_F^{(+)}$ but the amount of $\psi_F^{(-)}$ is left undetermined. This amount can be determined, however, from a knowledge of $\psi_{II}^{(-)}$ continued across the II, F boundary. Thus knowledge of solutions in I and II together with continuity conditions above determine the solutions in the entire space.

3. Green's functions

The spin half Green's functions in curved space-time satisfy the general equation (Boulware 1975b).

$$\beta\left(m+\frac{1}{i}\gamma^a\nabla_a\right)S(x,x')=\frac{\delta(x-x')}{\sqrt{-g}}. \tag{24}$$

This is easily recognized to be the generalization of the corresponding equation in ordinary flat space-time.

For the Rindler space, this equation reads

$$\beta\left[\frac{1}{i}\left\{\gamma^1\partial_x+\gamma^2\partial_y+\epsilon\left(\frac{\gamma^0}{\rho}\partial_\tau+\gamma^3\partial_\rho+\frac{\gamma^3}{2\rho}\right)\right\}+m\right]S(x,x')$$

$$=\frac{\delta(x-x')}{\rho}\text{ in } I, II \tag{25a}$$

$$\beta\left[\frac{1}{i}\left\{\gamma^1\partial_x+\gamma^2\partial_y+\epsilon\left(\frac{\gamma^3}{\rho}\partial_\tau+\gamma^0\partial_\rho+\frac{\gamma^0}{2\rho}\right)\right\}+m\right]S(x,x')$$

$$=\frac{\delta(x-x')}{\rho}\text{ in } F, P. \tag{25b}$$

Defining

$$S(x, x') = \int \frac{d\omega\, dk_1\, dk_2}{(2\pi)^3}\, e^{i\, k_1 (x-x')}\, e^{i\, k_2 (y-y')}\, e^{-i\omega(\tau-\tau')} \times \frac{S^{k_1 k_2}(\rho, \rho', \omega)}{\sqrt{\rho\rho'}}, \quad (26)$$

we get:

$$\beta \left[k_1\, \gamma^1 + k_2\, \gamma^2 + \epsilon \left(\frac{\gamma^3}{i}\, \partial_\rho - \frac{\omega\gamma^0}{\rho} \right) + m \right] S^{k_1 k_2}(\rho, \rho', \omega)$$

$$= \delta(\rho - \rho') \text{ in } I, II \quad (27a)$$

$$\beta \left[k_1\, \gamma^1 + k_2\, \gamma^2 + \epsilon \left(\frac{\gamma^0}{i}\, \partial_\rho - \frac{\omega\gamma^3}{\rho} \right) + m \right] S^{k_1 k_2}(\rho, \rho', \omega)$$

$$= \delta(\rho - \rho') \text{ in } F, P. \quad (27b)$$

We now proceed to construct Green's function in various regions.

<u>$\rho, \rho' \in I.$</u>

The boundary condition in region I is that S is well behaved as ρ or ρ' go to infinity. This implies that the dependence on the greater co-ordinate should be of the type of ψ_I^a (or $\psi_I^{a\dagger}$).

Hence in region I

$$S^{k_1 k_2}(\rho, \rho', \omega) = i\, \theta\, (\rho - \rho')\, \psi_I^a\, (\rho) \left[E_I^{s*}\; \psi_{Is}^{(+)\dagger}(\rho') + F_I^{s*}\; \psi_{Is}^{(-)\dagger}(\rho') \right]$$

$$- i\theta\, (\rho' - \rho) \left[G_I^s\; \psi_{Is}^{(+)}(\rho) + H_I^s\; \psi_{Is}^{(-)}(\rho) \right] \psi_I^{a\dagger}(\rho')$$

$$+ iK_I\; \psi_I^a\; (\rho)\, \psi_I^{a\dagger}(\rho'). \quad (28)$$

The last term in eq. (28) corresponds to the arbitrariness in the solution of the Green's function equation to within the addition of the solution of the homogenous equation. Substituting eq. (28) in eq. (27a) we get the identity

$$\beta\gamma^3 \Big[\psi_I^a(\rho) \left\{ E_I^{s*} \psi_{Is}^{(+)\dagger}(\rho) + F_I^{s*} \psi_{Is}^{(-)\dagger}(\rho) \right\} + \left\{ G_I^s\; \psi_{Is}^{(+)}(\rho) \right.$$

$$\left. + H_I^s\, \psi_{Is}^{(-)}(\rho) \right\} \psi_I^{a\dagger}(\rho) \Big] = I \quad (29)$$

The identity is true for arbitrary ρ. Taking the trace of this equation and evaluating the quantities in $\rho \to 0$ limit, we obtain

$$A_{aI}^s\, F_I^{s*} + A_{aI}^{s*}\, H_I^s + B_{aI}^s\, E_I^{s*} + B_{aI}^{s*}\, G_I^s = 0. \quad (30)$$

Thus the constants appearing in the solution, eq. (28) are constrained in terms of the expansion co-efficients of region I, A_{aI}^s and B_{aI}^s.

