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Initial-value problems and singularities in general relativity

BIDYUT KUMAR DATTA
S N Bose Institute of Physical Sciences, Calcutta University, Calcutta 700 009

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Abstract. Linearized solution of Datta in a non-symmetric and isentropic motion of a perfect fluid is studied by dealing with a Cauchy problem in co-moving coordinates in the framework of general relativity. The problem of singularities is discussed from the standpoint of a local observer both for rotating and non-rotating fluids. It is shown that, whatever the distribution of matter, a singularity which occurred in the past in both the rotating and non-rotating parts of the universe must occur again later after some finite proper time, if the universe is closed. A modification is incorporated in Penrose's theorem by explicitly exhibiting that the universe defined by Penrose can possess a Cauchy hypersurface.

Keywords. Cauchy problem; problem of singularities; Penrose's theorem; general relativity.

1. Introduction

For sometime it was thought (Lifshitz and Khalatnikov 1963) that the general relativistic singularity (see appendix A1) might be a consequence of a spherically symmetric collapse and consequently any deviation from spherical symmetry could halt a gravitational collapse to a point singularity. Oppenheimer and Snyder (1939) have shown that a spherically symmetric gravitational collapse of a cloud of dust, in fact, leads to a point singularity. Penrose (1965), Geroch (1966) and Hawking (1966) have shown that the space-time singularities are general features of Einstein's gravitation theory and that any isotropic models lead to the prediction of at least one moment in the history of the universe with an infinite density of matter. Ellis (1971) has pointed out that if one postulates a 'bounce' in an oscillating universe one must abandon either general relativity and similar theories such as Brans-Dicke theory of gravity (Brans and Dicke 1961) or the 'energy condition' (see appendix A2) or the 'causality condition' (see appendix A3). Nevertheless, Hawking (1967) has demonstrated that a violation of causality cannot prevent the occurrence of a singularity.

By introducing the idea of closed trapped surfaces, Penrose (1965) has made one of the epoch-making contributions in the history of general relativity. In physical language, Penrose's most remarkable theorem may be stated thus: If a star undergoes an asymmetric collapse with rotation, radiation, magnetic fields, shock waves, etc. in a universe having the following properties: (1) the universe is open, (2) the 'energy condition' and (3) the 'causality condition' are not violated and (4) general relativity is taken as the correct theory of gravitation, then the evolution of a trapped surface must lead to either or both of the following facts. (a) A space-time singularity occurs subsequently and (b) the universe so defined cannot possess an initial Cauchy hypersurface. One may draw conclusion with greater confidence that singularities necessarily follow the evolution of a trapped surface in our universe, if the following two modifications, as rightly suggested by Thorne (1966), in respect of the above theorem are done: (1) the elimination of condition (i) and (ii) the elimination of the possibility (b) as an alternative to (a). Penrose believes that it may, indeed, be possible to incorporate the first modification by a reformulation of the proof of the theorem; but he is, however, not optimistic about the second modification, though he has not explicitly shown it anywhere. One is also not aware if Penrose has done it elsewhere.

In the present paper it is proposed to examine whether the second modification, suggested by Thorne, can at all be incorporated in Penrose's theorem which establishes the intimate connection between closed trapped surfaces and singularities. We study the problem of singularities for a non-symmetric (appendix A5) and isentropic motion of a perfect fluid under the assumption of adiabatic thermodynamic processes from the standpoint of a local observer. For this purpose we deal with Cauchy problem (appendix A6) in co-moving coordinates in the framework of general relativity, impose a restriction on the equation of state (appendix A7) which is interesting from the point of view of singularity and define a singularity as a state with an infinite proper rest mass density. The possibility that the gravitational collapse might be halted by this restriction on the equation of state may be ruled out. The idea that a gravitational collapse may possibly be halted by a suitable equation of state has been pointed out and discussed by Gratton and Szamosi (1964), Chiu (1965), Marx and Németh (1965) and Gratton (1966). By investigating the problem of singularities, we have shown that, whatever the distribution of matter, there occurred a singularity in the past in both the rotating and non-rotating parts of the universe and it must occur again in future, if the universe is closed. This result was obtained earlier by Pachner (1970), Bera and Datta (1975) and Datta (1975-76) and is in complete agreement with the conclusions of Penrose (1965), Geroch (1966), Hawking and Ellis (1968) and Hawking and Penrose (1970).

