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Perturbation theory for large coupling constants

JYOTI CHOUBEY and L K SHARMA
Department of Applied Physics, Government Engineering College, Jabalpur 482 001

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Abstract. The perturbation technique for large coupling constants g^2 is used for obtaining the solutions of Schrödinger equation for a double exponential potential. In particular the solution valid for $g^2e^{-r/2} \gg 1$, is obtained in terms of confluent hypergeometric functions. A noteworthy aspect of this application is that the procedure developed can also be used for solving certain singular potentials.

Keywords. Perturbation theory; double exponential potential; single exponential potential; singular potentials.

1. Introduction

Large coupling solutions to various problems in particle physics have recently attracted considerable interest, particularly in view of the now almost classical desire to understand strong interactions in terms of purely field-theoretic formulations. Field theoretical models for large coupling constants are scarce except in the form of the Bethe-Salpeter equation (Cheng and Wu 1971a, b and Joos and Krammer 1971).

Although the ultimate aim of physicists is a fully relativistic particle theory, considerable insight has always been drawn from simpler, non-relativistic potential models for which complete or near complete solutions can be obtained. Apart from purely mathematical interest these non-relativistic models are directly applicable in nuclear or atomic physics and so justify separate investigation in their own right. Non-relativistic wave equations for large coupling constants of the potential have been studied by several authors. Cheng and Wu (1971a, b) calculated the approximate behaviour of Regge trajectories for the Yukawa potential. Müller (1965, 1968) investigated extensively the energy eigenvalues for the Yukawa potential. Other cases for which large coupling solutions have been derived are the Gauss potential (Müller 1970) and its generalized form (Sharma 1971, Simenog 1969).

In section 2, we investigate the solution of a double exponential (Morse type) potential for the case of large coupling constants. In particular we use a perturbation procedure which Müller (1965, 1968) has been using in his numerous other investigations. We then consider a single exponential potential which is a special case of a double exponential potential. In section 3, the procedure developed in section 2, is applied for solving two singular potentials with logarithmic singularities.

2. Perturbation theory and its application to double and single exponential potentials

We consider an s-wave radial Schrödinger equation ($\hbar=c=1$, $m=\frac{1}{2}$) for the potential

$$\nu(r)=-g^2\nu(r),\ g^2>0. \tag{1}$$

For imposing a minimum of restrictions on $\nu(r)$, the restrictions on $\nu(r)$ only are specified as and when they are required in the course of developing the theory. The radial Schrödinger equation now assumes the form:

$$\psi'' + [g^2\nu(r)-\omega^2]\,\psi = 0, \tag{2}$$

where $k^2 = -\omega^2$.

We now set $\psi = P(r)\exp\left[\pm ig \int^r \sqrt{\nu(r)}dr\right]$. (3)

The equation for $P(r)$ then takes the form

$$4\nu(r)P'(r)+\nu'(r)P(r) = \pm\frac{2\sqrt{\nu(r)}}{ig}[-P''(r)+\omega^2(r)P(r)]. \tag{4}$$

Note the need to study one of these equations. Obviously, the study of the second automatically follows by changing the sign of g.

Taking $$P(r) = \sum_{j=0}^{\infty} P(r)^{(J)}\frac{1}{(ig)^J}, \tag{5}$$

and substituting in eq. (4) one gets,

$$4\nu(r)P(r)^{(J)\prime}+\nu'(r)P(r)^{(J)} = \pm 2\sqrt{\nu(r)}\,[-P(r)^{(J-1)\prime\prime}+\omega^2P(r)^{(J-1)}]. \tag{6}$$

