Pramāṇa, Vol. 8, No. 5, 1977, pp. 471–477. © Printed in India.

Generalized Bethe-Faddeev wave functions

RAM K TRIPATHI
Tata Institute of Fundamental Research, Bombay 400005

MS received 7 January 1977

Abstract. Bethe-Faddeev wave functions are generalized to N-body clusters. A comparison made between the Bethe approximation and the Day approximation in the three body cluster indicates that the two approximations should be competitive for the higher body clusters.

Keywords. Brueckner theory; many body theory; nuclear matter.

1. Introduction

It was shown by Bethe (1965), Rajaraman (1963), and Rajaraman and Bethe (1967) that the Brueckner-Goldstone expansion should be evaluated in terms of the hole lines as the expansion in orders of the reaction matrix (G) was nonconvergent. The estimates of the 3-body and 4-body cluster calculations (Day 1966, 1969) show that the series is convergent in terms of the hole line expansion in nuclear matter and it does not seem worthwhile to calculate any higher order clusters. On the other hand in denser systems like ^{3}He, it was shown (Ostgaard 1968, 1969) that the series is not converging up to 3-body clusters. It may be necessary to go beyond the 4-body cluster contributions in such systems. The work involved in such calculations will, indeed, be tremendous. The formalism presented here is much simpler than that obtained by using Day's approximation (Day 1966), and it may be informative to use it first before trying the more difficult and elaborate formalism.

We generalize here the Bethe-Faddeev wave functions (Faddeev 1961, Bethe 1965, Rajaraman and Bethe 1967) to an N-body cluster. A comparison is made of the Day (1966) and the Bethe (1965) approximations within the 3-body clusters.

2. Theory

Our basic assumption is the presence of the two body force v_{12}. Then the reaction matrix between interacting particles 1 and 2 is defined (Bethe *et al* 1963) as,

$$g_{12} = v_{12} - v_{12} \frac{1}{e_{12}} g_{12}. \tag{1}$$

We are dropping out the Pauli operator from the above definition. This reference spectrum approximation can certainly be improved upon, particularly for

the 1st and the last interaction propagators. The energy denominator e_{12} is the difference between the particle and the hole energies.

Now consider a system of N nucleons all interacting through a two body interaction. There will be $N(N-1)/2$ pairs of nucleons. For simplicity let us denote these pairs by single indices 1, 2, ..., $P(=N(N-1)/2)$. We define the N-body reaction matrix as follows:

$$T = T_1 + T_2 + \ldots + T_i + \ldots + T_p, \tag{2}$$

where in our notation, T_i means the last interaction is between the pair i. These reaction matrices are connected by the P integral equations defined by,

$$T_i = g_i - g_i \frac{1}{e} \sum_{k \neq i} T_k, \tag{3}$$

where in the above equation, the sum on the RHS excludes the pair i. The energy denominator e is defined in the section 2·1. In analogy with the three body wave functions (Bethe 1965), we define the N-body perturbed wave functions ψ_i using the N-body unperturbed wave function ϕ (plane wave) as,

$$T_i \phi = g_i \psi_i \tag{4}$$

Equation (3) then yields,

$$\begin{aligned} \psi_i &= \phi - \frac{1}{e} \sum_{k \neq i} T_k \phi \\ &= \phi - \frac{1}{e} \sum_{k \neq i} g_k \psi_k, \end{aligned} \tag{5}$$

or equivalently we can define the N-body defect wave functions as

$$\begin{aligned} \phi - \psi_i &= \frac{1}{e} \sum_{k \neq i} g_k \psi_k, \\ &= \frac{1}{e} \sum_{k \neq i} T_k \phi, \\ &= \frac{1}{e} \sum_{k \neq i} \left\{ g_k - g_k \frac{1}{e} \sum_{m \neq k} T_m \right\} \phi, \end{aligned} \tag{6}$$

where the first sum does not include the pair i and there are, therefore, $(P-1)$ terms like $\{\ \}$. The second sum within the curly brackets does not include the pair k but does include the pair i, hence there are $(P-1)$ reaction metrices T_m excluding the reaction matrix T_k. We note that the main approximation used in deriving the wave functions ψ_i is that of neglecting the Pauli operator. This does not seem to be a serious approximation in the higher order cluster calculations, and has been made in the three body clusters by Bethe (1965). Our wave function ψ includes two, three... and N-body wave functions and is not just the N-body cluster wave function. One should pick up only its N-body cluster part while calculating the energy. This point will be elaborated on at a later time.

