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Dirac equation in Kerr space-time

B R IYER and ARVIND KUMAR
Department of Physics, University of Bombay, Bombay 400098

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Abstract. The weak field–low velocity approximation of Dirac equation in Kerr space-time is investigated. The interaction terms admit of an interpretation in terms of a 'dipole-dipole' interaction in addition to coupling of spin with the angular momentum of the rotating source. The gravitational gyro-factor for spin is identified. The charged case (Kerr-Newman) is studied using minimal prescription for electromagnetic coupling in the locally inertial frame and to the leading order the standard electromagnetic gyro-factor is retrieved. A first order perturbation calculation of the shift of the Schwarzschild energy level yields the main interesting result of this work: the anomalous Zeeman splitting of the energy level of a Dirac particle in Kerr metric.

Keywords. Kerr space-time; Dirac equation; anomalous Zeeman effect.

1. Introduction

The general formulation of Dirac equation in curved space-time has been known for a long time (Brill and Wheeler 1957). In this paper we take up the investigation of Dirac equation in the specific case of Kerr geometry which is supposed to represent the gravitational field of a rotating black hole or approximately the field outside of a rotating body. The corresponding problem of the scalar particle in Kerr metric has been recently studied by Ford (1975) and our purpose is to extend this work and obtain new results characteristic of the spin half case. In addition, using minimal prescription for electromagnetic coupling in the local Lorentzian frame we extend the results to Kerr-Newman geometry (charged rotating source).

The spin half case is complicated by the fact that the transformation properties of spinors (unlike tensors) are defined only in local Lorentzian frame. This necessitates the use of vierbein (tetrad) formalism (Boulware 1975). In the next section we evaluate the necessary vierbein components and 'spinor affinity' for the Kerr metric and take a systematic weak field–low velocity limit. The interaction terms admit of interpretation in terms of a 'dipole-dipole' interaction in addition to a direct coupling of spin with the angular momentum of the rotating source. This leads to the identification of the gravitational gyro-factor for spin. Extension to the charged case in section 3 retrieves, in the leading order, the standard electromagnetic gyro-factor. In sec. 4, we calculate in first-order perturbation theory, the splitting of the Schwarzschild energy level and, in contrast to the scalar case which gives normal ' Zeeman pattern', we obtain anomalous ' gravitational Zeeman pattern ' for a Dirac particle in Kerr geometry.

2. Spin half equation in Kerr metric

(Notation: In what follows, the Latin indices a, b, c, etc., run over 0, 1, 2, 3 and refer to local Lorentzian co-ordinates. The Greek indices α, β, μ, ν, etc.. run over the four general co-ordinates t, r, θ, ϕ.)

2.1. *The Kerr metric*

The Kerr metric in terms of Boyer-Lindquist co-ordinates is given by

$$ds^2 = \frac{\rho^2}{\triangle} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2Mr}{\rho^2} a^2 \sin^4\theta\right] d\phi^2$$
$$- \frac{4Mr}{\rho^2} a \sin^2\theta \, d\phi \, dt - \left(1 - \frac{2Mr}{\rho^2}\right) dt^2;$$
$$\triangle = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2\theta. \tag{1}$$

The contravariant $g^{\alpha\beta}$ can be read from

$$g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} = \rho^{-2} \left[\triangle \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \theta^2} + (\sin^{-2}\theta - a^2 \triangle^{-1}) \frac{\partial^2}{\partial \phi^2} \right.$$
$$\left. - \frac{4Mar}{\triangle} \frac{\partial^2}{\partial \phi \partial t} - \{\triangle^{-1}(r^2 + a^2)^2 - a^2 \sin^2\theta\} \frac{\partial^2}{\partial t^2} \right]. \tag{2}$$

The coefficients of affine connection

$$\Gamma^\alpha_{\beta\gamma} = \frac{g^{\alpha\delta}}{2}\left(\frac{\partial g_{\beta\delta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)$$

can be obtained directly. The expressions for the thirty-two non-vanishing $\Gamma^\alpha_{\beta\gamma}$ are lengthy and will not be given here.

The Kerr geometry is stationary and axially-symmetric with horizon at

$$r = r_+ = M + (M^2 - a^2)^{1/2}.$$

For $a = 0$ it reduces to the Schwarzschild geometry. It represents the geometry of a rotating black hole (of mass M and angular momentum $K = Ma$) in the region at and outside the horizon well after the collapse when gravitational radiation has died out. It also approximately describes the external gravitation field of a rotating star.

