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Cerenkov radiation and its polarization

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Abstract. General expression for the energy loss in Cerenkov radiation due to a charged particle possessing anomalous magnetic moment is obtained. The expressions include the spin-polarization of the particles. The contribution to the radiation due to anomalous magnetic moment is found to be small as compared to that due to charge. The interference term of charge and anomalous magnetic moment gives better contribution as compared to the term containing only anomalous magnetic moment. Polarization of the radiation as dependent on polarization of beam of particles is studied. The radiation has a dominant nature of linear polarization with small quantum corrections. The spin-flip also gives quantum correction to strong linear polarization and at threshold when phase velocity equals velocity of the particle.

Keywords. Cerenkov radiation; intensity; anomalous magnetic moment; spin-polarization; spin-flip.

1. Introduction

Cerenkov radiation given out by a charged particle moving through a medium with a velocity greater than the phase velocity of light in the medium has been given extensive theoretical as well as experimental study by many workers for quite a long time. After discovery by Cerenkov (1934), Frank and Tamm (1937) gave the classical theory and Ginzburg (1940) and Sokolov (1940) developed the quantum theory of Cerenkov radiation. Bolotovskii (1961) has reviewed the work on this topic. The theoretical studies are mostly classical and the interest lies in obtaining energy loss due to the radiation in transparent isotropic as well as uniaxial dielectric media. Quantum electrodynamical treatment for the radiation is developed by considering Dirac particle with quantised electromagnetic field. Quantum mechanical correction to the Cerenkov angle and also the energy loss have been obtained. Nemtan (1953) has obtained formulae for Cerenkov radiation as a first order transition induced by the interaction of charged particles with electromagnetic field of the radiation and the medium. In the first order transition the interaction of the charged particle with the radiation field is considered and that with the atoms of the medium is neglected since it gives rise to radiative transitions in third or higher orders. Further the interaction induces transitions between eigen states which are chosen to be the momentum states of the incident particle (Nemtan 1953).

The quantum electrodynamical treatment of Cerenkov radiation has been developed further by Sokolov and Loskutov (1957), Loskutov and Kukanov (1958) to consider polarization of Cerenkov radiation due to spin polarization of a beam of electrons. It is shown that the radiation consists of two parts; a polarized part which vanishes at threshold (i.e. when velocity of particles equals the phase velocity of light in the medium) and unpolarized part which does not vanish at the threshold. The unpolarized part of the radiation is accompanied by the spin-flip ($ss'=-1$) of the particle.

Kukanov (1961) has considered the motion of an uncharged Dirac particle with a normal magnetic moment and has obtained expressions for the energy loss by a polarized beam of particles and has also considered polarization of radiation. When a motion of the charged particle with a magnetic moment is considered by treating the particle by Dirac equation with the addition of Pauli term for anomalous magnetic moment, it is expected that the effect due to small value of anomalous magnetic moment will obviously be small. The major contribution to the radiation is due to the charge and also the normal magnetic moment of the particle. The product term of normal and anomalous magnetic moment will contribute more than that of the Pauli term.

We have considered the emission of Cerenkov radiation due to a charged particle possessing total magnetic moment (i.e. normal and anomalous) and the general expression for the energy loss has been obtained. The general expression includes all the descriptions of radiation emitted by polarized beam of particle. The polarization of radiation has been studied in some interesting cases.

2. General expression

For the process of Cerenkov radiation due to passage of charged particles with an anomalous magnetic moment through a medium with a velocity greater than the phase velocity of light in the medium, we consider interaction of charged particle with an electromagnetic field of the radiation and the medium with emission of photon.

The conservation laws for the process are:

$$E = E' + \hbar w$$

$$\mathbf{p} = \mathbf{p}' + \mathbf{x} \tag{1}$$

where $|\mathbf{x}| = \hbar w/c'$; $c'=c/n$ and n is the refractive index of the medium, $\mathbf{p}$ and $\mathbf{p}'$, are momentum vectors and $E=(p^2c^2+m^2c^4)^{\frac{1}{2}}$ and $E'=(p'^2c^2+m^2c^4)^{\frac{1}{2}}$ are energies of the particle before and after the emission, $\mathbf{x}=\hbar\mathbf{k}$ is the momentum and $\hbar w$ is the energy of the emitted photon.

