Pramāṇa, Vol. 9, No. 1, July 1977, pp. 71-77. © Printed in India.

Timelike and null geodesics in the Nordström field

NARESH DADHICH and P P KALE
Department of Mathematics and Statistics, University of Poona, Poona 411007

MS received 14 May 1976; revised 3 January 1977

Abstract. The timelike and null geodesics are investigated in the Nordström geometry and it is found that incoming geodesics always encounter a turning point at a finite radial distance. The limits for escape, bound and stable orbits are obtained and they are closer to the source as compared to their counterparts in the Schwarzschild's field.

Keywords. General relativity; geodesic motion; charged source.

1. Introduction

The geodesics in the Schwarzschild field have been extensively studied by Darwin (1959, 1961) and that in the Kerr field by Carter (1966), Felice (1968) and Felice and Calvani (1972) while motion in the Kerr-Newman field is, in general, outlined by Carter (1968). As far as we know geodesics in the Reissner-Nordström field have not been considered comprehensively except for some general remarks by Graves and Brill (1960), a causal study of radial geodesics by Brigman (1972) and circular orbits by Liang (1974) and by Armenti (1975).

The purpose of this paper is to investigate in detail geodesics in the Nordström field of a charged source.

In section 2 we write the equations of motion of a free particle and section 3 contains the general implications of these equations. In section 4 are given the equations of some interesting orbits while the section 5 considers photon orbits. The limits of escape, bound and stable orbits are obtained in section 6 and finally the section 7 has general results with discussion.

2. Equations of motion

The Reissner-Nordström field of a charged particle of mass m and charge e is described by the line-element,

$$ds^2 = \triangle dt^2 - \triangle^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)$$

$$\triangle = 1 - 2m/r + e^2/r^2. \tag{1}$$

We employ the relativistic units, $G=c=1$.

The first integrals of the equations of motion (with $\theta=\pi/2$) are as follows:

$$\triangle \frac{dt}{ds}=E \tag{2}$$

$$r^2 \frac{d\phi}{ds}=l \tag{3}$$

$$\triangle \left(\frac{dt}{ds}\right)^2-\triangle^{-1}\left(\frac{dr}{ds}\right)^2-r^2\left(\frac{d\phi}{ds}\right)^2=1 \tag{4}$$

Here E and l are energy and angular momentum of the test particle respectively. From the above equations, we readily obtain ($u=1/r$),

$$\left(\frac{du}{d\phi}\right)^2=f(u)=\frac{A}{l^2}+\frac{2m}{l^2}u-\left(1+\frac{e^2}{l^2}\right)u^2+2mu^3-e^2u^4 \tag{5}$$

where $A=E^2-1$.

If a particle has velocity V at infinity and starts on a line at a perpendicular distance p from the source, we could write,

$$E=(1-V^2)^{-1/2},\ l=pV(1-V^2)^{-1/2}$$
$$A=V^2(1-V^2)^{-1} \text{ and } l^2/A=p^2 \tag{6}$$

Now our interest is to study the eq. (5) which is a biquadratic ($e\neq 0$) and it becomes a cubic when $e=0$. One should not therefore expect to obtain the corresponding Darwin's (1959) results from the following by simply putting $e=0$.

3. General considerations

3.1. The orbits are determined by eq. (5). By the 'rule of signs' we note that the biquadratic $f(u)=0$ would have (i) when $A>0$, either three positive and one negative or one positive, one negative and a pair of complex conjugate roots, (ii) when $A<0$, all the four positive or two positive and a pair of complex conjugate or two pairs of complex conjugate roots.

A root signifies a turning point where $du/d\phi$ changes sign, *viz*, for a particle to have hyperbolic orbit, there should occur one positive root. A negative root has no physical meaning. If there is no positive root it means $du/d\phi$ is never zero and hence the particle falls into the source. Henceforth only positive roots would be of our interest.

3.2. Equation (5) could as well be written as

$$\left(\frac{du}{d\phi}\right)^2=f(u)=-\triangle u^2-\frac{\triangle}{l^2}+\frac{E^2}{l^2}. \tag{7}$$

It could be easily seen from the above equation that $(du/d\phi)^2$ would become negative as r goes arbitrarily close to zero. That means there would always occur a turning point at finite value of r below which no particle could penetrate.