2. The solution of a non-symmetric problem by dealing with Cauchy problem

We write out in brief the solution of a non-symmetric problem obtained by Datta by assigning Cauchy data g_{ik}, $g_{ik,4}$ on the hypersurface $x^4=0$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g_{\mu\nu} \equiv g_{\mu\nu}(x^1, x^2, x^3, x^4), \tag{1}$$

where

$$u_\nu = g_{\nu 4} = A_\nu (x^j) - x^4 \eta_{,\nu} \tag{2}$$

with

$$A_4 = -1, \; \eta(x^1, x^2, x^3) = \ln \mu, \;\; \mu = 1+\epsilon+p/\rho, \tag{3}$$

p = proper pressure, ρ = proper rest mass density, ϵ = proper internal energy per unit mass. $A_i\,(x^j)$ are three functions of space coordinates and may be determined by the initial condition. The functions A_ν which represent the covariant components of a four-vector are known as Coriolis potentials (Tauber and Weinberg 1961). By using coordinate transformations of the type:

$$\bar{x}^1 = \bar{x}^1\,(x^1, x^2, x^3), \qquad \bar{x}^2 = \bar{x}^2\,(x^2, x^3),$$

$$\bar{x}^3 = \bar{x}^3\,(x^3), \qquad \bar{x}^4 = x^4 + c, \tag{4}$$

one obtains [Pachner (1970), Bera and Datta 1974, 1975, Datta 1975-76]:

$$A_1 = A_2 = 0, \qquad A_3 = A_3(x^2, x^3), \qquad A_4 = -1, \tag{5}$$

if the vector field u_i and the proper rest mass density ρ vary sufficiently smoothly on $x^4 = 0$. We introduce a system of co-moving reference frame defined by

$$u^\nu = \delta^\nu{}_4, \tag{6}$$

where u^ν is the timelike four velocity.

The Greek indices run from 1 to 4, Latin indices from 1 to 3. The signature of the metric is +2. Comma followed by an index denotes ordinary partial differentiation, while semicolon followed by an index denotes covariant derivative.

In the reference frame defined by (6) the conservation law of baryon number is reduced to

$$\rho = (-g)^{-1/2} f(x^i), \tag{7}$$

where $f(x^i)$ is a function of space coordinates and may be determined by the initial distribution of matter. The energy-momentum tensor for a perfect fluid is defined by

$$T_\nu{}^\nu = \rho\mu u_\nu u^\nu + p\delta_\nu{}^\nu. \tag{8}$$

We consider the case where all thermodynamical processes are adiabatic and the following restriction in the equation of state is imposed

$$\epsilon + p/\rho = k(x^i), \tag{9}$$

where $k\,(x^i)$ is a function of space coordinates.

In terms of the Cauchy data g_{ik}, $g_{ik,4}$ on $x^4 = 0$, Datta's solution of the field equations

$$R_{ij} = -8\pi\,(T_{ij} - \tfrac{1}{2}Tg_{ij}) \tag{10}$$

is expressed by the power series:

$$g_{ik} = (g_{ik})_0 + x^4(g_{ik,4})_0 + \tfrac{1}{2}(x^4)^2(g_{ik,44})_0 + \cdots \tag{11}$$

The Cauchy data g_{ik}, $g_{ik,4}$ are chosen on the hypersurface $x^4 = 0$ such that

$$(g_{ik})_0 = (1+2\lambda)\delta_{ik}, \qquad |\lambda|^2 \ll 1, \tag{12}$$

$$(g_{ik,4})_0 = 2(\varphi_{i,k} + \varphi_{k,i}) - \delta_{ik}\,\varphi_{3,3}, \tag{13}$$

where

$$\lambda = \int [\rho(1+\epsilon)/r] dv, \tag{14}$$

$$\varphi_3 = \int (\rho\mu A_3/r) dv, \qquad \varphi_1 = \varphi_2 = 0. \tag{15}$$

$dv = dx^1 dx^2 dx^3$ and r is the spatial distance (in Euclidean metric) of dv from the point at which φ_i and λ are computed and the integration is over the hypersurface $x^4=0$.