$\rho, \rho' \in II.$

This case is analogous to the preceding case. All equations are the same as above with the index I replaced by II. In particular eq. (30) is replaced by

$$A^s_{aII}\ F^{s*}_{II} + A^{s*}_{aII}\ H^s_{II} + B^s_{aII}\ E^{s*}_{II} + B^{s*}_{aII}\ G^s_{II} = 0. \tag{31}$$

The next task is to continue the Green's functions in other regions.

$\rho \in F,\ \rho' \in I.$

Since the continuation is across the $\rho=0$, $\tau=+\infty$ boundary the second θ-function in eq. (28) contributes. Further $\psi^{(+)}_{Is}(\rho)$ will continue over to $\psi^{(+)}_{Fs}(\rho)$ with an unspecified amount of $\psi^{(-)}_{Fs}(\rho)$ (section 2). Therefore for this case

$$S^{k_1k_2}(\rho,\rho',\omega) = -i\Big[\Big(G^s_I - K_I B^s_{aI}\Big)\psi^{(+)}_{Fs}(\rho) + \eta^s_{IF}\psi^{(-)}_{Fs}(\rho)\Big]\psi^{a\dagger}_I(\rho') \tag{32}$$

η^s_{IF} stands for the unspecified amount of $\psi^{(-)}$ going from I to F. The positive frequency boundary condition in region F determines η^s_{IF} in terms of other constants. For, in region F, the time co-ordinate in ρ and $\rho \to \infty$ implies going to the remote future. The positive frequency boundary condition requires that the Green's function go like $e^{-iq\rho}$ as $\rho \to \infty$. This implies that the ρ-dependence in equation (32) is of the form $\psi^a_F(\rho)$. We then have

$$\eta^s_{IF} = \frac{A^s_{aF}}{B^s_{aF}}(G^s_I - K_I B^s_{aI}) \tag{33}$$

and $$S^{k_1k_2}(\rho,\rho',\omega) \sim \psi^a_F(\rho)\,\psi^{a\dagger}_I(\rho');\ \rho \in F,\ \rho' \in I \tag{34}$$

$\rho \in I,\ \rho' \in F.$

The same procedure as above gives

$$S^{k_1k_2}(\rho,\rho',\omega) = i\,\psi^a_I(\rho)\Big[\Big(E^{s*}_I + K_I\,B^{s*}_{aI}\Big)\,\psi^{(+)\dagger}_{Fs}(\rho') + \eta^{s*}_{IF}\,\psi^{(-)\dagger}_{Fs}(\rho')\Big]. \tag{35}$$

The positive frequency condition implies that

$$\eta^{s*}_{IF} = \frac{A^{s*}_{bF}}{B^{s*}_{bF}}(E^{s*}_I + K_I\,B^{s*}_{aI}) \tag{36}$$

and

$$S^{k_1k_2}(\rho,\rho',\omega) \sim \psi^a_I(\rho)\,\psi^{b\dagger}_F(\rho');\ \ \rho \in I,\ \rho' \in F \tag{37}$$

It is worth noting at this stage that the solution $G_I^s = E_I^s$, $K_I = -K_I^*$ is not possible for our case because eqs (33) and (36) would then imply that ψ_F^a and ψ_F^b are linearly dependent.

$\rho \in F,\ \rho' \in II.$

The procedure is as before but this time the boundary is different; here $\psi_{IIs}^{(-)}$ continues over to $\psi_{Fs}^{(-)}$ with an unspecified amount of $\psi_{Fs}^{(+)}$.

$$S^{k_1k_2}(\rho,\rho',\omega) = -i\Big[\Big(H_{II}^s - K_{II} A_{aII}^s\Big)\psi_{Fs}^{(-)}(\rho) + \eta_{IIF}^s \psi_{FS}^{(+)}(\rho)\Big]\psi_{II}^{a\dagger}(\rho'). \tag{38}$$

The positive frequency boundary condition implies

$$S^{k_1k_2}(\rho,\rho',\omega) \sim \psi_F^a(\rho)\psi_{II}^{a\dagger}(\rho');\ \rho \in F,\ \rho' \in II \tag{39}$$

The condition on η_{IIF}^s is analogous to eq. (33).