For $e^2\leqslant m^2$, $\triangle=0$ has two real roots u_- and u_+ ($u_-\geqslant u_+$) corresponding to r_- (antievent horizon) and r_+ (event horizon). Clearly no turning point lies in the singular region bounded by r_- and r_+ and one or three must lie below r_-, that is inside the black hole.

3.3. Let us write

$$\left(\frac{du}{d\phi}\right)^2 = f(u) = -e^2(u-a)(u-b)(u-c)(u-d) \tag{8}$$

where $a \geqslant b \geqslant c \geqslant d$ and a, b, c are positive, d could be positive or negative. We shall indicate radial distances corresponding to roots by $R_a = \frac{1}{a}$, $R_b = \frac{1}{b}$, etc. and they will have the reverse order as $R_a \leqslant R_b \leqslant R_c \leqslant R_d$ or ∞. It is clear that the motion is possible neither below R_a nor between R_b and R_c, it could only occur between R_a and R_b and R_c and R_d (or infinity). One could straightaway rule out the cases of all roots equal and of all complex since $(du/d\phi)^2 < 0$ throughout. There may however occur a trivial cirular orbit, corresponding to the four equal roots, at $r = 2r_0$ with $e^2/m^2 = 5/4$, $l^2 = 4e^4/m^2$ and $A = -m^2/4e^2$. Here $r_0 = e^2/m$ is the classical radius of the source particle.

On comparing the coefficients of (5) and (8), we have

$$a+b+c+d = \frac{2}{r_0} \tag{9}$$

$$ab+cd+(a+b)(c+d) = \frac{1}{e^2} + \frac{1}{l^2} \tag{10}$$

$$cd(a+b)+ab(c+d) = \frac{2}{l^2\, r_0} \tag{11}$$

$$abcd = -\frac{A}{l^2 e^2} \tag{12}$$

From the above relations we find that $\frac{1}{2}\, r_o < R_a < 2r_o$, $R_c > \frac{8}{2} r_o$ if $d > 0$ and $R_c > r_o$ if $d < 0$. For escape orbits $d < 0$ then $R_c > r_o$ and hence the particles having perihelia above r_o may only escape and no particle can come as close to the source as $\frac{1}{2}\, r_o$. We shall obtain a more precise limits on the orbits in section 6.

3.4. If $f(u) = 0$ has a pair of complex conjugate roots we should consider

$$\left(\frac{du}{d\phi}\right)^2 = f(u) = -e^2\,(u-a)(u-b)\,[(u-\alpha)^2+\beta^2] \tag{13}$$

where $a > 0$ and $a > b$ (b may be positive or negative) and $\alpha \pm i\beta$, a pair of complex conjugate roots. Again comparing (13) with (5) we get

$$a+b+2\alpha = 2/r_0 \tag{14}$$

$$ab+\alpha^2+\beta^2+2\alpha\,(a+b) = \frac{1}{e^2} + \frac{1}{l^2} \tag{15}$$

$$(\alpha^2+\beta^2)(a+b)+2\alpha ab = \frac{2}{l^2\, r_o} \tag{16}$$

$$ab(\alpha^2+\beta^2) = -\frac{A}{l^2 e^2}\,. \tag{17}$$

After much algebraic manipulations we find from the above equations, $R_a > \frac{1}{2} r_o$ if $b > 0$ and when $b < 0$ then $R_a \gtrless \frac{1}{2}\, r_o$ depending upon $R_a^2 \lessgtr p^2 (p^2 = l^2/A)$. In this case also the nearest turning point (R_a) would be above $\frac{1}{2} r_o$ except for a particle having large energy and small angular momentum (i.e., with small p^2, almost radial fall) may have its turning point below $\frac{1}{2}\, r_o$.

4. Equations of orbits

We have computed the equations of orbit in all the cases but we shall discuss below only the two cases in which the orbits may not be captured by black hole. The case $e^2>m^2$ is not physically realistic and hence is not of our interest.