The interaction Hamiltonian for the process is the current term with the multiplication of electromagnetic vector potential and can be written as (Muirhead 1968)

$$\langle i|H'|f\rangle = i\bar{\psi}(p')[F_1\gamma_\mu + iF_2\sigma_{\mu\nu}x_\nu]\psi(p)A_\mu \tag{2}$$

where x_ν is the photon four-vector$=[\mathbf{x}, \hbar w]$, $\sigma_{\mu\nu}=i[\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu]/2$, $F_1=e$, the charge on the particle; $F_2=\mu_a$, the anomalous magnetic moment in the units of $e\hbar/2mc$ (m=mass of the particle). Here the total magnetic moment of the particle is

$\mu=1+\mu_a$ which in our notation is $\mu=[F_1/e+F_2]$. For the electron $\mu=(1+\alpha/2\pi)\mu_B$ where α is the fine structure constant and μ_B is the Bohr magneton. For the proton $\mu=2\cdot79\ \mu_{\mathcal{N}}$ where $\mu_{\mathcal{N}}=e\hbar/2mc$ nuclear magneton. Here m =mass of nucleon and $F_1/e=1\ \mu_{\mathcal{N}}$; $F_2=1\cdot79\ \mu_{\mathcal{N}}$. For neutron

$$\mu=-1\cdot91\ \mu_{\mathcal{N}}\quad \text{i.e. } F_1/e=0;\ F_2=-1\cdot91\ \mu_{\mathcal{N}}.$$

The second term in eq. (2) is of the order of $F_2\,x$ and $F_2\approx 10^{-3}F_1$ for electron but it is comparable with F_1 for proton. $\bar{\psi}(p')$ and $\psi(p)$ are Dirac wave functions, $\gamma_\mu\equiv(\boldsymbol{\gamma},\gamma_4)$ are Dirac matrices. A_μ is the electromagnetic four-vector potential. Here

$$\bar{\psi}(p')=\sqrt{\frac{mc^2}{E'V}}\sum_{p'}\bar{u}\,(p')\exp i\left(\frac{\mathbf{p}'\cdot\mathbf{r}}{\hbar}-\frac{E't}{\hbar}\right) \tag{3a}$$

$$\psi(p)=\sqrt{\frac{mc^2}{EV}}\sum_{p}u\,(p)\ \exp\ i\left(\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}-\frac{Et}{\hbar}\right)$$

$$A_\mu=\sqrt{\frac{\hbar C'^2}{2wV}}\sum_{k}e_\mu\exp i\left(\frac{\mathbf{k}\cdot\mathbf{r}}{\hbar}-wt\right) \tag{3b}$$

and $p\equiv\left(\mathbf{p},\dfrac{iE}{c}\right)$ and $p'=\left(\mathbf{p}',\dfrac{i\,E'}{c'}\right)$ are the particle four-mementa before and after emission of photon. $e_\mu=(\mathbf{e},e_4)$ is a unit polarization four-vector of photon. Using the relation (Muirhead 1968).

$$e_\mu\,[F_1\,\gamma_\mu+iF_2\,\sigma_{\mu\nu}\,\chi_\nu]=[F_1+2mc\,F_2]\,e_\mu\,\gamma_\mu+iF_2\,e_\mu\,r_\mu \tag{4}$$

where

$$r_\mu=p_\mu+p'_\mu.$$

Substituting the Dirac wave function ψ and electromagnetic potential A_μ^- in eq. (2) and multiplying by complex conjugate we get

$$|\langle f|\,H'\,|\,i\,\rangle|^2=\frac{m^2c^4\hbar c'^2}{2EE'wV}\,\mathrm{T_r}\left[O\Lambda_p^{(+)}\tilde{O}\Lambda_{p'}^{(+)}\right]\delta_{pp\ell} \tag{5}$$

where spin projection operators are:

$$\Lambda_p^{(+)}=\left(\frac{mc-i\gamma.p}{2mc}\right)\left(\frac{1+i\gamma_5\gamma\cdot\omega}{2}\right) \tag{6a}$$

$$\Lambda_{p'}^{(+)}=\left(\frac{mc-i\gamma.p'}{2mc}\right)\left(\frac{1+i\,\gamma_5\cdot\gamma\cdot\omega'}{2}\right) \tag{6b}$$

and

$$O=[F_1+2mcF_2]e_\mu\gamma_\mu+iF_2e_\mu r_\mu$$

$$\tilde{O}=\gamma_4O^+\gamma_4$$

ω and ω' are spin four-vectors of the particle before and after radiation respectively. In the rest system of the particle ω is a unit vector with vanishing fourth component (Sakurai 1967).

$$[\omega]_{\text{rest frame}} = [\hat{s}, 0], \; [\omega']_{\text{rest frame}} = [\hat{s}', 0)$$

where $\hat{s}$ and $\hat{s}'$ are unit spin vectors of the charged particle before and after emission respectively.