In order to obtain an analytical expression for these wave functions in terms of the two body wave functions we have to make a few assumptions. Because of its simplicity we make the approximation proposed by Bethe (1965), which states that,

$$\begin{aligned} \frac{1}{e} g_i \psi_i &\simeq \eta(r_i)\psi_i && \text{for two body part} \\ &\simeq \zeta(r_i)\psi_i && \text{for three body part} \\ &\simeq \xi(r_i)\psi_i && \text{for four body part} \end{aligned} \tag{7}$$

which means that the effect of the two body operator $1/e\ g_i$ on a many body wave function is factorizable in terms of its two body part, leaving rest of the wave function undisturbed. This is, as will be shown later, a poor approximation near the hard core of the potential but competes with the more accurate Day (1966) approximation at long distances. We make a further approximation in (7) namely,

$$\zeta = \xi = \ldots \tag{8}$$

This approximation is based on the fact that for higher order clusters the defect wave function is quite small and hence off the energy shell interactions may be equated. Using these approximations, we can obtain an analytical expression for the defect wave functions. From (6) we get,

$$\phi - \psi_i = \sum_{k \neq i} \{\eta_k \phi - \zeta_k (\phi - \psi_k)\} \tag{9}$$

where we have written $\eta(r_k)$ as η_k and similarly for ζ_k. The RHS of (9) consists of $(P-1)$ terms and the term corresponding to i is missing. There are other $(P-1)$ similar equation like (9). So we have a set of P coupled equations, hence the P unknown wave functions $\phi - \psi_i$ can be solved, giving:

$$\phi - \psi_i = \frac{E_1 - \eta_i \{E_2 - (P-1) E_3\}}{E_4 - (P-1) E_5}. \tag{10}$$

In terms of $u_i = 1 - \zeta_i$ the symbols used in (10) are defined as:

$E_1 = \eta_1 u_2 \ldots u_{i-1} u_{i+1} \ldots u_P +$ $(P-2)$ terms obtained by cycling the suffixes $1, \ldots, i-1, i+1, \ldots P$. Notice the pair i is missing from this,

$E_2 = u_1 \ldots u_{i-1} u_{i+1} \ldots u_{P-1} +$ $(P-2)$ terms obtained by choosing $(P-2)$ u's at a time from $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_P$. Again the pair i is left out,

$E_3 =$ Product of all u's excluding u_i.

$E_4 = u_1 \ldots u_i \ldots u_{P-1} +$ $(P-1)$ terms obtained by choosing $(P-1)$ u's at a time from $u_1, \ldots, u_i, \ldots, u_P$. Note the pair i is included here,

$E_5 =$ Product of all u's including u_i.

In order to get some feeling about the nature of the solution of (10) we consider the following limiting case,

$$r_i = r_k = \ldots = r,$$

which has physical meaning only for the three and four particle correlations. Under the above approximations (10) reduces to,

$$\phi - \psi_i = \frac{(P-1)\,\eta\phi}{1+(P-1)\,\zeta}. \tag{11}$$

Since the above approximation leads to

$$\eta_i = \eta_k = \ldots = \eta$$

and

$$u_i = u_k = \ldots = u.$$

When all the distances are smaller than the core, then $\zeta = 1$. Assuming $\eta = 1$ then gives,

$$\psi_i = \frac{1}{P}\phi. \tag{12}$$

This result can be roughly understood as follows. When there are N nucleons within each other cores there will be $P(= N(N-1)/2)$ pairwise repulsive interactions. An uncorrelated treatment will destroy the wave function P times. Since the wave function in reality can be destroyed only once, a correlated treatment will give a contribution only $1/P^{\text{th}}$ of the uncorrelated one, hence the factor $1/P$ in (12).