3. The problem of singularities

In order to know the past and future of the proper rest mass density of a perfect fluid whose behaviour is governed by Einstein field equations, one must express the relation (7) in a differential form. One can then deal with the problem of singularities from the standpoint of a local observer.

Next, we introduce the quantity Φ^2 defined by the expression (Pachner 1970, Bera and Datta 1975):

$$\Phi^2 = \tfrac{1}{4}\,\gamma^{ia}\,\gamma^{kb}\,\gamma_{ik,4}\,\gamma_{ab,4} - \tfrac{1}{12}\,(\gamma^{ik}\,\gamma_{ik,4})^2, \tag{16}$$

where γ^{ik}, γ_{ik} are components of a three-dimensional tensor and are connected to the metric tensor components $g_{\mu\nu}$ by the relations (Landau and Lifshitz 1962)

$$\gamma_{ik} = g_{ik} - g_{i4}\,g_{k4}/g_{44}, \qquad \gamma^{ik} = g^{ik}. \tag{17}$$

The geometrical properties of space are defined by the positive definite metric

$$d\sigma^2 = \gamma_{ik}\, dx^i dx^k. \tag{18}$$

We take note of the fact that the quantity Φ^2 defined by (16) and expressing the influence of the shear is invariant with respect to the coordinate transformation (4). Thus without any ambiguity one may choose at any given moment a system of coordinates in which eq. (5) holds and whose spatial axes are at a particular point orthogonal to each other. The invariant Φ^2 in this system vanishes if the motion is isotropic at the point and is given in terms of the metric tensor $g_{\mu\nu}$ by the expression (Pachner 1970, Bera and Datta 1975 and Datta 1975-76):

$$\Phi^2 = \tfrac{1}{4} g^{ia}\, g^{kb}\, g_{ik,4}\, g_{ab,4} - \tfrac{1}{12}\,(g^{ik}\, g_{ik,4})^2 + \tfrac{1}{2}\, g^{i3}\, g^{k4}\, g_{ik,4}\, g_{34,4}$$
$$- \tfrac{1}{6} g^{ik}\, g^{34}\, g_{ik,4}\, g_{34,4} + \tfrac{1}{6}\,(g^{34}\, g_{34,4})^2. \tag{19}$$

Differentiating the relation (7) with respect to time, replacing $g^{\alpha\beta}\, g_{\alpha\beta,44}$ by the corresponding expression from the field equations, utilising the relation (19) and using the formula

$$ds = dt\,(-g_{44})^{1/2} = dt, \tag{20}$$

one obtains following Pachner (1970)

$$\frac{1}{\rho}\frac{\partial^2\rho}{\partial s^2} = 4\pi\rho\,(1+\epsilon+3p/\rho)/\mu^2+(2/\mu^4\rho^2)\,(dp/d\rho)^2\,(g^{ik}\,c_i c_k)+\tfrac{4}{3}\,(\rho_{,4}/\rho)^2$$

$$+\Phi^2+(-g)^{-1/2}\,[g^{\alpha k}c_k\,(dp/d\rho)\,(-g)^{1/2}/\rho\mu^3]_{,\alpha}-2\,|\,\Omega\,|^2/\mu^2, \tag{21}$$

where

$$c_i = \rho_{,i} - A_i\,\rho_{,4}\,, \tag{22}$$

$$|\,\Omega\,|^2 = g_{\mu\nu}\,\Omega^\mu\,\Omega^\nu. \tag{23}$$

The contravariant angular velocity is defined by the formula (Gödel 1948, Taub 1956):

$$\Omega^\nu = \tfrac{1}{2}(-g)^{-1/2}\,e^{\nu\alpha\beta\gamma}\,u_\alpha\,u_{\beta,\gamma}. \tag{24}$$

The differential relation (21), obtained here, will enable one to examine the problem of singularities.

