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The Dirac-Schwinger covariance condition in classical field theory

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Abstract. A straightforward derivation of the Dirac-Schwinger covariance condition is given within the framework of classical field theory. The crucial role of the energy continuity equation in the derivation is pointed out. The origin of higher order derivatives of delta function is traced to the presence of higher order derivatives of canonical coordinates and momenta in the energy density functional.

Keywords. Lorentz covariance; Poincaré group; field theory; generalized mechanics.

1. Introduction

It has been stated by Dirac (1962) and by Schwinger (1962, 1963a, b, c, 1964) that a sufficient condition for the relativistic covariance of a quantum field theory is the energy density commutator condition

$$[T^{oo}(x), T^{oo}(x')] = (T^{ok}(x) + T^{ok}(x')\ \partial_k \delta\,(\mathbf{x}-\mathbf{x}') + (b^k(x) + b^k(x')\ \partial_k \delta\,(\mathbf{x}-\mathbf{x}') + (c^{klm}(x)\ c^{klm}(x')\ \partial_k \partial_l \partial_m \delta\,(\mathbf{x}-\mathbf{x}') + \ldots \quad (1)$$

where $b_k = \partial^l\ \beta_{lk}$ and $\beta_{kl} - \beta_{lk} = \partial^m \gamma_{mkl}$ and the energy density is that obtained by the Belinfante prescription (Belinfante 1939). For a class of theories that Schwinger calls ' local ' the Dirac-Schwinger (DS) condition is satisfied in the simplest form with $b_k = c_{klm} = \ldots = 0$. Spin s $(s \leqslant 1)$ theories belong to this class. For canonical theories in which simple canonical commutation relations hold Brown (1967) has given two new proofs of the DS condition.

In a previous work we made use of the DS condition in Poisson-bracket (PB) form to demonstrate that in a few examples of pathological field theory, at the classical level, the formal Lorentz covariance of the Lagrangian does not ensure the correct Poincaré group structure relations for the generators (Babu Joseph and Sabir 1976). In this note we examine the significance of the DS condition for a classical field theory in a general setting. Considering a PB realization of the Poincaré group Lie algebra, we derive the DS condition starting from the basic definition of PB for the two functionals of canonical field coordinates and momenta. In the simple case where the energy density is a functional of the canonical co-ordinates and momenta and their first derivatives only the first derivative of delta function appears on the right hand side of (1) and the general form of the coefficient of this term can be identified by appealing to the energy continuity equation. The Poincaré group structure relations serve only to determine the quantities b_k in (1).

The present investigation reveals that at the classical level the absence of higher derivatives of delta function on the r.h.s. of (1) is not in any way directly related to the spin of the field and that such terms inevitably appear when the energy density is a functional of higher derivatives of fields and momenta. A derivation of the DS condition in the case of a generalized mechanics with higher derivatives of canonical coordinates and momenta is sketched.

2. DS condition for simple systems

For a simple system described by a set of independent dynamical variables ψ_α, π_α ($\alpha=1 \ldots N$) we assume that the energy density derived from a symmetric energy-momentum tensor, is a functional only of ψ_α, π_α and their first derivatives. The PB of the energy densities at two distinct points at equal times is

$$\{T^{oo}(x), T^{oo}(x')\} = \sum_\alpha \int d^3y \left(\frac{\delta T^{oo}(x)}{\delta \psi_\alpha(y)} \frac{\delta T^{oo}(x')}{\delta \pi_\alpha(y)} - \frac{\delta T^{oo}(x)}{\delta \pi_\alpha(y)} \frac{\delta T^{oo}(x')}{\delta \psi_\alpha(y)}\right) \tag{2}$$