2.2. *Dirac equation in curved space-time*

The generalization of ordinary flat space Dirac equation to curved space-time is more complicated than the corresponding problem of scalar or vector equations because Dirac spinors are defined relative to Minkowski co-ordinates. One way to handle this situation is to introduce local Lorentzian frames with respect to which the transformation properties of Dirac spinor are known (Boulware *op. cit*). Let $\bar{e}_a(x)$ be a set of orthonormal basis vectors defined at every point of space-time:

$$\bar{e}_a \cdot e_b = \eta_{ab};$$
$$-\eta_{00} = \eta_{11} = \eta_{22} = \eta_{33} = 1. \tag{3}$$

The Lorentz transformation of the local frame is given by

$$\bar{e}_a(x) = \Lambda_a^{\ b}(x)\, e_b(x)$$

with

$$\Lambda_a^{\ b}\, \Lambda_{a'}^{\ b'}\, \eta_{bb'} = \eta_{aa'}.$$

Under infinitesimal Lorentz transformations

$$\psi(x) \to \psi'(x) = \psi(x) + \tfrac{1}{2} i\, \delta\omega_{ab} S^{ab} \psi(x)$$

where

$$S^{ab} = \frac{i}{4}[\gamma^a, \gamma^b].$$

The transformation between the co-ordinate basis vectors $\bar{\omega}_\mu$ and the local vectors e_a is given by

$$e_a(x) = e_a^{\ \mu}(x)\, \bar{\omega}_\mu \tag{4}$$

where $e_a^{\ \mu}(x)$ are called the vierbein components. It is easily seen that

$$e_a^{\ \mu} = \eta_{ab}\, g^{\mu\nu} \frac{\partial \xi^b}{\partial x^\mu} \tag{5}$$

where ξ^b are the locally ineitial co-ordinates.

In terms of these vierbein components it is possible to generalize the ordinary derivative of a spinor to one that is covariant under transformation of the local Lorentzian frame. The details of this procedure are available in the literature (Boulware *op. cit*) and will not be repeated here. The covariant derivative of a spinor ψ is of the form:

$$\bar{\nabla}_\nu \psi = (\partial_\nu + \tfrac{1}{2} i\, S^{ab}\, \omega_{ab\nu})\, \psi \tag{6}$$

where the 'spinor affinity' is given by

$$\omega_{ab\nu} = e_a^{\ \mu}\, e_{b\mu;\nu}$$
$$e_{b\mu;\nu} = \partial_\nu\, e_{b\mu} - \Gamma^\lambda_{\mu\nu}\, e_{b\lambda}. \tag{7}$$

In terms of the covariant derivative, Dirac equation in curved space takes the form

$$\left(\frac{\gamma^a}{i}\, \bar{\nabla}_a + m\right)\psi = 0 \tag{8}$$

where

$$\bar{\nabla}_a = e_a^{\ \mu}\, \bar{\nabla}_\mu. \tag{9}$$

To write eq. (8) explicitly for Kerr space-time we need the vierbein components and spinor affinities for this case. After a straightforward computation, we obtain the results given in tables 1 and 2. Using these, the Dirac equation in Kerr space-time is given by

$$\Bigg[m + \frac{1}{iW}\gamma^0\,\partial_t - \frac{2Mar\sin\theta}{i\rho^2\sqrt{\triangle}\,W}\gamma^3\,\partial_t + \frac{\sqrt{\triangle}}{i\rho}\,\gamma^1\,\partial_r + \frac{1}{i\rho}\,\gamma^2\,\partial_\theta$$
$$+ \frac{W}{i\sqrt{\triangle}\sin\theta}\,\gamma^3\,\partial_\phi - \frac{M - r\left(1 + \frac{\triangle}{\rho^2}\right)}{2i\rho\sqrt{\triangle}}\,\gamma^1$$
$$- \frac{Mar\sqrt{\triangle}\cos\theta}{\rho^5 W^2}\,\gamma^1\,\gamma_5 + \frac{\cot\theta}{2i\rho}\left(1 - \frac{a^2\sin^2\theta}{\rho^2}\right)\gamma^2$$
$$+ \frac{Ma\sin\theta}{2\rho^3 W^2}\left(1 - \frac{2r^2}{\rho^2}\right)\gamma^2\,\gamma_5\Bigg]\psi = 0;$$
$$W = \left(1 - \frac{2Mr}{\rho^2}\right)^{1/2}. \tag{10}$$

It is easily verified that for $a = 0$ this equation reduces to the known Dirac equation in Schwarzschild space-time.

