Pramāṇa, Vol. 9, No. 3, September 1977, pp. 239–247, © Printed in India.

Renormalizable theories from nonrenormalizable interactions

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MS received 11 May 1977

Abstract. By using Kikkawa's method the equivalence of the nonrenormalizable pair interaction $\bar{\psi}\psi\phi^2$ to a renormalizable theory is proved. Equivalence relationships between a few other nonrenormalizable and renormalizable interactions are also indicated.

Keywords. Quantum field theory; renormalizable and nonrenormalizable interactions; composite particles; pair interaction; ϕ^6 coupling.

1. Introduction

It is well-known that although an infinite variety of interacting field theories can be constructed at the classical level, the class of quantum field theories of practical utility is severely limited by the requirement of renormalizability. Of course the recent discovery of renormalizable gauge theories has enlarged this class to cover physically interesting interactions such as the weak interactions of particles. Nevertheless, there still remain very many theories which are apparently nonrenormalizable and hence not usable in practice. It would be desirable to bring more and more theories of this type into the more respectable class of usable quantum field theories.

A recent development which has its origins in the theory of superconductivity appears to be an important step in this direction. This is the work of Kikkawa (1976) and Eguchi (1976) which is based on the earlier ideas of Eguchi and Sugawara (1975), Chakrabarti and Hu (1975), and Bjorken (1963). Kikkawa as well as Eguchi considered the nonlinear spinor theory involving nonrenormalizable four-fermi couplings of the type $(\bar{\psi}\psi)^2$. By introducing a suitable bose field ϕ representating a fermion-antifermion composite they showed that the above coupling is equivalent to the renormalizable Yukawa interaction $\bar{\psi}\psi\phi$. Similar equivalence holds between $(\bar{\psi}\gamma_\mu\psi)^2$ and $\bar{\psi}\gamma_\mu\psi A_\mu$ where A_μ is a vector field that can be chosen to be massless. One may say that the formation of the composite particle partly exhausts the strength of the original nonrenormalizable coupling so that the residual interaction is renormalizable. Terazawa *et al* (1976) and Saito and Shigemoto (1976) have gone further and constructed suitable four-fermi interactions which are equivalent to the renormalizable unified gauge theories of weak, electromagnetic and strong interactions. (For a brief review of these developments, the reader is referred to Rajasekaran 1977).

In this paper we report on the equivalence of some more nonrenormalizable theories to renormalizable ones. First we treat the nonrenormalizable pair interaction $\bar{\psi}\,\psi\,\phi^2$ (where ψ is a fermi field and ϕ is a bose field) and by an explicit calculation show its equivalence to the renormalizable Yukawa interaction. Whereas the composite field occurring for the four-fermi coupling was bosonic, in the present case of $\bar{\psi}\,\psi\,\phi^2$ coupling, the composite field is fermionic. We then consider a few more nonrenormalizable interactions and give arguments to show their equivalence to corresponding renormalizable theories. Thus it is clear that this type of equivalence relationship is not a special property of the nonlinear spinor theory or of the fermion-antifermion system, but is rather a general feature of quantum field theories.

2. Equivalence of the pair and Yukawa interactions

We start with the Lagrangian of a fermi field ψ_1 and a bose field ϕ with a pair interaction:

$$L_1=\bar{\psi}_1\,(i\gamma.\partial-m_1)\,\psi_1+\tfrac{1}{2}\,(\partial_\mu\phi)^2\,-\tfrac{1}{2}\mu^2\phi^2+G\bar{\psi}_1\psi_1\phi^2. \tag{1}$$

It is easy to show this Lagrangian is equivalent to

$$L_2=\bar{\psi}_1\,(i\gamma\,.\,\partial-m_1)\,\psi_1+\tfrac{1}{2}\,(\partial_\mu\phi)^2-\tfrac{1}{2}\mu^2\phi^2-\frac{1}{G}\,\bar{\psi}_2\psi_2+(\bar{\psi}_2\psi_1+\bar{\psi}_1\psi_2)\,\phi. \tag{2}$$

For, the Euler-Lagrange variational equations for ψ_2 and $\bar{\psi}_2$ obtained from L_2 are

$$\psi_2=G\psi_1\phi;\ \bar{\psi}_2=G\bar{\psi}_1\phi \tag{3}$$

and these substituted in (2) give (1). Of course, the auxiliary fermi field ψ_2 introduced in (2) is not equipped with a kinetic energy term.