For any frame of reference, general expression for ω is given by

$$[\omega]_{\text{general}} \equiv (\vec{\omega}, \omega_4) \equiv \left[\hat{s} + \frac{\mathbf{p}\,(\mathbf{p}\cdot\hat{s})}{m(E+mc^2)}, \; \frac{i\,|\,p\,|\,s}{mc} \right]. \tag{7}$$

Carring out the covariant trace calculations we get the general expression for $|\langle f|\,H'\,|\,i\,\rangle|^2$ as

$$\begin{aligned} |\langle f|\,H'\,|\,i\,\rangle|^2 = \frac{m^2c^4\hbar c'^2}{2EE'Vw} \Big\{ & \frac{F_1^2}{4m^2c^2} \{2(ep)[(e^*p)(1+(\omega\omega'))+(e^*\omega)(p\omega')] \\ & +(px)[2(e\omega)(e^*\omega')-(ee^*)(1+(\omega\omega'))] \\ & +(x\omega)[(ee^*)(p\omega')-2(ep)(e^*\omega')]\} \\ & + \frac{F_1F_2}{mc} \{(ee^*)(px)(1+(\omega\omega'))+2(px)(e\omega)(e^*\omega')-(ee^*)(p\omega')(x\omega) \\ & +(e^*p)(x\omega)(e\omega')-(ep)(e^*\omega)(p\omega')\} \\ & - \frac{F_2^2}{m^2c^2} \{(ep)(e^*p)(px)(1+(\omega\omega'))-2m^2c^2(px)(e\omega)(e^*\omega') \\ & -(p\omega')(x\omega)(m^2c^2+(ep)(e^*p))+m^2c^2(px)(1+(\omega\omega'))\} \Big\}. \end{aligned} \tag{8}$$

By using energy-momentum conservation the angle θ of the Cerenkov cone i.e. angle between $\mathbf{p}$ and $\mathbf{x}$ is given by (Kunkanov 1961).

$$\cos\theta = \frac{1}{\beta n} + \frac{n\hbar W}{2pc}\left(1 - \frac{1}{n^2}\right), \tag{9}$$

where the second term represents the quantum correction to the usual formula. In eq. (9) and afterwards we use $p \equiv |\,\mathbf{p}\,|$ and $x \equiv |\,\mathbf{x}\,|$ for simplicity.
By the perturbation theory, transition probability per unit time is:

$$\widetilde{W} = \frac{2\pi}{\hbar}\,\rho(k)\,|\langle f|\,H'\,|\,i\,\rangle|^2.$$

By considering photon emitted into solid angle $d\Omega = \sin\theta\, d\theta\, d\phi = 2\pi \sin\theta\, d\theta$ wherein we used azimuthal symmetry, the energy density of the final state is:

$$\rho(k) = \frac{Vk^2}{(2\pi)^3}\left(\frac{dE_f}{dk}\right)^{-1} d\Omega = \frac{VE'}{2\pi^2 pc^2 \hbar}\, kdk.$$

Therefore, energy radiated per unit time is given by

$$W = \int \hbar w\, \tilde{w} = \int \frac{V}{\pi} \frac{E'w}{pc^2 \hbar}\, kdk\, | \langle f | H' | i \rangle |^2.$$

Using $k = \dfrac{w}{c'}$

$$W = \int \frac{V}{\pi} \frac{E'w^2}{pc^2 \hbar} \frac{dw}{c'^2} | \langle f | H' | i \rangle |^2. \tag{10}$$

If $F_2 = 0$ i.e. if we consider charged particle without anomalous magnetic moment and retain the higher order terms of the type p^2 and neglect terms like px, x^2 we get.

$$W = \frac{F_1^2}{4\pi Ep} \int_0^{w_{\max}} (\mathbf{e} \cdot \mathbf{p})\, (\mathbf{e} \cdot \mathbf{p})\, [1 + \vec{\omega} \cdot \vec{\omega}' + \vec{\omega}_4 \vec{\omega}_4'] \, wdw. \tag{11}$$

Averaging over the initial and summing over the final spin states we get:

$$W = \frac{F_1^2}{4\pi Ep} \int_0^{w_{\max}} (\mathbf{e} \cdot \mathbf{p})\, (\mathbf{e} \cdot \mathbf{p})\, wdw$$

$$= \frac{e^2 v}{4\pi c^2} \int_0^{w_{\max}} \sin^2 \theta\, w\, dw, \tag{12}$$

which is obtained by Nemtan (1953).
Energy radiated per unit time per unit length

$$W' = \frac{e^2}{4\pi c^2} \int_0^{w_{\max}} \sin^2 \theta\, w\, dw, \tag{13}$$

which is a Frank and Tamm's (1937) relation when $\hbar \to 0$.