On equating coefficients of the same powers of (ig) we find

$$P(r)^{(0)} = [\nu(r)]^{-1/4}. \tag{7}$$

Again putting

$$Q(r)^{(J-1)} = \frac{1}{2\sqrt{\nu(r)}}\left[-P(r)^{(J-1)\prime\prime}+\omega^2P(r)^{(J-1)}\right], \tag{8}$$

we have $$P(r)^{(J)} = \frac{1}{\{\nu(r)\}^{1/4}}\int^r Q(r)^{(J-1)}\nu(r)^{1/4}dr, \tag{9}$$

where

$$Q(r)^{(0)} = \frac{1}{2\sqrt{\nu(r)}}\left[\frac{1\nu''(r)}{4\nu(r)} - \frac{5}{16}\left\{\frac{\nu'(r)}{\nu(r)}\right\}^2 + \omega^2\right]P(r)^{(0)}. \tag{10}$$

Equation (10) yields a general expression

$$Q(r)^{(J)} = \frac{1}{2\sqrt{\nu(r)}}\Big[-Q(r)^{(J-1)'} + \frac{1}{4}\frac{\nu(r)'}{\nu(r)}Q(r)^{(J-1)} + \Big\{\frac{1}{4}\frac{\nu''(r)}{\nu(r)}$$

$$-\frac{5}{16}\Big[\frac{\nu'(r)}{\nu(r)}\Big]^2 + \omega^2\Big\}\frac{1}{\{\nu(r)\}^{1/4}}\int Q(r)^{(J-1)}\{\nu(r)\}^{1/4}dr\Big]. \quad (11)$$

The above relation is now used for solving the double exponential potential viz:

$$V(r) = -g^2(e^{-2r} - ae^{-r}) \quad (12)$$

which for $a = 2$ reduces to the well known Morse (1929) potential.
For this potential we find

$$P(r) = (e^{-2r} - ae^{-r})^{-1/4}\Big[1 - \frac{\omega^2}{iga}\Big\{e^{r/2}(e^{-r} - a)^{1/2}\Big\}$$

$$+ e^{-r/2}\frac{(e^{-r} - a)^{-1/2}}{iga} - \frac{1}{4ig}e^{r/2}(e^{-r} - a)^{1/2} + \ldots\Big] \quad (13)$$

The solution of (3) is therefore given by:

$$\psi(r) = C_{\pm}\exp\{\mp ig(e^{-2r} - ae^{-r})^{1/2}\}\{e^{-r/2} + (e^{-r} - a)^{1/2}\}^{iga}$$

$$(e^{-2r} - ae^{-r})^{-1/4}\Big[1 - \frac{\omega^2}{iga}\{e^{r/2}(e^{-r} - a)^{1/2}\} + \frac{e^{-r/2}(e^{-r} - a)^{-1/2}}{iga}$$

$$-\frac{1}{4ig}e^{r/2}(e^{-r} - a)^{1/2} + \ldots\Big] \quad (14)$$

where $C_{\pm}$ are suitable normalization constants. Now for the choice

$$|ge^{-r/2}| \gg |\omega|, \; |ge^{-r/2}| \gg 1 \text{ and } -\pi < \arg(ge^{-r/2}) < \pi, \quad (15)$$

equation (14) reduces to

$$\psi = C_{\pm}(e^{-2r} - ae^{-r})^{-1/4}\exp\{\mp ig(e^{-2r} - ae^{-r})^{1/2}\} \times$$

$$e^{\mp igar/2}\{1 + (1 - ae^{-r})^{1/2}\}^{\pm iga}. \quad (16)$$

The conditions mentioned in (15) imply that the expansions are valid only for large values of $|g|$ and $r/2 | \ll |\log g|$. We could have directly transformed equation (2) into confluent hypergeometric equation for the double exponential potential, but our aim here is to find solutions for whole class of potentials, of which (12) is only a simple example. The asymptotic behaviour of confluent hypergeometric function is given by (Erdelyi 1953)

$$\phi\,(a, p; z) = \frac{\Gamma p}{\Gamma a}\, e^{z} \cdot z^{a-p} \cdot [1 + O(z)^{-1}] \tag{17}$$