2.3. *The weak-field low-velocity limit*

Consider the solutions of eq. (10):

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} e^{-iEt}. \tag{11}$$

Table 1. Vierbein components $e_a^{\ \mu}$ for Kerr–Newman metric. Here a stands for the Lorentz index and μ for the co-ordinate index. The axes of the local frames are oriented parallel to the co-ordinate axes except for the third direction. For Kerr metric put $Q = 0$ and for Schwarzschild metric $Q = a = 0$.

$a \backslash \mu$	t	r	θ	ϕ
0	$\frac{1}{W}$	0	0	0
1	0	$\frac{\sqrt{\triangle}}{\rho}$	0	0
2	0	0	$\frac{1}{\rho}$	0
3	$\frac{a\sin\theta}{\sqrt{\triangle}}\left(W - \frac{1}{W}\right)$	0	0	$\frac{W}{\sqrt{\triangle}\sin\theta}$

Table 2. Spinor affinity $\omega_{ab\mu}$ for Kerr–Newman metric given as matrix elements $(\omega_\mu)_{ab}$. The unlisted ones may be obtained from the antisymmetry of $\omega_{ab\mu}$: $\omega_{ab\mu} = -\omega_{ba\mu}$

$$(\omega_\mu)_{01} = \frac{\sqrt{\triangle}}{\rho^3 W}[M + r(W^2 - 1)][\delta_\mu{}^0 - a\sin^2\theta\;\delta_\mu{}^3]$$

$$(\omega_\mu)_{02} = \frac{a\sin 2\theta}{2\rho^3}\left(W - \frac{1}{W}\right)[(r^2 + a^2)\,\delta_\mu{}^3 - a\,\delta_\mu{}^0]$$

$$(\omega_\mu)_{03} = \frac{a}{\rho^2 W^2\sqrt{\triangle}}[M\sin\theta\;\delta_\mu{}^1 + (W^2 - 1)(r\sin\theta\;\delta_\mu{}^1 - \triangle\cos\theta\;\delta_\mu{}^2)]$$

$$(\omega_\mu)_{12} = -\frac{\sqrt{\triangle}}{\rho^2}\left[\frac{a^2\sin 2\theta}{2\triangle}\,\delta_\mu{}^1 + r\delta_\mu{}^2\right]$$

$$(\omega_\mu)_{13} = \frac{\sin\theta}{\rho W}\left[\frac{a}{\rho^2}(M - r + rW^2)(\delta_\mu{}^0 - a\sin^2\theta\;\delta_\mu{}^3) - rW^2\,\delta_\mu{}^3\right]$$

$$(\omega_\mu)_{23} = \frac{\sqrt{\triangle}\cos\theta}{\rho W}\left[\frac{a}{\rho^2}(W^2 - 1)(a\sin^2\theta\;\delta_\mu{}^3 - \delta_\mu{}^0) - \delta_\mu{}^3\right]$$

The two-component spinors u and v are found to satisfy the equations:

$$mu - \frac{E}{W}u + \frac{2MaEr\sin\theta}{\rho^2\sqrt{\triangle}\,W}\sigma^3 v + \frac{\sqrt{\triangle}}{i\rho}\sigma^1\partial_r v$$
$$+ \frac{1}{i\rho}\sigma^2\partial_\theta v + \frac{W}{i\sqrt{\triangle}\sin\theta}\sigma^3\partial_\phi v - \frac{M - r\left(1 + \frac{\triangle}{\rho^2}\right)}{2i\rho\sqrt{\triangle}}\sigma^1 v$$
$$+ \frac{\cot\theta}{2i\rho}\left(1 - \frac{a^2\sin^2\theta}{\rho^2}\right)\sigma^2 v - \frac{Mar\sqrt{\triangle}\cos\theta}{\rho^5 W^2}\sigma^1 u$$
$$+ \frac{Ma\sin\theta}{2\rho^3 W^2}\left(1 - \frac{2r^2}{\rho^2}\right)\sigma^2 u = 0 \qquad (12)$$