On evaluation of the functional derivatives we obtain this in the form

$$\begin{aligned}\{T^{oo}(x), T^{oo}(x')\} = \sum_\alpha \Bigg[\Bigg(&\frac{\partial T^{oo}(x)}{\partial \pi_\alpha(x)} \frac{\partial T^{oo}(x')}{\partial \partial'_k \psi_\alpha(x')} - \frac{\partial T^{oo}(x)}{\partial \psi_\alpha(x)} \frac{\partial T^{oo}(x')}{\partial \partial'_k \pi_\alpha(x')} \\ &+ \frac{\partial T^{oo}(x')}{\partial \pi_\alpha(x')} \frac{\partial T^{oo}(x)}{\partial \partial_k \psi_\alpha(x)} - \frac{\partial T^{oo}(x')}{\partial \psi_\alpha(x')} \frac{\partial T^{oo}(x)}{\partial \partial_k \pi_\alpha(x)}\Bigg) \partial_k \delta(\mathbf{x}-\mathbf{x}') \\ &+ \left(\frac{\partial T^{oo}(x)}{\partial \partial_k \pi_\alpha(x)} \frac{\partial T^{oo}(x')}{\partial \partial'_l \psi_\alpha(x')} - \frac{\partial T^{oo}(x)}{\partial \partial_k \psi_\alpha(x)} \frac{\partial T^{oo}(x')}{\partial \partial'_l \pi_\alpha(x)}\right) \partial_k \partial_l \delta(\mathbf{x}-\mathbf{x}')\Bigg].\end{aligned} \tag{3}$$

Making use of the Schwinger ideniity (Schwinger 1964)

$$(f(x)\, g(x') + f(x')\, g(x))\, \partial_k \delta(\mathbf{x}-\mathbf{x}') = (f(x)\, g(x) + f(x')\, g(x'))\, \partial_k \delta(\mathbf{x}-\mathbf{x}') \tag{4}$$

and an easily verified analogous identity involving the second derivatives of the delta function

$$\begin{aligned}(f_k(x)\, g_l(x') - g_k(x)\, f_l(x'))\, \partial^k \partial^l \delta(\mathbf{x}-\mathbf{x}') = -\, (&\partial^k f_k(x) \cdot g_l(x) - \partial^k g_k(x') \cdot f_l(x) \\ &+ \partial^{k'} f_k(x') \cdot g_l(x') - \partial^{k'} g_k(x') \cdot f_l(x'))\, \partial^l \delta(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{5}$$

the expression on the r.h.s. of (3) simplifies to

$$\{T^{oo}(x), T^{oo}(x')\} = (f^{ok}(x) + f^{ok}(x'))\, \partial_k \delta(\mathbf{x}-\mathbf{x}')$$

where

$$\begin{aligned}f^{ok} = \sum_\alpha \Bigg(&\frac{\partial T^{oo}}{\partial \pi_\alpha} \frac{\partial T^{oo}}{\partial \partial_k \psi_\alpha} - \frac{\partial T^{oo}}{\partial \psi_\alpha} \frac{\partial T^{oo}}{\partial \partial_k \pi_\alpha} \\ &+ \frac{\partial T^{oo}}{\partial \partial_k \pi_\alpha} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \psi_\alpha} - \frac{\partial T^{oo}}{\partial \partial_k \psi_\alpha} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \pi_\alpha}\Bigg)\end{aligned} \tag{7}$$

Integration of (7) over x' yields

$$\{T^{oo}(x), P^o\} = \partial_k f^{ok}(x) \tag{8}$$

where $P^o = \int T^{oo}(x)\, d^3x$ is the total energy.

If we assume that the field system has space-time translational invariance, the equation for T^{oo} (x) can be cast into the form

$$d_o T^{oo}(x) + \partial_k f^{ok} = 0 \tag{9}$$

where d_o represents the total time derivative. From (9) we can infer that

$$f^{ok} = T^{ok} + b^k \tag{10}$$

with $\partial^k b_k = 0$.
Hence we write the energy density PB in the form

$$\{T^{oo}(x),\ T^{oo}(x')\} = (T^{ok}(x) + T^{ok}(x')\ \partial_k \delta(\mathbf{x}-\mathbf{x}') + (b^k(x) + b^k(x')\partial_k \delta(\mathbf{x}-\mathbf{x}') \tag{11}$$