Now for the choice

$$z = \mp\, ig\,(e^{-2r} - \alpha e^{-r})^{1/2}$$

$$a = -\tfrac{1}{2} \text{ and}$$

$$p = \mp\, 2\,ik$$

$$\psi_{\pm} = A_{\pm}(k)\phi[-\tfrac{1}{2}, \mp 2\,ik; \mp ig\,(e^{-2r} - \alpha e^{-r})^{1/2}]\{(e^{-r} - \alpha)\}^{\mp ik} \cdot e^{\pm - if(k)r} \tag{18}$$

where

$$(k \mp g\alpha/2) = f(k),$$

$$A_{\pm}(k) = C_{\pm}(\mp ig)^{1/2 \mp 2ik}\, \frac{\Gamma(-1/2)}{\Gamma(\mp\, 2\, ik)}. \tag{19}$$

We now investigate the solution (19) for Morse potential in particular. The s-wave solution for the Morse potential is obviously a linear combination of the solutions $\psi_{\pm}(\alpha=2)$, i.e.

$$\psi_{\pm} = A_{\pm}(k)\phi\ [-\tfrac{1}{2}, \mp 2ik; \mp ig(e^{-2r} - 2e^{-r})^{1/2}]$$
$$\{(e^{-r} - 2)\}^{\mp ik} \cdot e^{\pm if(k)r}. \tag{20}$$

Asymptotically

$$\psi_{\pm} = D_{+}(k)e^{if(k)r} + D_{-}(k)e^{-if(k)r} \tag{21}$$

where

$$D_{\pm}(k) = A_{\pm}(k)(-2)^{\mp ik}. \tag{22}$$

The domain of r when $r \to \infty$ will be upper half of a circle of radius r. This together with the requirement $\psi(r) = 0$ for $r = 0$ yields the s-matrix explicitly in the form

$$s(k) = \frac{D_{+}(k)}{D_{-}(k)}\, z^{2ik}\, \frac{\phi(-\tfrac{1}{2}, -2ik; +g)}{\phi(-\tfrac{1}{2}, +2ik; -g)}. \tag{23}$$

The bound states are given by the zeros of the confluent hypergeometric function, i.e.

$$\phi(-\tfrac{1}{2}, -2ik; +g) = 0 \tag{24}$$

Particular case (single exponential potential)

The potential (12) for $\alpha = 0$ is reduced to a simple exponential potential viz:

$$v(r) = e^{-2r}. \tag{25}$$

For this potential we find that eq. (16) reduces to

$$\psi_\pm = C_\pm \exp\{\mp ige^{-r}\} \,.\, e^{r/2} \tag{26}$$

This solution again is obviously valid for large values of $|g|$ and $r \ll |\log g^2|$.
Under the suitable choice of

$$C_\pm = \sqrt{\frac{2}{\pi g}}\, e^{\mp i\left(\frac{-\omega\pi}{2} - \frac{\pi}{4}\right)}$$

(26) can be written as:

$$\psi_\pm = H_\omega^{(2,\,1)}\,(ge^{-r/2}), \tag{27}$$

where $H_\omega^{(2,\,1)}$ are Hankel functions.
The s-wave Jost functions $f(0,k), f(0,-k)$ are given as:

$$f(0, \pm k) = \lim_{r\to 0} f\,(0, \pm k;\, r) \tag{28}$$

where

$$f(0, \pm k;\, r) \approx e^{\pm ikr} \text{ for } r\to\infty. \tag{29}$$

The Jost solutions $f(0, \pm k;\, r)$ are therefore linear combinations of $H_\omega^{(1,2)}$, i.e.

$$J_\omega\,(ge^{-r}) = \tfrac{1}{2}\,[H_\omega^{(1)}\,(ge^{-r}) + H_\omega^{(2)}\,(ge^{-r})]. \tag{30}$$

Now $J_\omega\,(ge^{-r})$ for $r\to\infty$ has the behaviour

$$J_\omega\,(ge^{-r}) \simeq \frac{(\frac{1}{2}ge^{-r})^\omega}{\omega!} + O(ge^{-r})^{\omega+2}. \tag{31}$$