For small ζ one can expand (11) giving

$$\phi - \psi_i = (P-1)\,\eta\,\phi\,\{1-(P-1)\,\zeta + (P-1)^2\,\zeta^2 + \ldots\} \tag{13}$$

This series represents the N-body cluster contribution order by order. For ζ small, namely $\zeta < 1/(P-1)$, the series is convergent and one may calculate the N-body cluster energy in orders of interaction. This will be true for the long-range part of the force. For $\zeta > 1/(P-1)$ the series does not converge order-by-order although the whole series has a closed contribution given by (11).

2.1. *The energy denominator*

The energy denominator e is given by the difference between the particle and hole energies just after the interaction has taken place. Consider N particles a_1, $a_2 \ldots$, a_N and N holes h_1, h_2, $\ldots$, h_N. The energy denominator e is defined by,

$$e = [E(a_1) + E(a_2) + \ldots + E(a_N) - (E(h_1) + E(h_2) + \ldots + E(h_N))]. \tag{14}$$

Using the reference spectrum method (Bethe *et al* 1963), we have

$$E(a) = \frac{k_a^2}{2m^*} + A_3$$

and

$$E(h) = \frac{k_h^2}{2m^*} + A_1, \tag{15}$$

where m^* is the effective mass. Defining the gap between the particle and the hole spectrum as

$$A_2 - A_1 = (\triangle/m^*)\, k_F^2, \tag{16}$$

one gets,

$$e=(1/2m^*)\,[k_{a_1}^2 + k_{a_2}^2 + \ldots + k_{a_N}^2 - k_{h_1}^2 - k_{h_2}^2 \ldots - k_{h_N}^2 + 2N\Delta\, k_F^2]. \tag{17}$$

If the interacting particles were a_1 and a_2, one can write

$$e = (1/m^*)\,[k_{a_1a_2}^2 + P_{a_1a_2}^2 + \tfrac{1}{2}\,(k_{a_3}^2 + \ldots + k_{a_N}^2) - \tfrac{1}{2}\,(k_{h_1}^2 + k_{h_2}^2 + \ldots + k_{h_N}^2) + N\,\Delta\, k_F^2] \tag{18}$$

where k and P are the relative and centre of mass momenta for the interacting particles. Conservation of momentum and averaging over the angles, leads to (Rajaraman and Bethe 1967):

$$e = (1/m^*)\,[k_{a_1a_2}^2 + \tfrac{3}{4}\,(k_{a_3}^2 + \ldots + k_{a_N}^2) - \tfrac{1}{4}\,(k_{h_1}^2 + k_{h_2}^2 + \ldots + k_{h_N}^2) + N\triangle k_F^2]. \tag{19}$$

One can write it in a reduced form,

$$e = (1/m^*)\,[-\nabla_{a_1a_2}^2 + \gamma^2], \tag{20}$$

with

$$\gamma^2 = \tfrac{3}{4}\,(k_{a_3}^2 + \ldots + k_{a_N}^2) - \tfrac{1}{4}\,(k_{h_1}^2 + \ldots + k_{h_N}^2) + N\triangle k_F^2. \tag{21}$$

In order to obtain some numerical estimate of γ^2 we make the following approximations (Rajaraman and Bethe 1967),

$$\langle k_{h_1}^2\rangle = \langle k_{h_2}^2\rangle = \ldots = 0{\cdot}6k_F^2$$

$$\langle k_{a_3}^2\rangle = \langle k_{a_4}^2\rangle = \ldots = (\pi/2c)^2 \approx 16F^{-2}$$

$$\triangle = 0{\cdot}55 \tag{22}$$

where c is the hard core radius. Using the above approximations

$$\gamma^2 \approx 12\,(N-2) + N\,(0{\cdot}4)\,k_F^2 \tag{23}$$

where N represents the number of excited particles.