Up to this point we have not used the concept of relativistic covariance in any manifest way. According to Schwinger only the two following PB's

$$\{J^o{}_k, P^o\} = -P_k \tag{12a}$$

$$\{J^o{}_k, J^o{}_l\} = J_{kl} \tag{12b}$$

pertain to the Lorentz invariance of the theory, the rest of the Poincaré group structure relations following from three dimensional invariance and the vector property of the three boost-generators J^{ok}. The generators in (12) may be expressed in terms of the energy-momentum tensor as follows:

$$\begin{aligned} P_k &= \int d^3x\, T^o{}_k\,(x) \\ J^o{}_k &= x^o P_k - \int d^3x\ x_k\ T^{oo}(x) \\ J_{kl} &= \int d^3x\ (x_k T^o{}_l(x) - x_l T^o{}_k(x)\,). \end{aligned} \tag{13}$$

Relation (12a) is already allowed by the general form of (7) and in order that (12b) be satisfied b_k must obey the condition

$$\int d^3x(x_k b_l(x) - x_l b_k(x)) = 0.$$

In order that this volume integral vanishes, the integral must be of the form of a divergence i.e.

$$x_k b_l - x_l b_k = \partial^m\ a_{klm}. \tag{15}$$

From (15) we deduce, with the aid of the fact that b_k is divergenceless, that b_k must itself be a divergence

$$b_k = \partial^l\ \beta_{lk} \tag{16}$$

where $\beta_{lk} = x_l b_k + \partial^m\ a_{klm}$.

It also follows that the antisymmetric part of β_{kl} is a divergence.

3. DS condition for non-conservative systems

We now consider the case when the Lagrangian of the system has an explicit space-time dependence. Equation (9) is in this instance replaced by

$$d_o T^{oo} + \partial_k f^{ok}(x) = \partial_o T^{oo}(x) \tag{17}$$

where we have made a slight change of notation in that ∂_o now denotes explicit differentiation with respect to time whereas ∂_k has still the usual meaning. Energy-momentum conservation does not hold in this case and we have

$$d_o T^{oo} + \partial_k T^{ok} = \partial_o \mathscr{L}. \tag{18}$$

From (17) and (18) we conclude that f^{ok} must be of the form

$$f^{ok} = T^{ok} + b^k$$

with

$$\partial^k b_k = \partial_o (T^{oo} - \mathscr{L}) \tag{19}$$

since b_k is not divergenceless the structure relation (12a) is satisfied only if

$$\int d^3x \; b_k(x) = 0 \tag{20}$$

and this implies $b_k = \partial^l \beta_{kl}$ where β_{kl} is arbitrary. The relation (12b) imposes on β_{kl} the restriction

$$\beta_{kl} - \beta_{lk} = \partial^m \gamma_{mkl}. \tag{21}$$

Thus the conditions that b_k has to satisfy are, in the present case, same as those for b_k in (1).

4. DS condition in higher derivative field theories

A Hamiltonian formulation of a generalized mechanics with higher order derivatives of field variables has been considered by Coelho de Souza and Rodrigues (1969). They have shown that the PB of two functionals A and B is to be defined as

$$\{A, B\} = \sum_{\alpha=1}^{N} \sum_{m=0}^{S-1} \int d^3x \left(\frac{\triangle A}{\triangle \psi_\alpha^{(m)}} \frac{\triangle B}{\triangle \pi_{\alpha/m+1}} - \frac{\triangle A}{\triangle \pi_{\alpha/m+1}} \frac{\triangle B}{\triangle \psi_\alpha^{(m)}} \right) \tag{22}$$

where $\psi_\alpha^{(m)} = \dfrac{d^m}{dx^m{}_0} \psi_\alpha$ and $\pi_{\alpha/m+1}$ is the momentum conjugate to $\psi_\alpha^{(m)}$; s is the order of the highest derivative considered. If $F = \int \mathscr{F} d^3x$, then by definition,

$$\frac{\triangle F}{\triangle \psi_\alpha} = \frac{\partial \mathscr{F}}{\partial \psi_\alpha} - \partial_{i_1} \frac{\partial \mathscr{F}}{\partial \partial_{i_1} \psi_\alpha} + \ldots + (-1)^s \partial_{i_1} \cdots \partial_{i_s} \frac{\partial \mathscr{F}}{\partial \partial_{i_1} \cdots \partial_{i_s} \psi_\alpha} \tag{23}$$