$$mv + \frac{E}{W}v - \frac{2MaEr\sin\theta}{\rho^2\sqrt{\triangle}\,W}\sigma^3 u - \frac{\sqrt{\triangle}}{i\rho}\sigma^1\partial_r u$$
$$- \frac{1}{i\rho}\sigma^2\partial_\theta u - \frac{W}{i\sqrt{\triangle}\sin\theta}\sigma^3\partial_\phi u + \frac{M - r\left(1 + \frac{\triangle}{\rho^2}\right)}{2i\rho\sqrt{\triangle}}\sigma^1 u$$
$$- \frac{\cot\theta}{2i\rho}\left(1 - \frac{a^2\sin^2\theta}{\rho^2}\right)\sigma^2 u + \frac{Mar\sqrt{\triangle}\cos\theta}{\rho^5 W^2}\sigma^1 v$$
$$- \frac{Ma\sin\theta}{2\rho^3 W^2}\left(1 - \frac{2r^2}{\rho^2}\right)\sigma^2 v = 0. \qquad (13)$$

In the extreme weak-field low-velocity limit $r \to \infty$, $E \to m$ the spinor $v \to 0$. Thus in the far-away region v is the smaller component of ψ as expected. To proceed further, notice that eq. (13) has no derivative term in v. We invert the matrix coefficient of v up to order $1/r^3$ and eq. (13) then gives v in terms of u

and its derivatives up to this order. This solution for v is substituted in eq. (12) and retaining terms up to the leading order, the equation for u is obtained. The actual calculation based on the above procedure is considerably lengthy but straightforward. Thus in the weak-field low-velocity limit Dirac equation for Kerr space-time is found to reduce to

$$(E - m)\,u = -\frac{\nabla^2}{2m}\,u - \frac{mM}{r}\,u + \frac{2Ma}{ir^3}\,\partial_\phi u + \frac{3}{2}\,\frac{Ma\sin\theta}{r^3}\,\sigma^2\,u \tag{14}$$

Eq. (14) is the non-relativistic Schrödinger equation for a spin $\frac{1}{2}$ particle with interaction energies provided by the gravitational field. The first three terms on the right are the same as obtained for the spin zero case (Ford *op. cit*). The last term is a new interaction term characteristic of spin $\frac{1}{2}$ particle.

2.4. *Interaction terms*

The term $(-mM)/r$ is the familiar gravitational Coulomb potential energy. To interpret the other two interaction terms of eq. (14), we note that associated with the rotating mass will be a gravitational vector potential, (Weinberg 1970)

$$\bar{\mathscr{G}} = \frac{1}{r^3}\,(\bar{K} \times \bar{r})$$

where

$$\bar{K} = Ma\,\hat{z}\,. \tag{15}$$

The 'gravitational dipole field' $\bar{G}$ may be defined as

$$\bar{G} = \bar{\nabla} \times \bar{\mathscr{G}}. \tag{16}$$

The components of $\bar{G}$ are:

$$G_r = \frac{2Ma\cos\theta}{r^3},$$

$$G_\theta = \frac{Ma\sin\theta}{r^3}\,,\; G_\phi = 0. \tag{17}$$

It turns out that the dipole field $\bar{G}$ defined above is equal to the precessional frequency of an inertial compass with respect to an inertial frame far away from the rotating mass (Zeldovich and Novikov 1971).

A little manipulation brings eq. (14) in the following form:

$$(E - m)\,u = \left[-\frac{\nabla^2}{2m} - \frac{mM}{r} - 2\bar{G}\,.\,\bar{L} + \bar{G}\,.\,\bar{S} - \frac{2\bar{K}\,.\,\bar{S}}{r^3}\right]u \tag{18}$$

The last term in the above equation is the direct coupling term of spin with the angular momentum of the source. The terms $\bar{G} \cdot \bar{L}$ and $\bar{G} \cdot \bar{S}$ represent the coupling of orbit and spin with the gravitational dipole field $\bar{G}$. In analogy with coupling of spin with magnetic field, we can identify gravitational gyro-factor for spin as:

$$|g s^G| = 1. \tag{19}$$

The gravitational gyro-factor for spin has been earlier discussed by DeOliveira and Tiomno (1962) in the weak-field non-relativistic limit of a gravitational field. Our explicit solution for the Kerr case thus confirms this value.