3. Photon polarization

We shall now consider two different cases of polarization namely (i) particle longitudinally polarized (ii) particle transverely polarized and find the polarization effects in expression (10). For convenience we shall consider separately the contribution of each term containing F_1^2, F_1F_2 and F_2^2 in the general expression for W. For this purpose we split W into three parts namely, $W_{F_1^2}$, $W_{F_1F_2}$ and $W_{F_2^2}$ and write eq. (10) in the form.

$$W = \frac{1}{2\pi Ep} \int_0^{w_{\max}} [W_{F_1^2} + W_{F_1F_2} + W_{F_2^2}]\, w\, dw. \tag{14}$$

In order to describe photon polarization we shall assume that the photon is moving along Z-direction ($\mathbf{Z} \parallel \mathbf{x}$) and particle momentum $\mathbf{p}$ is in YZ-plane. Obviously $\mathbf{e}$ will be in the XY-plane and

$$e_\mu \equiv (\cos\alpha,\ \sin\alpha\ e^{i\delta},\ 0,\ 0) \tag{15}$$

where α is the angle between three-vector $\mathbf{e}$ and X-axis and δ is the phase angle. The orientation of transverse vector $\mathbf{e}$ or the values of α and δ decide the nature of polarization of the radiation.

Case (i) Particle longitudinally polarized

For this case, four-spin vectors of the particle in the state before and after radiation are:

$$\omega = \left[\frac{sE}{mc^2}\hat{p}, \frac{i|p|s}{mc}\right];\ \omega' = \left[\frac{s'E'}{mc^2}\hat{p}', \frac{i|p'|s'}{mc}\right] \tag{16}$$

where s, $s'=\pm 1$ and positive sign indicates that particle spin is in the direction of motion and negative sign in the opposite direction of motion.

On substituting the expression for four-spin vector ω and photon polarization vector e_μ in the expression (14) we obtain

$$W_{F_1^2} = \frac{F_1^2}{4}\left\{2p^2\sin^2\theta\sin^2\alpha(1+ss') + x^2\left(1-\frac{1}{n^2}\right)[ss'\sin^2\theta\sin^2\alpha - \tfrac{1}{2}(1+ss')]\right\} \tag{17a}$$

$$W_{F_1F_2} = F_1F_2mc\left\{\frac{x^2}{2}\left(1-\frac{1}{n^2}\right)[(1+ss') - \frac{2ss'E^2}{m^2c^4}\sin^2\theta\sin^2\alpha] - \frac{ss'x^2\cos^2\theta}{E^2}(m^2c^2 + p^2\sin^2\theta\sin^2\alpha)\right\} \tag{17b}$$

$$W_{F_2^2} = F_2^2m^2c^2\left\{\frac{x^2}{2}\left(1-\frac{1}{n^2}\right)\left[\left(1+\frac{p^2}{m^2c^2}\sin^2\theta\sin^2\alpha\right)(1+ss') - \frac{2ss'E^2}{m^2c^4}\sin^2\theta\sin^2\alpha\right]\right\} \tag{17c}$$

In the above expressions we have retained terms proportional to p^2, px and x^2 and neglected the terms of higher order of smallness. The first term in (17a) is a dominant one since it is proportional to p^2 and the terms in the brackets of (17b) and (17c) are of the order of x^2.

If $ss'=-1$ i.e. when there is a spin-flip, the terms containing $(1+ss')$ in (17) cancel out but the rest of the terms give small contribution of the order of x^2 and is purely of quantum nature.

For $\alpha=\pi/2$, the expression (17) gives the usual dominant linear polarization perpendicular to the surface of the cone i.e. along Y-axis (W_2), but if $\alpha=0$ the expressions give linear polarization along X-axis (W_3), i.e. parallel to the surface of the cone and the contribution is quite small. Further $\alpha=\pi/4$ gives contribution due to circular polarization of the radiation. Since phase angle δ from photon polarization vector does not enter in the above expression we have equal contribution from left (W_{-1}) and right (W_{+1}), circular polarization. Hence $W_{-1}=W_{+1}$. The equal contribution of left and right circular polarization shows th at the radiation is linearly polarized and the total intensity of radiation is $W = W_2 + W_3$ as given by Sokolov and Loskutov (1957) for the special case of $F_2 = 0$.