For an incoherent fluid eq. (21) reduces to

$$\frac{1}{\rho}\frac{\partial^2\rho}{\partial s^2} = 4\pi\rho + \tfrac{4}{3}\,(\rho_{,4}/\rho)^2 + \Phi^2 - 2\,|\,\Omega\,|^2, \tag{25}$$

where Φ^2 is given by

$$\Phi^2 = \tfrac{1}{4}g^{ia}\,g^{kb}\,g_{ik,4}\,g_{ab,4} - \tfrac{1}{12}\,(g^{ik}\,g_{ik,4})^2 \tag{26}$$

and one thus obtains Raychaudhuri's formula (Raychaudhuri 1955) for incoherent matter.

In the case of irrotational motion,where all A_i's may be reduced to zero by a suitable coordinate transformation, eq. (21) assumes the form:

$$\frac{1}{\rho}\frac{\partial^2\rho}{\partial s^2} = 4\pi\rho\,(1+\epsilon+3p/\rho)/\mu^2+(2/\mu^4\rho^2)\,(dp/d\rho)^2\,(g^{ik}\,\rho_{,i}\,\rho_{,k})$$

$$+\tfrac{4}{3}\,(\rho_{,4}/\rho)^2 + \Phi^2+(-g)^{-1/2}\,[g^{\alpha k}\,\rho_{,k}\,(dp/d\rho)\,(-g)^{1/2}/\rho\mu^3]_{,\alpha}. \tag{27}$$

One can thus deduce the condition of equilibrium for non-rotating matter

$$[4\pi\rho(1+\epsilon+3p/\rho)/\mu^2 + (2/\mu^4\rho^2)(dp/d\rho)^2 (g^{ik}\rho_{,i}\rho_{,k})](-g)^{1/2}$$

$$= -[g^{ik}\rho_{,k}(dp/d\rho)(-g)^{1/2}/\rho\mu^3]_{,i}\,, \tag{28}$$

which represents a relativistic generalization of the classical equation

$$4\pi G\rho = -\text{div}\,[(1/\rho)\ \text{grad}\ p], \tag{29}$$

where G is the Newtonian constant of gravitation.

It is evident from eq. (27) that in the absence of rotation, the gravitational attraction created by the rest mass, the internal energy, the pressure and the square of the pressure gradient cannot be counterbalanced by the elastic forces on the right hand side of (28) if the mass of the fluid exceeds a certain critical limit. Consequently, the fluid starts contracting. As contraction proceeds, the increasing pressure in the first term of the second member of (27) and the kinetic energy in the third term accelerate the collapse to a singularity which is reached in a finite interval of proper time (Oppenheimer and Volkoff 1939, Landau 1932). Moreover, any anisotropy in the isentropic motion of a non-rotating ideal fluid further accelerates the collapse (Pachner 1970, Bera and Datta 1975 and Datta 1975-76).

The applicability of the above argument in cosmology is ensured by our introduction of co-moving reference frame. One may thus state that, whatever the distribution of matter, there occurred a singularity in the past in the non-rotating parts of the universe and it must occur again in future, if the universe is closed.

As many stars and galaxies exhibit visible rotation and even the possibility of rotation of the universe in the large is admitted (Wolfe 1970), the problem of gravitational collapse in rotational motion in the formalism of general relativity has drawn considerable attraction. We examine the problem of singularities when the motion is rotational.

In view of (2) and (24), the formula (23) reduces to

$$|\,\Omega\,|^2 = -[g_{11}(A_3\eta_{,2}+A_{3,2})^2+g_{22}(A_3\eta_{,1})^2+(x^4 A_{3,2}\eta_{,1})^2 \tag{30}$$

$$-2g_{12}A_3\eta_{,1}(A_3\eta_{,2}+A_{3,2})]/(4g).$$

The elastic forces for rotating matter

$$[g^{ik}c_k(dp/d\rho)(-g)^{1/2}/\rho\mu^3]_{,i}\,, \tag{31}$$

occuring in the fifth term of the second member of eq. (21) oppose contraction.