Using the definition (22) to evaluate the PB $\{T^{oo}(x), T^{oo}(x')\}$ where T^{oo} is a functional of $\psi_\alpha^{(m)}$, $\pi_{\alpha/m+1}$ and their derivatives up to order s we arrive at the general result

$$\{T^{oo}(x), T^{oo}(x'),\} = (a_k(x) + a_k(x') \, \partial^k \delta(\mathbf{x} - \mathbf{x}')$$
$$+ (c_{klm}(x) + c_{klm}(x')) \, \partial^k \partial^l \partial^m (\mathbf{x} - \mathbf{x}') + \ldots \tag{24}$$

The coefficients a_k, c_{klm},... appearing in (24) may be evaluated directly in any particular case. When the highest order of derivatives in T^{oo} is the second, we obtain

$$a^k = \sum_{m=0}^{1} \sum_{\alpha} \Bigg[\Bigg(\frac{\partial T^{oo}}{\partial \pi_{\alpha/m+1}} \frac{\partial T^{oo}}{\partial\, \partial_k \psi_\alpha^{(m)}} - \frac{\partial T^{oo}}{\partial \psi_\alpha^{(m)}} \frac{\partial T^{oo}}{\partial \partial_k \pi_{\alpha/m+1}} +$$

$$\frac{\partial T^{oo}}{\partial \partial_k \pi_{\alpha/m+1}} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \psi_\alpha^{(m)}} - \frac{\partial T^{oo}}{\partial \partial_k \psi_\alpha^{(m)}} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \pi_{\alpha/m+1}} - \Bigg(\partial_j \frac{\partial T^{oo}}{\partial \partial_k \psi_\alpha^{(m)}} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \partial_j \pi_{\alpha/m+1}}$$

$$+ \partial_j \frac{\partial T^{oo}}{\partial \partial_j \psi_\alpha^{(m)}} \partial_i \frac{\partial T^{oo}}{\partial \partial_i \partial_k \pi_{\alpha/m+1}} + \partial_j \frac{\partial T^{oo}}{\partial \partial_i \psi_\alpha^{(m)}} \partial_l \frac{\partial T^{oo}}{\partial \partial_j \partial_k \pi_{\alpha/m+1}} \Bigg) + \Bigg(\partial_j \frac{\partial T^{oo}}{\partial \partial_k \pi_{\alpha/m+1}}$$

$$\partial_l \frac{\partial T^{oo}}{\partial \partial_l \partial_j \psi_\alpha^{(m)}} + \partial_j \frac{\partial T^{oo}}{\partial_j \pi_{\alpha/m+1}} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \partial_k \psi_\alpha^{(m)}}, + \partial_j \frac{\partial T^{oo}}{\partial \partial_i \pi_{\alpha/m+1}} \partial_l \frac{\partial T^{oo}}{\partial \partial_l \partial_k \psi_\alpha^{(m)}} \Bigg) \Bigg] \tag{25}$$

$$c^{ijk} = \sum_{m=0}^{1} \sum_{\alpha} \Bigg(\frac{\partial T^{oo}}{\partial \partial_i \psi_\alpha^{(m)}} \frac{\partial T^{oo}}{\partial \partial_j \partial_k \pi_{\alpha/m+1}} - \frac{\partial T^{oo}}{\partial \partial_i \pi_{\alpha/m+1}} \frac{\partial T^{oo}}{\partial \partial_j \partial_k \psi_\alpha^{(m)}} \Bigg) \tag{26}$$