4.1. The case $a>b>c>d$: In this case motion may occur between R_a and R_b and R_c and R_d (or ∞) but the former would be a capture orbit (at least one root lying inside the black hole) and we shall hence consider the latter. We obtain the solution in terms of the elliptic functions as

$$u=\frac{d\,(a-c)+a(c-d)\,sn^2\zeta}{(a-c)+(c-d)sn^2\zeta} \quad \begin{matrix}(\text{mod } K)\\ (R_c<r<R_d \text{ or } \infty)\end{matrix} \tag{18}$$

where

$$K^2=\frac{(a-b)\,(c-d)}{(a-c)\,(b-d)} \tag{19}$$

$$\left(\frac{d\zeta}{d\phi}\right)^2=\frac{e^2}{4}\,(a-c)\,(b-d). \tag{20}$$

For a hyperbolic orbit $d<0$, a particle starts at $r=\infty$ with $sn^2\zeta_1=-d\,(a-c)/a\,(c-d)$, turns back when $\zeta=K$ and finally reaches infinity again when $\zeta=2K-\zeta_1$. For an elliptic orbit, particle starts from R_d when $\zeta=0$, reaches R_c when $\zeta=K$ and then goes back to R_d when $\zeta=2K$.

4.2. The case $a>b=c>d$: Here again the interesting orbit occurs between R_c and R_d and it is given by

$$u=\frac{d(a-b)+b(a-d)\sinh^2\psi}{(a-b)+(a-d)\sinh^2\psi}, \quad (R_c\leq r\leq R_d \text{ or } \infty) \tag{21}$$

$$\psi=\tfrac{1}{2}[(e^2(a-b)\,(b-d)]^{\frac{1}{2}}\,(\phi-\phi_0) \tag{22}$$

where ϕ_0 is an arbitrary constant.

If $d>0$, the particle may start at R_d when $\phi=\phi_0$ and will fall asymptotically to R_c as $\phi\to\infty$, when $d<0$, the particle starts from infinity with $\sinh^2\psi_1=-d(a-b)/b\,(a-d)$ and falls to R_c as $\phi\to\infty$.

As an illustration we consider for $e^2/m^2=343/375$, a particle with $p^2=(1125/28)\,r_0^2$ and $A=8\,m^2/7\,e^2$ would have

$$r=\frac{15\,r_0}{2}\left[\frac{7+11\sin h^2\psi}{7+33\sin h^2\psi}\right], \quad \left(\frac{5}{2}\,r_0\leq r\leq\frac{15}{2}\,r_0\right) \tag{23}$$

$$\psi=\frac{2\sqrt{15}}{21}\,(\phi-\phi_0). \tag{24}$$

Here the perihelion lies outside the black hole and below 3 m.

5. Light rays

Light rays could be regarded as propagation of photons with $V=1$. In this case $ds=0$ and the orbit equation could similarly be obtained but we shall alternatively write it from (5) by letting A and l become infinite while $A/l^2 = 1/p^2$ is finite. So (5) would now read as

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{p^2} - u^2 + 2m\,u^3 - e^2\,u^4 \tag{25}$$

This would have either three positive and one negative or one positive, one negative and a pair of complex conjugate roots. The analogoue of (7) is

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{p^2} - u^2 \triangle. \tag{26}$$

The photon orbits would be similar to the particle orbits as discussed in the sections 3 and 4. As an example let us consider for $e^2 = m^2$, a photon having a hyperbolic orbit with perihelion at $R_c = 5/2\,m$, $p = 25m/6$, $R_a = 5m/6$, $R_b = 5m/3$ and $R_d = -5m$. The orbit is given by

$$r = \frac{5m}{2}\left[\frac{4 + 3\,sn^2\,\psi}{-2 + 9\,sn^2\,\psi}\right] \quad (\text{mod } 3/4) \tag{27}$$

$$\psi = \tfrac{2}{5}\,(\phi + \phi_0).$$

6. Escape, bound and stable orbits

6.1. *Escape orbit:* The condition for escape (hyperbolic) orbit is that $f'(r) > 0$ at the perihelion point and $1 - E^2 < 0$. These conditions read as

$$\frac{r^2 - 3mr + 2e^2}{r^2(mr - e^2)} > \frac{1}{l^2}, \quad \frac{r^2 - 2mr + e^2}{r^2(2mr - e^2)} > \frac{1}{l^2}. \tag{28}$$

Since $l^2 > 0$, we should have

$$r^2 - 3mr + 2e^2 > 0 \tag{29}$$

which gives

$$r_e > \frac{3m}{2}\left[1 + \sqrt{1 - 8e^2/9m^2}\right]. \tag{30}$$

This gives the value of r for the closest perihelion. For photon we should consider equality in (30).