$\rho \in II,\ \rho' \in F.$

In the same manner

$$S^{k_1k_2}(\rho,\rho',\omega) = i\,\psi_{II}^a(\rho)\Big[\Big(F_{II}^{s*} + K_{II} A_{aII}^{s*}\Big)\psi_{Fs}^{(-)\dagger}(\rho') + \eta_{IIF}^{s*}\ \psi_{Fs}^{(+)\dagger}(\rho')\Big] \tag{40}$$

$$\sim \psi_{II}^a(\rho)\psi_F^{b\dagger}(\rho');\ \rho \in II,\ \rho' \in F. \tag{41}$$

$\rho,\ \rho' \in F.$

Consider eq. (32) and let ρ' go across the boundary to F. $\psi_I^{(+)\dagger}(\rho')$ continues over to $\psi_F^{(+)\dagger}(\rho')$ and there is an unspecified amount of $\psi_F^{(-)\dagger}(\rho')$. The latter contribution is however determined from a knowledge of Green's function in II, eq. (38). The two together determine the Green's function in F for $\rho > \rho'$. In a similar manner using eqs (35) and (40), one can determine the Green's function for $\rho < \rho'$. We then have

$$S^{k_1k_2}(\rho,\rho',\omega)$$

$$= -i\,\theta(\rho-\rho')\Big[\Big\{\Big(G_I^s - K_I B_{aI}^s\Big)\psi_{Fs}^{(+)}(\rho) + \eta_{IF}^s\ \psi_{Fs}^{(-)}(\rho)\Big\} \times B_{aI}^{t*}\ \psi_{Ft}^{(+)\dagger}(\rho')$$

$$+ \left\{ \left(H^{s}_{II} - K_{II}\, A^{s}_{aII}\right) \psi^{(-)}_{Fs}(\rho) + \eta^{s}_{IIF} \cdot \psi^{(+)}_{Fs}(\rho) \right\}$$

$$\times A^{t*}_{aII}\, \psi^{(-)\dagger}_{Ft}(\rho') \Big]$$

$$+ i\theta(\rho'-\rho) \Big[B^{s}_{aI}\, \psi^{(+)}_{Fs}(\rho) \left\{ \left(E^{t*}_{I} + K_{I}\, B^{t*}_{aI}\right) \psi^{(+)\dagger}_{Ft}(\rho') \right.$$

$$\left. + \eta^{t*}_{IF}\, \psi^{(-)\dagger}_{Ft}(\rho') \right\}$$

$$+ A^{s}_{aII}\, \psi^{(-)}_{Fs}(\rho) \left\{ \left(F^{t*}_{II} + K_{II}\, A^{t*}_{aII}\right) \psi^{(-)\dagger}_{Ft}(\rho') \right.$$

$$\left. + \eta^{t*}_{IIF}\, \psi^{(+)\dagger}_{Ft}(\rho') \right\} \Big]. \tag{42}$$

Employing the positive frequency conditions imposed already on the coefficients appearing above (eqs (33), (36), etc.) we get

$$S^{k_1 k_2}(\rho,\rho',\omega) = -i\theta(\rho-\rho')\, \psi^{a}_{F}(\rho) \Big[M^{s}_{IF}\, \psi^{(+)\dagger}_{Fs}(\rho')$$

$$+ N^{s}_{IIF}\, \psi^{(-)\dagger}_{Fs}(\rho') \Big]$$

$$+ i\theta(\rho'-\rho) \left[O^{s}_{IF}\, \psi^{(+)}_{Fs}(\rho) + R^{s}_{IIF}\, \psi^{(-)}_{Fs}(\rho) \right] \psi^{b\dagger}_{F}(\rho');$$

$$\rho, \rho' \in F \tag{43}$$

where M^{s}_{IF}, etc. depend on the expansion co-efficients of regions I and F. Similarly for $\rho, \rho' \in P$ we have

$$S^{k_1 k_2}(\rho,\rho',\omega) = -i\theta(\rho-\rho')\, \psi^{a}_{p}(\rho) \left[M^{s}_{IIp} \psi^{(+)\dagger}_{ps}(\rho') + N^{s}_{Ip} \psi^{(-)\dagger}_{ps}(\rho') \right]$$

$$+ i\theta(\rho'-\rho) \left[O^{s}_{IIp} \psi^{+}_{ps}(\rho) + R^{s}_{Ip} \psi^{(-)}_{ps}(\rho) \right] \psi^{b\dagger}_{p}(\rho'). \tag{44}$$

The method given above can be repeated for all the remaining cases. The full scheme of Green's functions in all regions of the extended Rindler space is presented in table 2.

To demonstrate the consistency of our construction we first note that the Green's functions were required to satisfy the inhomogenous equations in *regions I and II only.* A consequence of this is the identities eqs (30) and (31). A consistency-check on our

Table 2. The Green's functions $S^{k_1k_2}(\rho, \rho', \omega)$ for ρ, ρ' in the four sectors of the extended Rindler space.