The important step is to show that the kinetic energy term $\bar{\psi}_2\,i\gamma\,.\,\partial\psi_2$ is generated by divergent radiative corrections. For this purpose, let us use the generating functional obtained by functional integration (Kikkawa 1976):

$$W_2=\int\mathscr{D}\psi_1\,\mathscr{D}\bar{\psi}_1\,\mathscr{D}\phi\,\exp\,[i\int d^4x\,\{L_2+\bar{\eta}\psi_1+\bar{\psi}_1\eta+\chi\phi\}] \tag{4}$$

where η and χ are the source functions for ψ_1 and ϕ respectively. We change the integration variables from ψ_1, $\bar{\psi}_1$, to

$$\psi'_1=\psi_1+S_F\,(\eta+\psi_2\phi)$$

$$\bar{\psi}'_1=\bar{\psi}_1+\,(\bar{\eta}+\bar{\psi}_2\phi)\,S_F \tag{5}$$

where

$$S_F=(i\gamma\,.\,\partial-m_1)^{-1}$$

and carry out the integration over ψ'_1 and $\bar{\psi}'_1$. We get

$$W_2=\int \mathscr{D}\phi \exp\, [i\int d^4x \{\tfrac{1}{2}(\partial_\nu\phi)^2-\tfrac{1}{2}\mu^2\phi^2+\chi\phi-\frac{1}{G}\bar{\psi}_2\psi_2$$

$$-(\bar{\eta}+\bar{\psi}_2\phi)\, S_F\,(\eta+\psi_2\phi)\}] \tag{6}$$

where we have ignored a normalization constant. The ϕ integration is then performed after another change of variable

$$\phi'=\phi+(\bar{\eta}S_F\psi_2+\bar{\psi}_2S_F\eta+\chi)\, M^{-1}\triangle_F \tag{7}$$

where

$$\triangle_F=(\Box+\mu^2)^{-1}$$

and M denotes another integral operator with the following kernel:

$$M(x,y)=\delta^4(x-y)+\int d^4z\,\triangle_F(x-z)\{\bar{\psi}_2(z)S_F(z-y)\psi_2(y)+(z\leftrightarrow y)\}. \tag{8}$$

The result is (apart from a normalization constant)

$$W_2=(\mathrm{Det}\, M)^{-1}\exp\, i\int d^4x\{-\bar{\eta}S_F\eta-\frac{1}{G}\bar{\psi}_2\psi_2$$

$$+\tfrac{1}{2}(\bar{\eta}S_F\psi_2+\bar{\psi}_2S_F\eta+\chi)M^{-1}\triangle_F(\bar{\eta}S_F\psi_2+\bar{\psi}_2S_F\eta+\chi)\}. \tag{9}$$

We reexpress the determinantal factor as follows:

$$(\mathrm{Det}\, M)^{-1}=\exp\{-\mathrm{Tr}\ln M\}$$

$$\exp\left\{\mathrm{Tr}\sum_n\frac{(-1)^n}{n}(2\triangle_F\bar{\psi}_2S_F\psi_2)^n\right\} \tag{10}$$

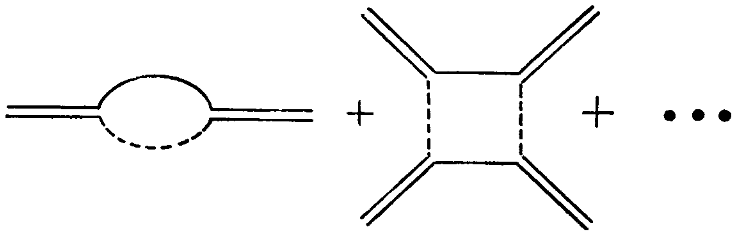
where Det and Tr refer respectively to the determinant and trace operations on the space-time point x. Pictorially, the trace expression in (10) corresponds to the series depicted in figure 1, namely, diagrams with one closed loop involving ψ_1 and ϕ propagators and arbitrary number n of pairs of external ψ_2 lines. These are divergent for $n=1$ and convergent for $n\geqslant 2$. The divergent part of the $n=1$ term is calculated to be (see appendix)

$$i\int d^4x\{m_1I_1\bar{\psi}_2\psi_2+I_2\bar{\psi}_2 i\gamma.\partial\psi_2\} \tag{11}$$