6.2. *Bound orbit:* For bound orbit, $f'(r) > 0$ at the perihelion and $1 - E^2 > 0$ which give us

$$\frac{r^2 - 3mr + 2e^2}{r^2(mr - e^2)} > \frac{1}{l^2} > \frac{r^2 - 2mr + e^2}{r^2(2mr - e^2)} \tag{31}$$

that means

$$r^3 - 4mr^2 + 4e^2 r - e^4/m > 0. \tag{32}$$

For $e^2 \ll m^2$, it approximates to

$$r_b > 4m(1 - e^2/4m^2). \tag{33}$$

6.3. *Stable orbit:* In this case $f'(r)=0$ and $f''(r) \leqslant 0$ and hence we get

$$r^3-6mr^2+9e^2r-4e^2/m \geqslant 0 \tag{34}$$

which approximates for $e^2 \lll m^2$, to give

$$r_s \geqslant 6m(1-e^2/4m^2). \tag{35}$$

The limits for the closest, bound and stable circular orbit as obtained by Liang (1974) and by Armenti (1975) are the same as given above for general orbit. The circular orbits can exist below r_b but they are unbound in energetics in the sense of Wilkins (1972) since $1-E^2<0$. Any forward displacement for such orbits would result into the particle flying to infinity while a backward displacement would carry it into the source. Here r_e, r_b and r_s are closer to the source as compared to their counterparts in the Schwarzschild case and the latter would result from the above inequalities when $e=0$.

7. Discussion

1. The distinguishing feature of this field in contrast to the Schwarzschild's field is that there always occurs a turning point $r=R_a$, the region lying below R_a is inaccessible to timelike and null geodesics. R_a is greater than $\frac{1}{2}$ ro except for almost radial motion (when $p^2=l^2/A$ is very small). However R_a would always lie inside the black hole.

2. As is clear from the considerations in section 3, r could only range between R_a and R_b or R_c and R_d or infinity. The non-capture orbits (orbits lying outside the black hole) occur in (i) when all roots are distinct (ii) when $a>b=c>d$ ($R_a<R_b=R_c<R_d$ or ∞). In both cases, the non-capture orbit would lie between R_c and R_d or infinity.

3. No turning point can occur in the singular region $r_-\leqslant r\leqslant r_+$ and one or three must lie below r_-, i.e., inside the blackhole. This is because, here the singular region is also bounded below (it is again regular below r_-) while in the Schwarzschild's case it is all through singular below $r=2m$ and hence no turning point occurs inside the black hole there.

4. As could be seen from section 6, the limits of escape, bound and stable orbits are closer to the source as compared to their counterparts in the Schwarzschild's case.

The timelike and null geodesics always encounter a turning point at a finite radial distance and closer limits for escape, bound and stable orbits suggest that presence of charge on the source particle weakens its gravitational field. This becomes evident if one considers radial force on a free particle; the field changes sign at $r=r_0$ being attractive for $r>r_0$ and repulsive for $r<r_0$. It goes on increasing (in the repulsive sense) without limit for $r<r_0$.

Acknowledgement

We are thankful to Profs J V Narlikar and P C Vaidya for many helpful suggestions and for critically reading the manuscript. Thanks are also due to the referees for their constructive criticism.

Referenees

Armenti A Jr. 1975 *Nuovo Cimento* **B25** 442
Brigman G H 1972 *Tensor* **25** 267
Carter B 1966 *Phys. Rev.* **141** 1242
Carter B 1968 *Phys. Rev.* **174** 1559
Darwin C 1959 *Proc. R. Soc. (London)* **A249** 180
Darwin C 1961 *Proc. R. Soc. (London)* **A263** 39
Felice F De 1968 *Nuovo Cimento* **B57** 351
Felice F De and Calvani M 1972 *Nuovo Cimento* **B10** 447
Graves J C and Brill D R 1960 *Phys. Rev.* **120** 1507
Liang E P T 1974 *Phys. Rev.* **D9** 3257
Wilkins D C 1972 *Phys. Rev.* **D5** 814