	$\rho' \in I$	$\rho' \in P$	$\rho' \in II$	$\rho' \in F$
$\rho \in I$	eq. (28)	$\sim \psi_I{}^a(\rho)\, \psi_P{}^{b\dagger}(\rho')$	$\sim \psi_I{}^a(\rho)\, \psi_{II}{}^{a\dagger}(\rho')$	$\sim \psi_I{}^a(\rho)\, \psi_F{}^{b\dagger}(\rho')$
$\rho \in P$	$\sim \psi_P{}^a(\rho)\, \psi_I{}^{a\dagger}(\rho')$	Eq. (44)	$\sim \psi_P{}^a(\rho)\, \psi_{II}{}^{a\dagger}(\rho')$	$\sim \psi_P{}^a(\rho)\, \psi_F{}^{b\dagger}(\rho')$
$\rho \in II$	$\sim \psi^a{}_{II}(\rho)\, \psi_I{}^{a\dagger}(\rho')$	$\sim \psi_{II}{}^a(\rho)\, \psi_P{}^{b\dagger}(\rho')$	Eq. (28) with I replaced by II	$\sim \psi_{II}{}^a(\rho)\, \psi_F{}^{b\dagger}(\rho')$
$\rho \in F$	$\sim \psi_F{}^a(\rho)\, \psi_I{}^{a\dagger}(\rho')$	$\sim \psi_F{}^a(\rho)\, \psi_P{}^{b\dagger}(\rho')$	$\sim \psi_F{}^a(\rho)\, \psi_{II}{}^{a\dagger}(\rho')$	Eq. (43)

scheme will be to see if the analogous identities following from the inhomogeneous equations in F and P hold good. If $S^{k_1k_2}(\rho, \rho', \omega)$; $\rho, \rho', \epsilon F$ satisfies the Green's function eq. (27b) with $\epsilon=1$, we have the identity analogous to eq. (29)

$$\begin{aligned}
&[(G_I^s - K_I B_{aI}^s)\psi_{Fs}^{(+)}(\rho) + \eta_{IF}^s \psi_{Fs}^{(-)}(\rho)] B_{aI}^{t*} \psi_{Ft}^{(+)\dagger}(\rho)\\
&+[(H_{II}^s - K_{II} A_{aII}^s)\psi_{Fs}^{(-)}(\rho) + \eta_{IIF}^s \psi_{Fs}^{(+)}(\rho)] A_{aII}^{t*} \psi_{Ft}^{(-)\dagger}(\rho)\\
&+B_{aI}^s \psi_{Fs}^{(+)}(\rho)[(E_I^{t*} + K_I B_{aI}^{t*})\psi_{Ft}^{(+)\dagger}(\rho) + \eta_{IF}^{t*} \psi_{Ft}^{(-)\dagger}(\rho)]\\
&+A_{aII}^s \psi_{FS}^{(-)}(\rho)[(F_{II}^{t*} + K_{II} A_{aII}^{t*})\psi_{FT}^{(-)\dagger}(\rho) + \eta_{IIF}^{t*} \psi_{FT}^{(+)\dagger}(\rho)]\\
&=-I.
\end{aligned} \tag{45}$$

Again as the identity is true for arbitrary ρ we evaluate its trace as $\rho\to 0$. This gives

$$G_I^s B_{aI}^{s*} + H_{II}^s A_{aII}^{s*} + B_{aI}^s E_I^{s*} + A_{aII}^s F_{II}^{s*} = -2. \tag{46}$$

Similarly if $S^{k_1k_2}(\rho, \rho', \omega)$; $\rho, \rho' \epsilon P$ satisfies eq. (27b) with $\epsilon=-1$ we have

$$H_I^s A_{aI}^{s*} + G_{II}^s B_{aII}^{s*} + A_{aI}^s F_I^{s*} + B_{aII}^s E_{II}^{s*} = 2. \tag{47}$$

The identities eqs (30), (31), (46) and (47) admit of a consistent solution:

$$A_{aI}^s F_I^{s*} + A_{aI}^{s*} H_I^s = B_{aII}^s E_{II}^{s*} + B_{aII}^{s*} G_{II}^s = 1 \tag{48a}$$

$$B_{aI}^s E_I^{s*} + B_{aI}^{s*} G_I^s = A_{aII}^s F_{II}^{s*} + A_{aII}^{s*} H_{II}^s = -1 \tag{48b}$$

4. Conclusion

The η-prescription of Boulware has been shown to yield a consistent scheme of Green's functions for the spin half field in the extended Rindler space. However since the full analytic solutions of the equation are not known the equivalence of our scheme with the Minkowski-Green's function is still an open question.

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