The first term of the second member of (21) representing the gravitational attraction created by the rest mass, the internal energy and the pressure and the second term representing the square of the pressure-gradient support contraction. Once the contraction begins when the mass of the fluid exceeds a certain critical limit, the increasing pressure and the kinetic energy in the third term lend support to contraction. Moreover, any anisotropy in the motion of a rotating fluid in the fourth term and given by (19) accelerates the contraction.

Now it follows from (21) that the effect of attraction is of the order ρ, while that of rotation which opposes contraction is given by (30). Thus, in spite of the fact that the rotation together with the elastic forces opposes contraction, it is evident that the square of the pressure-gradient, the kinetic energy and the anisotropy in the motion of rotating fluid together with the gravitational attraction might be able to make

$$\frac{\partial^2 \rho}{\partial s^2} > 0, \tag{32}$$

during the contraction of space. Hence the collapse of a rotating fluid is inevitable if the universe possesses an initial Cauchy hypersurface.

Let us examine the problem of singularities more closely for rotational motion. During contraction of a rotating fluid, when the proper rest mass density becomes sufficiently high, one must use the relativistic equation of state:

$$p=\rho(1+\epsilon)/3. \tag{33}$$

One may get in this case

$$1+\epsilon=(\rho/\rho_0)^{1/3},$$

$$\rho/p=\tfrac{1}{3}(\rho/\rho_0)^{1/3}, \tag{34}$$

where ρ_0 denotes the proper rest mass density with a zero internal energy. It is easily seen from (34) that the internal energy density becomes negative when $\rho < \rho_0$. Hence the relativistic equation of state (33) may be applied for $\rho \gg \rho_0$ and ρ_0 may be treated as a constant.

In the case under consideration we can derive as before the second derivative of the proper rest mass density with respect to the proper time and obtain in view of (7), (30) and (34)

$$\frac{1}{\rho}\frac{\partial^2 \rho}{\partial s^2} = \frac{9}{2}\pi\, \rho_0{}^{1/3}\rho^{2/3} + (2/\mu^4\rho^2)\left(\frac{dp}{d\rho}\right)^2 (g^{ik}c_i c_k) + \frac{4}{3}(\rho_{,4}/\rho)^2 + \Phi^2$$

$$+ (-g)^{-1/2}\left[g^{\alpha k}c_k\left(\frac{dp}{d\rho}\right)(-g)^{1/2}/\rho\mu^3\right]_{,\alpha} + \frac{9}{32g}\rho_0{}^{2/3}\rho^{-2/3}\,Q, \tag{35}$$

where

$$Q = g_{11}(A_3\eta_{,2} + A_{3,2})^2 + g_{22}(A_3\eta_{,1})^2 + (x^4 A_{3,2}\eta_{,1})^2$$

$$-2g_{12}(A_3\eta_{,1}(A_3\eta_{,2} + A_{3,2}). \tag{36}$$

From eqs (35) and (36) one notices that high pressure creates additional attraction of the other $\rho^{2/3}$, while the effect of rotation which plays a predominant part in opposing contraction decreases as ρ increases and becomes vanishingly small when the proper rest mass density is infinite. Hence, it is evident from (35) that the collapse of a rotating fluid is inevitable when the relativistic equation of state is considered.

One may finally state that, whatever the distribution of matter might be, there occurred a singularity in the past in both the rotating and non-rotating parts of the universe and it must occur again in future, if the universe is closed. Thus our result agrees with the conclusions arrived at by Penrose (1965), Geroch (1966), Hawking (1966), Hawking and Ellis (1968), Hawking and Penrose (1970) and Ellis (1971).

4. Conclusions

Hawking and Ellis (1968), Hawking and Penrose (1970) and Ellis (1971) have elegantly discussed the problem of singularities in great detail and have shown that a singularity must occur in the universe. They have shown, on the basis of reasonable physical assumptions, that the inhomogeneity and anisotropy of the real universe could not prevent the existence of the singularity in the past.