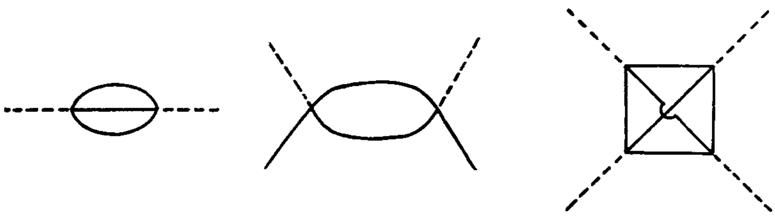
where

$$I_1=\frac{-2i}{(2\pi)^4}\int_0^1 dz\int d^4k\frac{1}{[k^2-(m_1{}^2-\mu^2)z-\mu^2]^2},$$

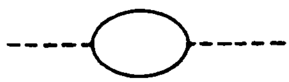
$$I_2=\frac{-2i}{(2\pi)^4}\int_0^1 dz\int d^4k\frac{(1-z)}{[k^2-(m_1{}^2-\mu^2)z-\mu^2]^2}. \tag{12}$$



(1)



(2)



(3)

Figure 1. Closed loops for the pair interaction. Double lines, single lines and dotted lines denote ψ_2, ψ_1 and ϕ respectively.

Figure 2. Closed loops for the ϕ^6 interaction. Full lines and dotted lines represent ϕ and σ respectively.

Figure 3. Closed loop for the ϕ^4 interaction. Full lines and dotted lines denote ϕ and σ respectively.

Hence, (9) becomes

$$W_2=\exp\Big[\, i\int d^4x\Big\{ I_2\bar{\psi}_2 i\gamma.\partial\psi_2-\Big(\frac{1}{G}-m_1I_1\Big)\bar{\psi}_2\psi_2-\bar{\eta}S_F\eta$$

$$+\tfrac{1}{2}(\bar{\eta}S_F\psi_2+\bar{\psi}_2S_F\eta+\chi)M^{-1}\Delta_F(\bar{\eta}S_F\psi_2+\bar{\psi}_2S_F\eta+\chi)\Big\}+F(\psi_2,\bar{\psi}_2)\Big] \quad (13)$$

where $F(\psi_2, \bar{\psi}_2)$ is the finite part of the trace expression in (10). We have thus generated the kinetic energy for ψ_2.

Alternatively, let us now consider the Lagrangian with a Yukawa interaction:

$$L_3=\bar{\psi}_1(i\gamma.\partial-m_1)\psi_1+\tfrac{1}{2}(\partial\mu\phi)^2-\tfrac{1}{2}\mu^2\phi^2+\bar{\psi}_2(i\gamma.\partial-m_2)\psi_2$$

$$+g(\bar{\psi}_2\psi_1+\bar{\psi}_1\psi_2)\phi. \quad (14)$$

With this Lagrangian, the same one-loop integrations as before yield

$$W_3=\exp[i\int d^4x\{(1+g^2I_2)\overline{\psi}_2 i\gamma.\partial\psi_2-(m_2-g^2m_1I_1)\overline{\psi}_2\psi_2-\overline{\eta}S_F\eta$$
$$+\tfrac{1}{2}(g\overline{\eta}S_F\psi_2+g\overline{\psi}_2S_F\eta+\chi)M^{-1}\triangle_F(g\overline{\eta}S_F\psi_2+g\overline{\psi}_2S_F\eta+\chi)\}$$
$$+F(g\psi_2,g\overline{\psi}_2)] \tag{15}$$

where

$$M(x,y)=\delta^4(x-y)+g^2\int d^4z\,\triangle_F(x-z)\{\overline{\psi}_2(z)S_F(z-y)\psi_2(y)+(z\leftrightarrow y)\}.$$

Note that the finite functional F has precisely the same form as before.

One can see that both the generating functionals in (13) and (15) are the same apart from certain infinite coefficients. But these infinite coefficients in the case of L_3 are absorbed by the usual renormalization or rescaling conditions:

$$(1+g^2I_2)^{1/2}\psi_2=\psi_2^R$$
$$(m_2-g^2m_1I_1)(1+g^2I_2)^{-1}=m_2^R$$
$$g(1+g^2I_2)^{-1/2}=g_R. \tag{16}$$

And the same thing can be done for L_2 also:

$$I_2^{1/2}\psi_2=\psi_2^R$$
$$\left(\frac{1}{G}-m_1I_1\right)I_2^{-1}=m_2^R \tag{17}$$
$$I_1^{-1/2}=g_R.$$

Once the rescaling conditions (16) and (17) are implemented, W_2 and W_3 become identical. Thus, L_2 is equivalent to L_3 and hence L_1 is equivalent to L_3; in other words, we have converted the nonrenormalizable pair interaction into an equivalent renormalizable Yukawa interaction.