For electron $F_2 \ll F_1$ and the contribution due to terms containing F_2^2 and F_1F_2 will be very small. However, the contribution due to the interference term $W_{F_1F_2}$ will be larger than that of $W_{F_2^2}$.

Case (ii) Particle transversely polarized

For this case the four-spin vectors of the particle before and after radiation are:

$$\omega = [\hat{s}, 0]; \quad \omega' = [\hat{s}', 0] \tag{18}$$

where $\hat{s}$ is a unit spin vector perpendicular to the direction of motion of the particle. For convenience let us take $\hat{s}$ and $\hat{s}'$ along X-axis then $ss' = +1$ will mean $s = +1 = s'$ or $s = -1 = s'$ i.e., there is no spin-flip. The spin vector of the particle is perpendicular to the plane of $\mathbf{p}$ and $\mathbf{x}$ and is along or opposite to X-axis.

In this case we get the following expressions:

$$W_{F_1^2} = \frac{F_1^2}{4}\left\{2p^2 \sin^2\theta \sin^2\alpha\,(1+ss') + \frac{x^2}{2}\left(1-\frac{1}{n^2}\right)(2ss'\cos^2\alpha - (1+ss'))\right\} \tag{19a}$$

$$W_{F_1F_2} = F_1 F_2\, mc \left\{\frac{x^2}{2}\left(1-\frac{1}{n^2}\right)(1+ss') - 2ss'\cos^2\alpha\right\} \tag{19b}$$

$$W_{F_2^2} = F_2^2\, m^2c^2 \left\{\frac{x^2}{2}\left(1-\frac{1}{n^2}\right)\left[((1+ss') - 2ss'\cos^2\alpha) + \frac{p^2}{m^2c^2}\sin^2\theta\sin^2\alpha\,(1+ss')\right]\right\} \tag{19c}$$

In this case of radiation by transversly polarized beam, the Cerenkov radiation is strongly linearly polarized as indicated by first term in (19a). The spin-flip ($ss' = -1$) contributes to the radiation and is of the order of x^2. This is purely a quantum effect.

If $\alpha = 0$ and $ss' = -1$ the contribution of linear photon polarization will be only along X-axis. Similarly if $\alpha = 0$ and $ss' = +1$; $W = 0$ and hence there is no radiation with linear photon polarization along X-axis. A case of no radiation exists when $\alpha = \pi/2$ and $ss' = -1$. But if $\alpha = \pi/2$ and $ss' = +1$ the radiation will be only along Y-axis i.e. totally linearly polarized. In this case also the sum of linear polarizations along X-axis, W_3, and along Y-axis, W_2, will give total intensity $W = W_2 + W_3$. If $\alpha = \pi/4$ the equal contribution of left, W_{-1}, and right, W_{+1} circular polarization gives total intensity $W = W_{-1} + W_{+1}$.

From expressions (19) it follows that the Cerenkov radiation vanishes at threshold i.e. at $\cos\theta = 1$ and for $\alpha = 0$.

$W(ss' = +1) = 0$ and

$$W(ss' = -1) \quad \frac{1}{2\pi Ep}\int_0^{w_{\max}} x^2\left(1-\frac{1}{n^2}\right) w\,dw \left(-\frac{F_1^2}{4} + F_1F_2\,mc + F_2^2\,m^2c^2\right)$$

which is again a purely quantum effect i.e. radiation is partially polarized and non-vanishing at threshold. This result obtained for general case matches with the result of Loskutov and Kukanov (1958) for their special case ($F_2 = 0$).

The expressions and the discussion given above is applicable to the radiation given by all the charged particles with spin half. In the case of stable particles like electrons and protons the contributions due to the terms containing F_2^2 and F_1F_2 in eqs. (17) and (19) depend on the relative value of anomalous magnetic moment of the particle. For electron $F_2 \simeq 10^{-3} F_1$ and the contribution for $W_{F_1F_2}$ is larger than $W_{F_2}{}^2$. However, for proton, $F_1 \sim F_2$ and the term $W_{F_1F_2}$ gives contribution to the radiation comparable to that of $W_{F_2}{}^2$. The neutron gives out Cerenkov radiation due to its anomalous magnetic moment and the radiation is given by the terms containing F_2^2.

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