Thus for $\omega = -ik$, we have

$$J_{-ik}\,(ge^{-r}) = \frac{(ge^{-r/2})^{-ik}}{(-ik)!} = \left(\frac{2}{g}\right)^{ik} \frac{e^{ikr}}{(-ik)!}.$$

Hence $f(0, \pm k;\, r) = (\mp ik)!\,(g/2)^{\pm ik} J_{\mp ik}(ge^{-r})$. (32)
The s-matrix and the phase shift δ_0 are thus given by:

$$s = e^{2i\delta_0} = \frac{f(0,+k)}{f(0,-k)} = \frac{(-ik)!(g/2)^{2ik}\, J_{-ik}(g)}{(+ik)!\, J_{+ik}\,(g)}. \tag{33}$$

Here the bound states are obviously given by the zeros of the Bessel function i.e.

$$J_{-ik}(g)=0 \tag{34}$$

3. Solution of a singular potential

The interest in the studies of highly singular potentials in both relativistic and non-relativistic theories derives from many indications that interactions responsible for high energy elementary particle reactions are of more singular nature than the Yukawa potential.

Recently, potentials with singularities of the pole and logarithmic type have been investigated (Aly *et al* 1964, 1965; Wu 1964 and Gale 1966). In this section we solve a potential with a logarithmic singularity viz:

$$V(r)=-g^2\left\{\frac{\log(r/R)}{r^2}-\frac{a}{r}\right\},\ R>0 \tag{35}$$

setting $x=k/g$ and $\rho=xr$. (36)

We find that radial Schrödinger equation with $l=(\lambda-\frac{1}{2})$ is transformed to

$$\frac{d^2\psi}{d\rho^2}+\left[\frac{k^2}{x^2}-\frac{(\lambda^2-\frac{1}{4})}{\rho^2}+\frac{g^2}{\rho^2}\left(\log\frac{\rho}{xR}-\frac{\alpha\rho}{x}\right)x^2\right]\psi=0. \tag{37}$$

Taking $$1+\frac{\log\frac{\rho}{x}-\frac{\alpha p}{x}}{\rho^2}=v(\rho)$$

and $$\mu^2=\omega^2-\tfrac{1}{4},\ \text{where } \omega^2=\lambda^2+g^2\log R, \tag{38}$$

one can write eq. (37) as:

$$\frac{d^2\psi}{d\rho^2}+\left[g^2v(\rho)-\frac{\mu^2}{\rho^2}\right]\psi=0. \tag{39}$$

If $$v=-\log\rho/x \text{ and } \psi=e^{-v}F \tag{40}$$

equation (39) can be transformed to:

$$F''+\left[-\frac{\omega^2}{x^2}+g^2(e^{-2v}-\alpha e^{-v}-v)\right]F=0. \tag{41}$$

For $x^2=1$, (41) has the same form as an s-wave Schrödinger equation containing the modified double exponential potential viz:

$$(e^{-2x}-\alpha e^{-v}-v).$$

It may be of interest to note that for $\alpha=0$ and $x^2=1$, the singular potential considered in eq. (35) is reduced to the form

$$V(r)=-g^2\frac{(\log r/R)}{r^2},\ R>0. \tag{42}$$

In this case eq. (41) will obviously be similar to an s-wave Schrödinger equation containing the modified exponential potential $(e^{-2v}-v)$.

Recently, there have been some attempts to obtain meaningful finite results in nonrenormalizable field theories. In this connection, the singular potential given in (42) has attracted considerable attention as it provides a model for field theoretic interactions. Bertocchi *et al* (1965) have discussed an analogy between a renormalizable field theory and scattering on a potential behaving in the origin as $\{gr^{-2}\log^{\gamma}(R/r)\}\gamma\geqslant 0$. Arbuzov *et al* (1964) have shown that, in the case $\gamma=1$, the wave function for this problem contains an essential singularity at $g=0$. Calogero *et al* (1964) have proved that the essential singularity of the wave function is also present in the scattering parameters. Finally, this potential is of special interest, because Mandelstam representation has also been proved for it by Brander (1969).

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