For the sake of completeness, we should include the $\lambda\phi^4$ term in L_3, for this is needed for renormalizability. Correspondingly, let us introduce the $\lambda\phi^4$ term in L_1 and hence in L_2. It is however not necessary to include this $\lambda\phi^4$ in the ϕ integration; instead we may use the identity:

$$\int \mathscr{D}\phi \exp i\int d^4x\,\{L_2-\lambda\phi^4+\chi\phi\}$$
$$=\exp\left\{-i\lambda\int d^4x\left(-i\frac{\delta}{\delta\chi(x)}\right)^4\right\}\int D\phi\exp i\int d^4x\,\{L_2+\chi\phi\}. \tag{18}$$

It is clear that this modification does not affect the proof of equivalence.

In closing, we may draw attention to the following point in connection with eqs (17). The Yukawa coupling constant g_R is essentially determined by the ultraviolet cut-off occurring in I_2; in particular it is independent of the strength of the original pair coupling constant G.

3. Further equivalence relationships

We shall now briefly consider a few more nonrenormalizable interactions and indicate corresponding renormalizable theories.

3.1. *Four-fermi cum boson-pair interaction*

Consider the Lagrangian containing both the four-fermi and boson pair couplings

$$L_1=\bar{\psi}\,(i\gamma.\partial-m)\,\psi+\tfrac{1}{2}\,(\partial_\mu\phi)^2-\tfrac{1}{2}\,\mu^2\phi^2+G_1\,(\bar{\psi}\psi)^2-G_2\bar{\psi}\psi\phi^2. \tag{19}$$

This is equivalent to

$$L_2=\bar{\psi}\,(i\gamma.\partial-m)\,\psi+\tfrac{1}{2}\,(\partial_\mu\phi)^2-\tfrac{1}{2}\,\mu^2\phi^2-\frac{1}{4G_1}\,\sigma^2+\bar{\psi}\psi\sigma$$

$$+\frac{G_2}{2G_1}\,\phi^2\sigma-\frac{G_2^{\,2}}{4G_1}\phi^4 \tag{20}$$

in view of the following Euler-Lagrange eq. for the auxiliary field σ:

$$\sigma=2\,G_1\bar{\psi}\psi+G_2\phi^2. \tag{21}$$

The couplings in L_2 are all renormalizable. Again, the kinetic and potential terms $(\partial_\mu\sigma)^2$ and σ^4 can be generated by radiative corrections and thus, the Lagrangian given by (19) is equivalent to a renormalizable theory. Equation (21) suggests that σ is partly a fermion-antifermion composite and partly a di-boson composite. For $G_2=0$, this theory reduces to the four-fermi theory considered by Kikkawa and others.

3.2. ϕ^6 *Coupling*

The Lagrangian of a bose field ϕ with a ϕ^6 self-interaction is

$$L_1=\tfrac{1}{2}\,(\partial_\mu\phi)^2-\tfrac{1}{2}\,\mu^2\phi^2-\alpha\phi^6 \tag{22}$$

where α is a constant. Again this Lagrangian can be replaced by

$$L_2=\tfrac{1}{2}\,(\partial_\mu\phi)^2-\tfrac{1}{2}\,\mu^2\phi^2+\frac{1}{4\alpha}\,\sigma^2-\sigma\phi^3 \tag{23}$$

where σ is an auxiliary field with the following variational equation:

$$\sigma = 2\, a\phi^3. \tag{24}$$

The Lagrangian L_2 contains only renormalizable couplings in contrast to L_1. The kinetic term $(\partial_\mu\sigma)^2$ as well as the renormalizable couplings such as $\sigma^2\phi^2$, $\sigma^3\phi$ and σ^4 can be generated by radiative corrections involving internal ϕ lines. Some of the closed loop diagrams contributing to these terms are shown in figure 2. The explicit calculation of all these radiatively generated terms is quite involved in this case and will be reported in a subsequent publication.

Finally, we may also note that the renormalizable ϕ^4 theory given by

$$L_1 = \tfrac{1}{2}\,(\partial_\mu\phi)^2 - \tfrac{1}{2}\,\mu^2\phi^2 - \lambda\phi^4 \tag{25}$$

is equivalent to

$$L_2 = \tfrac{1}{2}\,(\partial_\mu\phi)^2 - \tfrac{1}{2}\,\mu^2\phi^2 + \frac{1}{4\lambda}\,\sigma^2 - \sigma\phi^2; \quad \sigma = 2\lambda\phi^2 \tag{26}$$

which contains only the super renormalizable coupling $\sigma\phi^2$. However, it is not possible to generate the kinetic term $(\partial_\mu\sigma)^2$ the same way as before, for, the one-loop diagram shown in figure 3 gives a finite contribution to $(\partial_\mu\sigma)^2$.