3. Extension to Kerr-Newman space-time

The Kerr-Newman metric is given by

$$dS^2 = \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2) \sin^2\theta + \frac{2Mr - Q^2}{\rho^2} a^2 \sin^4\theta \right] d\phi^2$$

$$- \frac{2a \sin^2\theta}{\rho^2} (2Mr - Q^2)\, d\phi\, dt - \left(1 - \frac{2Mr - Q^2}{\rho^2}\right) dt^2;$$

$$\Delta = r^2 + a^2 - 2Mr + Q^2 \tag{20}$$

which for $Q = 0$ reduces to the Kerr metric. Q represents the charge of the rotating source. Far from the source the electric and magnetic fields have components (Misner *et al* 1973)

$$E_r = \frac{Q}{r^2}$$

$$B_r = \frac{2Qa}{r^3} \cos\theta$$

$$B_\theta = \frac{Qa}{r^3} \sin\theta \tag{21}$$

which correspond to a four-vector potential A^μ given by

$$A^\mu = \left(\frac{Q}{r}\ ;\ 0, 0, \frac{Qa \sin\theta}{r^2}\right). \tag{22}$$

To write down Dirac equation in Kerr-Newman space-time we use the minimal prescription for electromagnetic coupling in the local Lorentzian frame:

$$\left[\gamma^a \left(\frac{1}{i} \bar{\nabla}_a - qA_a\right) + m\right] \psi = 0 \tag{23}$$

where A_a are the components of the four vector potential along the local axes $\bar{e}_a$. Using $e_a{}^\mu$ and $\omega_{ab\mu}$ for the Kerr-Newman case (tables 1 and 2) the equation for ψ is:

$$\Big[m + \frac{1}{iW}\gamma^0 \partial_t - \frac{a \sin\theta\,(2Mr - Q^2)}{i\rho^2 \sqrt{\Delta}\, W}\gamma^3 \partial_t + \frac{\sqrt{\Delta}}{i\rho}\gamma^1 \partial_r$$

$$+ \frac{1}{i\rho}\gamma^2 \partial_\theta + \frac{W}{i\sqrt{\Delta}\sin\theta}\gamma^3 \partial_\phi - \frac{M - r\left(1 + \frac{\Delta}{\rho^2}\right)}{2i\rho\sqrt{\Delta}}\gamma^1$$

$$- \frac{a\cos\theta\,\sqrt{\Delta}\,(2Mr - Q^2)}{2\rho^5 W^2}\gamma^1\gamma_5 + \frac{\cot\theta}{2i\rho}\left(1 - \frac{a^2\sin^2\theta}{\rho^2}\right)\gamma^2$$

$$+ \frac{a\sin\theta}{2\rho^3 W^2}\left\{M - \frac{r}{\rho^2}(2Mr - Q^2)\right\}\gamma^2\gamma_5 - q\gamma^0 A_0 - q\gamma^3 A_3\Big]\psi$$

$$= 0; \qquad (24)$$

$$W = \sqrt{\left(1 - \frac{2Mr - Q^2}{\rho^2}\right)}.$$

The weak-field low-velocity limit of this equation is obtained by following the same procedure as for the Kerr case. Up to the leading order, the equation for the large component u is:

$$(E - m)\,u = -\frac{\nabla^2}{2m}u - \frac{mM}{r}u + \frac{2Ma}{ir^3}\partial_\phi u + \frac{3}{2}\frac{Ma\sin\theta}{r^3}\sigma^2 u$$

$$- \frac{qA_3}{imr\sin\theta}\partial_\phi u - \frac{q}{2m}\left(\frac{\sigma^1}{r}\partial_\theta A_3 - \sigma^2\partial_r A_3\right)u - qA_0\,u. \qquad (25)$$

The gravitational coupling terms are the same as for the Kerr case; the additional terms are the electromagnetic coupling terms. We note that the local vector $\bar{e}_3$ is aligned in the direction of $\hat{\phi}$ in the asymptotic region. Thus

$$A_3 \simeq A_\phi.$$

It is then seen that the above equation can be put in the form:

$$(E - m)\,u = \Big[\frac{(\bar{p} - q\bar{A})^2}{2m} - \frac{mM}{r} - \frac{2\bar{K}\cdot\bar{S}}{r^3} - 2\bar{G}\cdot\bar{L}$$

$$+ \bar{G}\cdot\bar{S} + q\phi - \frac{q}{2m}\bar{\sigma}\cdot(\bar{\nabla}\times\bar{A})\Big]u \qquad (26)$$

where $\bar{\nabla}\times\bar{A}$ is to be evaluated in the local frame. We thus retrieve the standard electromagnetic gyro-factor :

$$|g_e^{em}| = 2. \qquad (27)$$

The corrections to electromagnetic coupling terms due to gravitation turn out to be of higher order and therefore do not appear in the above equation. The higher order terms also contain the geodesic (Thomson) precession factors (Weinberg *op. cit.*).