Here we have made use of the identity

$$\begin{aligned}
&(f_i\,(x)g_{jk}(x') + g_{jk}(x) f_i(x')\partial^i\partial^j\partial^k\, \delta(\mathbf{x}-\mathbf{x}') \\
&= (f_i(x)g_{jk}(x) + f_i(x')g_{jk}(x'))\partial^i\partial^j\partial^k\, \delta(\mathbf{x}-\mathbf{x}') \\
&-(\partial^j f_i(x)\partial^k g_{jk}(x) + \partial^j f_j(x)\partial^k g_{ik}(x) + \partial^j f_k(x)\partial^k g_{ij}(x) \\
&+\partial_j' f_i(x')\partial^{k'} g_{jk}(x') + \partial^{j'} f_j(x')\ .\ \partial^{k'} g_{ik}(x') + \partial^{j'} f_k(x')\partial^{k'} g_{ij}(x')) \\
&\quad \times \partial^i \delta\,(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{27}$$

When still higher derivatives are involved, a_k, c_{klm}... will have additional contributions from these higher derivatives and they may be evaluated using identities similar to (27) for terms with higher derivatives of delta function.

Integrating (24) over x' we have

$$\{T^{oo}(x), P^o\} = \partial^k a_k + \partial^k \partial^l \partial^m c_{klm} + \ldots \tag{28}$$

Because of the assumed energy-momentum conservation for the system this becomes

$$\partial_o T^{oo} + \partial^k a_k + \partial^k \partial^l \partial^m c_{klm} + \ldots = 0 \tag{29}$$

from which it may be inferred that

$$a_k + \partial^l \partial^m c_{klm} + \ldots = T^o_k + b_k \tag{30}$$

where $\partial^k b_k = 0$.

Hence the energy density PB assumes the form

$$\{T^{oo}(x), T^{oo}(x')\} = (T^{ok}(x) + T^{ok}(x')\, \partial_k \delta\, (\mathbf{x}-\mathbf{x}')$$
$$+(B^k(x)+B^k(x')\partial_k\delta(\mathbf{x}-\mathbf{x}')+(c^{klm}(x)+c^{klm}(x')\times\partial_k\partial_l\partial_m\delta(\mathbf{x}-\mathbf{x}')+\ldots \quad (31)$$

where

$$B_k = b_k - \partial^l \partial^m c_{klm} - \ldots \quad (32)$$

A look at the group structure relations (12a, b) now reveals that while (12a) is allowed by the form of the PB (32), (12b) will be satisfied only if B_k is restricted by the condition

$$\int (x_k B_l(x) - x_l B_k(x)) d^3x = 0 \quad (33)$$

in consequence of which

$$\int (x_k b_l(x) - x_l b_k(x)) d^3x = 0 \quad (34)$$

and this implies as we have seen in section 2 that $b_k = \partial_l \beta^l k$ and the antisymmetric part of β_{kl} is again a divergence.

5. Concluding remarks

We have presented a derivation of the DS covariance condition within the framework of classical field theory. The general form of the condition is determined by the energy continuity equation. For simple cases this takes on a form involving the first derivatives of delta function while the presence of higher derivatives in the energy-density invites the appearance of higher derivatives of delta function. This makes clear the fact that for these higher order terms to appear it is not necessary that a higher spin ($s \geqslant 1$) be associated with the field system. On the other hand the appearance of higher derivatives of delta function must be taken to mean that the energy density functional in such theories contain higher derivatives of canonical co-ordinates and momenta.

While it is the translational invariance of the theory that determines the general form of the DS condition, Lorentz covariance of the theory finds expression through the restrictions imposed on the coefficients b_k. Here we may note a point of difference between the classical DS condition and the quantum version. In the classical case both for simple systems and higher derivative systems the coefficient b_k is divergenceless whereas there is no such restriction in the quantum case. However, when the energy density has an explicit space-time dependence b_k obeys the same conditions as in the quantum case. Despite these differences which may not be too significant, the quantum-classical analogy would permit us to extend to quantum field theory the result herein obtained in respect of the correspondence between higher order Lagrangians and higher order derivatives of delta function in the DS condition.

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