2.2. *Comparison of the Bethe and the Day approximations*

In case of the three body clusters $N = 3$ and $P = 3$ our expressions reduce to those given by Bethe (1965). We emphasize that the approximations used to get the Bethe-Faddeev equations in three body clusters are the same as used by him. It is interesting in this case to compare Bethe's approximation (1965) with that of Day (1966). This comparison has been made numerically by Rajaraman and Bethe (1967). We shall in the following make a simple analytical comparison of the two wave functions which bears the same conclusions. This can be done roughly if, we compare the total wave function of both these approximations. One can then write,

$$\psi^D_{\text{tot}} = 1 - \tfrac{1}{2}(Z^D_{12} + Z^D_{23} + Z^D_{31}) \tag{24}$$

and

$$\psi^B_{\text{tot}} = 1 - \tfrac{1}{2}(Z^B_{12} + Z^B_{23} + Z^B_{31}) \tag{25}$$

where $Z = \phi - \psi$ is the defect wave function and the superscripts B and D refer to the Bethe and the Day approximations respectively. With purpose we have retained the notation used by them. Substituting the Bethe's (1965) and Day's wave functions one gets,

$$\psi^B_{\text{tot}} = \frac{\psi^D_{\text{tot}}}{1 - \zeta_{12}\,\zeta_{13} - \zeta_{12}\zeta_{23} - \zeta_{13}\zeta_{23} + 2\zeta_{12}\zeta_{13}\zeta_{23}} \tag{26}$$

In the limit, when the denominator on the RHS goes to unity both the approximations give identical wave functions. This will be true when at least two of the $\zeta \to 0$, which means at least two of the relative distances are very large ($r_{ij} \gg C$). When $(\zeta_{12}\zeta_{13} + \zeta_{12}\zeta_{23} + \zeta_{13}\zeta_{23} - 2\zeta_{12}\zeta_{13}\zeta_{23})$ is smaller than unity, one can expand the denominator. In that case the Bethe approximation gives the Day approximation as its major component. It is only when this expression approaches unity, that the Bethe function has discontinuities. That happens when all the particles are close together and their relative distances are of the order of the hard core. With increasing number of particles (higher N), there is less probability of all the particles getting together, hence the denominator in eq. (26) will be (1 + small factor) and then the two approximations should give similar results. Unfortunately no direct comparison can be made like this for fourth and higher body clusters, since the formalism developed by Day (1969) is different from ours. Our wave functions, as pointed out earlier, have two, three and higher-body components mixed, while Day (1969) separates them out. It is possible, however to extract only the four body component from our wave functions and then make the comparison, but we shall not pursue it here.

3. Conclusions

We have generalized the Bethe-Faddeev wave functions to a system of N particles. The three-body cluster comparison indicates that for higher body clusters the Day approximation and the Bethe approximation should be competitive. It may, therefore, be worthwhile to use the formalism developed here first, because of its simplicity, before trying more elaborate and laborious calculations.

Acknowledgement

The author acknowledges the discussions with the colleagues at the Tata Institute of Fundamental Research, Bombay.

References

Bethe H A 1965 *Phys. Rev.* **138B** 804
Bethe H A, Brandow B H and Petschek A G 1963 *Phys. Rev.* **129** 225
Day B D 1966 *Phys. Rev.* **151** 826
Day B D 1969 *Phys. Rev.* **187** 1269
Faddeev L D 1961 *JETP* (*Sov. Phys.*) **12** 1014
Ostgaard E 1968 *Phys. Rev.* **176** 351
Ostgaard E 1969 *Phys. Rev.* **180** 263
Rajaraman R 1963 *Phys. Rev.* **129** 265
Rajaraman R and Bethe H A 1967 *Rev. Mod. Phys.* **39** 745