4. First-order energy shift over the Schwarzschild ($a = 0$) case

We first need to define the norm of a state of positive energy in Schwarzschild space. For the scalar case (Ford *op. cit*) this is expressed as

$$\langle \Psi_0 \mid \Psi_0 \rangle = \int -g^{oo} \ \psi_0{}^* \psi_0 \sqrt{-g}\, d^3x$$

where the integration over r is from $2M$ to ∞. The inner product defined above is positive definite. The natural generalization to spin $\frac{1}{2}$-case is

$$\langle \Psi_0 \mid \Psi_0 \rangle = \int -g^{00}\, \psi_0{}^+ \psi_0 \sqrt{-g}\, d^3x. \tag{28}$$

It is not difficult to verify that two positive energy solutions $\psi_{E'K'J'M'}$ and ψ_{EKJM} of Dirac equation in Schwarzschild space satisfy the orthogonality relation with respect to the above inner product:

$$-\int g^{00} \psi^{\dagger}{}_{E'K'J'M'}\, \psi_{EKJM} \sqrt{-g}\, d^3x \propto \delta_{E'E}\, \delta_{K'K}\, \delta_{J'J}\, \delta_{M'M}.$$

($EKJM$ refer to the complete set of operators required to specify a Dirac energy spinor). This observation supports the above generalization, eq. (28).

We next obtain the change in Schwarzschild energy levels when the source acquires angular momentum ($a \neq 0$). The hamiltonian of eq. (14) is

$$H = H_0 + \frac{2K}{r^3} L_z + \frac{3}{2} \frac{K \sin\theta}{r^3} \sigma^2 \tag{29}$$

where H_0 is the hamiltonian for Schwarzschild case:

$$H_0\, \psi_0 = E_0\, \psi_0. \tag{30}$$

The first-order energy shift is

$$E - E_0 = \frac{\left\langle \Psi_0 \left| \frac{2K}{r^3} L_z + \frac{3}{2} \frac{K \sin\theta}{r^3} \sigma^2 \right| \Psi_0 \right\rangle}{\langle \Psi_0 | \Psi_0 \rangle}$$

$$= \frac{\int \left(1 - \frac{2M}{r}\right)^{-1} u_0{}^{\dagger} \left[\frac{2K}{r^3} L_z + \frac{3}{2} \frac{K \sin\theta}{r^3} \sigma^2\right] u_0 \sqrt{-g}\, d^3x}{\int \left(1 - \frac{2M}{r}\right)^{-1} u_0{}^+ u_0 \sqrt{-g}\, d^3x} \tag{31}$$

where u_0 is the two-component spinor.

For u_0 that does not vanish at the horizon, the dominant contribution to the integral in eq. (31) comes from $r = 2M$.

$$E - E_0 \simeq \frac{K m_l}{4M^3} + \frac{\int \left(1 - \frac{2M}{r}\right)^{-1} u_0{}^{\dagger} \left(\frac{3}{2} \frac{K}{r^3} \sigma^2 \sin\theta\right) u_0 \sqrt{-g}\, d^3x}{\int \left(1 - \frac{2M}{r}\right)^{-1} u_0{}^+ u_0 \sqrt{-g}\, d^3x} \tag{32}$$

To evaluate the second term in eq. (32) we note that

$$\sigma^2 = \sigma_x \cos\theta \cos\phi + \sigma_y \cos\theta \sin\phi - \sigma_z \sin\theta$$

so that for a spinor χ_{m_s} with z-axis as the axis of quantization

$$\langle \sigma^2 \rangle = -2m_s \sin\theta \tag{33}$$

$$\therefore E - E_0 \simeq \frac{Km_l}{4M^3} - JK\, m_s \int Y^*_{lm_l}\, 3\sin^2\theta\; Y_{lm_l}\, d\Omega$$

where

$$J = \int \left(1 - \frac{2M}{r}\right)^{-1} \frac{R_0^* R_0}{r^3}\, r^2\, dr \Big/ \int \left(1 - \frac{2M}{r}\right)^{-1} R_0^* R_0\, r^2\, dr.$$