4. Discussion

Kikkawa and Eguchi showed the equivalence of nonlinear spinor models with corresponding renormalizable theories. By following the same approach we have brought the nonrenormalizable pair interaction as well as the ϕ^6 coupling into the fold of renormalizable theories.

As was already mentioned in the introduction, the theory of superconductivity exerted a great influence on the formulation of these equivalence relationships. We now wish to draw attention to the remarkable fact that this outcome of the superconductivity approach has in fact a close connection to the results conjectured many years ago on the nature of composite particles in quantum field theory, namely, the vanishing of the wavefunction renormalization constant Z of the composite particle (see for instance Lurie 1968). In particular, the equivalence relationship proved by Kikkawa and Eguchi is connected to the discussion in Lurie's book; the equivalence between the pair and Yukawa interactions shown in the present paper is related to a similar equivalence among static solvable models discussed by Vaughn *et al* (1960).

However, in earlier works, the emphasis was on the $Z=0$ condition itself whereas we now focus our attention on the generation of equivalent renormalizable Lagrangians. Further the present approach appears to be superior to the earlier work on composite particles in a number of ways: 1. The earlier work was either based on approximation schemes such as chain approximation or on solvable models. In contrast, in the present approach, we deal with fully relativistic quantum field theories and we do not restrict ourselves to any approximation. 2. In the earlier work, not all the couplings of the composite particle (as for instance, the ϕ^4 coupling of the composite boson occurring in the nonlinear spinor theory) could be calculated whereas in the present approach, the values of these coupling constants are also determined. 3. The procedure adopted in the earlier approach was to calculate the S matrix elements in two models and then show their equality under certain conditions. Thus, one could actually show equivalence only in the sectors in which the S matrix elements were calculable. In the present approach, attention is focussed on the complete generating function and the entire renormalizable Lagrangian including the kinetic energy for the composite particle is obtained. Thus, equivalence is established for the complete theory. In short one may say the present procedure completes the earlier programme and puts it on a firmer footing.

As a final remark, we may mention that the connecting equations (17) are meaningful only if the ultraviolet cut-off is regarded as finite, which presumably implies a nonlocal field theory. Thus in establishing such equivalence relationships one may be going beyond the realm of local field theory. However, even in conventional renormalizable field theory, it is questionable whether the cut-off can really be taken to be infinite without affecting the consistency of the theory. In any case, even though we may be going beyond the framework of local field theory, the Kikkawa-Eguchi procedure provides at least a consistent method of taming the nonrenormalizable interactions and bringing them into the class of renormalizable field theories.

Acknowledgement

One of the authors (VS) would like to thank the University Grants Commission for financial support.

Appendix

Separation of the divergent part of the closed loop

The $n=1$ term of the exponent in eq. (10) is

$$-\mathrm{Tr}\,\{2\triangle_F \bar{\psi}_2\, S_F\, \psi_2\}$$

$$=-2 \int\int d^4x\, d^4y\, \triangle_F(y-x)\, \bar{\psi}_2(x)\, S_F(x-y)\, \psi_2(y)$$

$$=2i \int d^4x\, \bar{\psi}_2(x) \int d^4y \left\{ \frac{1}{(2\pi)^4} \int d^4p\, e^{-ip.(x-y)}\, \Sigma(p) \right\} \psi_2(y) \qquad \text{(A.1)}$$

where $\Sigma(p)$ is the lowest-order fermion-self-energy operator given by

$$\Sigma(p) = \frac{-i}{(2\pi)^4} \int d^4k \frac{1}{\{\gamma.(p-k)-m_1\}\,(k^2-\mu^2)}. \qquad \text{(A.2)}$$

Introduction of the Feynman parameter z and the usual manipulations lead to

$$\Sigma\,(p)=\frac{-i}{(2\pi)^4}\int_0^1 dz\int d^4k\,\frac{\gamma.p\,(1-z)+m_1}{[k^2+p^2\,z\,(1-z)-(m_1{}^2-\mu^2)\,z-\mu^2]^2} \tag{A.3}$$

$$=\tfrac{1}{2}\,m_1\,I_1+\tfrac{1}{2}\,\gamma.p\,I_2+\text{convergent parts} \tag{A.4}$$

where I_1 and I_2 are the divergent integrals defined in eq. (12). Substitution of (A.4) into (A.1) gives (11).

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