The conclusion arrived at the previous section should be discussed from the point of view of Penrose's remarkable theorem (Penrose 1965) on the existence of a trapped surface. Penrose has pointed out that if a trapped surface evolves during stellar collapse, the universe defined by him cannot possess a Cauchy hypersurface. If the singularities which are predicted in every sufficiently large co-moving volume in the universe are time-like, one can conceive that a contracting phase in the universe could have changed to an expanding phase with most of the matter in the universe passing between isolated singularities. Consequently complete knowledge of the state of the universe on any space-section would be insufficient to determine its complete time development. In other words, one may state that the universe cannot possess an initial Cauchy hypersurface. As we obtain a solution corresponding to a linear approximation by assigning Cauchy data on the hypersurface $x^4=0$ and by assuming that the vector field u_i and the proper rest mass density ρ vary sufficiently smoothly on $x^4=0$, the occurrence of the evolution of a trapped surface in asymmetric regions of space-time might be possible during gravitational collapse. Apparently it turns out that in our case trapped surfaces are necessarily a feature peculiar to non-symmetric universe having an initial Cauchy hypersurface. Thus our result explicitly exhibits that the universe defined by Penrose can possess a Cauchy hypersurface if the vector field u_i and the proper rest mass density ρ vary sufficiently smoothly on the Cauchy hypersurface. Our result also contradicts in no way the conclusions arrived at by Penrose-Geroch-Hawking-Ellis. However, their conclusions are examined in a non-symmetric universe having an initial Cauchy hypersurface by our study and we are in a position to conclude that the modification, as rightly suggested by Thorne (1966)—the elimination of the condition that the universe defined by Penrose (1965) cannot possess an initial Cauchy hypersurface—can, in fact, be incorporated in Penrose's theorem about which Penrose himself has expressed divergent view (Thorne 1966).

Trautman (1972a, b; 1973), Kopczyński (1972, 1973) and Tafel (1973) have also studied the problems on the basis of Einstein-Cartan theory of gravitation and have shown that rotation may prevent the occurrence of a singularity.

Appendix

A1. Singularity

In the mathematical apparatus of the general theory of relativity developed here a local observer who considers himself as being at rest defines a state to be singular when the proper rest mass density of a perfect fluid is infinite.

A2. Energy condition

We assume that for every time-like vector u^{α} at each point

$$R_{\alpha\beta}u^{\alpha}u^{\beta} \geqslant 0.$$

In the case under consideration this is equivalent to the restriction one would put on the matter that its energy density is positive.

A3. Causality condition

In special relativity the assumption of the speed of light as the limiting speed of any particle or wave packet leads to the fact that an event can causally influence only events in or on its future light cone, and can be causally influenced only by events within or on its past light cone. In a general curved space-time the field equations cannot guarantee causal connections between events; an observer in Gödel's universe can travel into, and influence, his own history. From physical point of view we assume that space-time obeys the 'strong causality condition'; 'every point in space-time is contained in a small open neighbourhood such that every time-like (and null) curve that leaves this neighbourhood never re-enters it.'

A4. Closed trapped surface

The fact that the ' radius function ' of metric increased monotonically from a singularity a finite time previously may possibly lead to the existence of a minimum angular diameter observed for sources of the same metric size. This implies that there are two surfaces, called ' closed trapped surfaces ', on which the past null geodesics generating the observer's light cone are re-converging.

A5. Non-symmetric problem

Problem of an isentropic motion of a perfect fluid studied in the framework of general relativity without assuming any symmetry e.g. spherical, axial, etc. at all is termed here a non-symmetric problem.

A6. Cauchy problem

In general relativity the field equations in a matter-free region contain ten second order quasi-linear differential equations in the four space-time variables and thus allow one to formulate the initial-value problem in the following manner. Firstly, we prescribe a three-dimensional hypersurface S oriented in space and choose a coordinate system such that the hypersurface S is represented by $x^4=0$. Physically, S may be interpreted as the space at the given time $x^4=0$. Since the space derivatives of $g_{\mu\nu}$ in S may be determined by differentiation of the given $g_{\mu\nu}$ in S, we simply assign the values of $g_{\mu\nu}$, $g_{\mu\nu}$, 4 on the hypersurface $x^4=0$ (Cauchy data). These Cauchy data together with the field equations form Cauchy initial-value problem which represents the causal development of a physical system from initial data.

A7. Equation of state

In order to get a physical picture we specify equations restricting ρ, p, ϵ and these furnish further properties of the energy-momentum tensor.

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