Thus

$$E - E_0 = \frac{KM_l}{4m^3} - \frac{Km_s}{4M^3}\left[1 - \int Y^*_{lm_l} P_2\, Y_{lm_l}\, d\Omega\right]$$

$$= \frac{Km_l}{4M^3} - \frac{Km_s}{4M^3}\left[1 - \langle\, lm_l\, 20 \mid lm_l \rangle \langle\, l\, 020 \mid 0\, \rangle\right] \tag{34}$$

where the right side involves Clebsch-Gordon coefficients in an obvious notation.

Equation (34) is the main new result of this work. It describes the 'gravitational Zeeman pattern' of a spin $\frac{1}{2}$ particle in Kerr metric. The first term agrees with that obtained by Ford (*op. cit*) for the scalar case. We see that the degeneracy of Schwarzschild levels with respect to both m_l and m_s is lifted in the presence of the rotating source. Notice that the first term in eq. (34) will give normal Zeeman pattern while the second term typical of spin $\frac{1}{2}$ particle gives rise to unequal (*i.e.*, depending on angular momentum quantum numbers of the levels) energy splittings. We have then obtained the *anomalous Zeeman pattern* due to Kerr gravitational field. This result has come about for much the same reason that gives rise to the familiar Zeeman effect in atomic physics, namely the characteristic way in which spin couples with the external field.

An interesting prediction follows immediately from eq. (34). Since the bracketted term on the right of this equation is non-negative the *energy of a Dirac particle in the presence of a rotating source is lower when its spin is aligned parallel to the angular momentum of the source than when it is antialigned.* Consider the case $m_l = \pm\, l$. This will correspond to a particle roughly localized on the equatorial plane of the source, particularly for large l. Eq. (34) gives for this case

$$E - E_0 = \pm \frac{Kl}{4M^3} - \frac{3Km_s}{4M^3}\frac{l+1}{2l+3} \tag{35}$$

$$\simeq \pm \frac{Kl}{4M^3} - \frac{3Km_s}{8M^3} \quad \text{for large } l. \tag{36}$$

Thus (for a given l) the energy split in the equatorial plane of the source is given by

$$\triangle E \equiv E_{\downarrow} - E_{\uparrow} = \frac{3K}{8M^3} \tag{37}$$

where $\uparrow$ ($\downarrow$) refers to the spin of the particle being parallel (antiparallel) to the angular momentum of the source.

Finally for S-state ($l = 0$) we again have a simple result.

$$E - E_0 = -\frac{Km_s}{4M^3} \tag{38}$$

5. Discussion

Although the analogue of Zeeman pattern in gravitational field has been discussed on general grounds in the literature (*see, e.g.*, Zeldovich and Novikov *op. cit*) the explicit quantum mechanical calculation for Kerr metric reported in this paper is new. This has yielded, among other things, the first-order gravitational anomalous Zeeman effect for a Dirac particle. The actual Zeeman splittings are of course expected to be exceedingly small. As an example we consider a supermassive star of mass $M = 10^9 M_\odot$ and angular momentum $K = 10^{68}$ ergs sec. From eq. (37), the splitting turns out to be of the order of 10^{-18} eV. The results thus have more theoretical interest than immediate observational consequences. There is, however, one context in which the results assume some significance. From eq. (38) it is seen that for sufficiently large angular momentum of the source, the energy E of the particle may turn negative even though E_0 is positive. This is the first qualitative hint that the single particle theory may break down for sufficiently large K. A similar situation obtains for the scalar case (Ford *op. cit*). These indications are consistent with the second quantized treatments in Kerr metric for spin zero (Ford *op. cit*) and massless spin $\frac{1}{2}$ (Unruh 1974) cases which yield unstable vacua exhibiting thereby the failure of ordinary single particle theory. It is thus conceivable that a Kerr black hole will emit massive fermion pairs in addition to other particles and steadily lose its angular momentum. A rigorous discussion of this process requires the formalism of spin half quantum field theory in Kerr metric. An essential step in this direction is the recent work of Chandrasekhar (1976) who has demonstrated the exact separation of Dirac equation in Kerr metric.* The calculation of pair production rate on the basis of these developments will be the subject of a future investigation.

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