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New sum rules for $\pi\pi$ scattering using Roskies' amplitudes

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Abstract. On the basis of crossing symmetry a new set of exact relations involving the $\pi\pi$ partial wave amplitudes have been derived, using Roskies' amplitudes.

Keywords. Roskies' amplitudes; crossing symmetry; $\pi\pi$ sum rules.

In constructing models of low energy pion-pion amplitudes, one approximates these by a finite number of partial waves since it is for partial waves that unitarity is most simple. However, crossing is a property of the full scattering amplitude with apparently complicated consequences for individual partial waves. Balchandran and Nuyts (1968) made the important discovery that if we consider the amplitude in the unphysical Euclidean region then we can impose exact consequences of crossing symmetry on a finite number of partial waves. This observation was utilised by Roskies (1969) and Deo and Patnaik (1971) to establish integral relations involving $\pi\pi$ partial wave amplitudes. The latter authors, in particular, have written down two simple general expressions which readily give the desired set of relations, once the highest partial wave is specified. For example, there are just two relations involving S-waves only and S-waves satisfying these two relations *exactly* are guaranteed to be the $1 = 0$ waves of a crossing symmetric amplitude.

However, when these finite subset of exact equations are put to practical use (Basdevant *et al* 1969, Krinsky 1970, Bonnier and Gauron 1972, Deo and Mohapatra 1975) for constructing models or doing phenomenology they are never satisfied exactly (in the mathematical sense). Thus one is led to many equally acceptable forms of the same partial waves satisfying these relations within a reasonable percentage of violation.+ In order to optimise these seemingly equivalent solutions one uses additional inputs like the mass and width of the Rho meson or certain axiomatic results in the form of inequalities (Martin 1967, Pennington 1971) or searches for other equalities, in the remaining subset, involving higher waves and the waves under consideration. These equalities may be exact or may be suitable approximations in view of the fact that (i) the primary exact

+In fact the minimum violation obtained so far (Deo and Mohapatra 1975) for the fifth Roskies' sum rule is as high as 20%.

relations are only approximately satisfied when put to practical use and (ii) low and intermediate energy pion-pion scattering is dominated by partial waves with $l \leqslant 2$.

In this paper we first derive two sets of exact relations using Roskies' amplitudes (Roskies 1970). Then restricting ourselves to only S, P, D waves, we write down a few usable simple relations.

Crossing symmetry and the symmetry of the Mandelstam triangle for the $\pi\pi$ case leads one to write

$$\iint_{\Delta} ds\, du\, h(s, u) = \iint_{\Delta} ds\, du\, h(u, s). \tag{1}$$

In general $h(s, u)$ can be written as

$$h(s, u) = g(s, u)\, G_k(s, u) \tag{2}$$

where $g(s, u)$ is a polynomial of finite order in s and u and $G_k(s, u)$ is any one of the Roskies' amplitudes given below.

$$G_0(s, u) = (F^0(s, t, u) + 2F^2(s, t, u)) \tag{3}$$

$$G_1(s, u) = \frac{F^1(s, t, u)}{t - u} + \frac{F^1(t, u, s)}{u - s} + \frac{F^1(u, s, t)}{s - t} \tag{4}$$

$$G_2(s, u) = \left[\frac{F^1(s, t, u)}{t - u} - \frac{F^1(t, s, u)}{s - u}\right]\frac{1}{s - t} + \left[\frac{F^1(t, u, s)}{u - s} - \frac{F^1(u, t, s)}{t - s}\right]\frac{1}{t - u} + \left[\frac{F^1(u, s, t)}{s - t} - \frac{F^1(s, u, t)}{u - t}\right]\frac{1}{u - s}. \tag{5}$$

The $F^I(s, t, u)$ are the $\pi\pi$ iso-spin amplitudes obeying the Legendre expansion,

$$F^I(s, t, u) = \sum_{l=0}^{\infty} (2l + 1) f_l^I P_l(\cos\theta). \tag{6}$$

Taking for $g(s, u)$, a typical term like $s^N u^M$, where M and N are integers and noting that in units of $m\pi$

$$u = (4 - s)(1 + z)/2; \quad z = \cos\theta \tag{7}$$

we obtain from eq. (1)

$$\int_0^4 ds\, s^N \frac{(4 - s)^{M+1}}{2^{M+1}} \int_{-1}^{+1} dz\, (1 + z)^M G_k(s, t, u) = \int_0^4 ds\, s^M \frac{(4 - s)^{N+1}}{2^{N+1}} \int_{-1}^{+1} dz\, (1 + z)^N G_k(s, t, u). \tag{8}$$

To evaluate the z integral, we use the following expansion,

$$(1 + z)^M = \sum_{n=0}^{M} a_n P_n(z) \tag{9}$$

where, the coefficients a_n are

$$a_n = (2n + 1)\, 2^M \lambda_{M,\, n} \tag{10}$$

obtained by using the formula (Bateman 1953),

$$\int_{-1}^{+1} dz\, (1 + z)^m P_l(z) = 2^{m+1} \lambda_{m,\, l} \tag{11}$$

where,

$$\lambda_{m,\, l} = \frac{(\Gamma(m+1))^2}{\Gamma(m+l+2)\,\Gamma(m-l+1)}\,. \tag{12}$$

Now using (Bateman 1953),

$$\int_{-1}^{+1} \frac{P_l(z)\, P_n(z)}{(x - z)}\, dz = 2P_l(x)\, Q_n(x);\; l \leqslant n \tag{13}$$

we obtain the following two sets of sum rules corresponding to $G_1(s, t, u)$ and $G_2(s, t, u)$, respectively.

$$\int_0^4 ds\, s^N (4 - s)^M \sum_{n=0}^{M} (2n + 1)\, \lambda_{M,\, n} \left[\sum_{l=0}^{n} (2l + 1)\, A_{l,\, n} + \sum_{l=n+1}^{\infty} (2l + 1)\, B_{n,\, l} \right]$$
$$= \int_0^4 ds\, s^M (4 - s)^N \sum_{n=0}^{N} (2n + 1)\, \lambda_{N,\, n} \left[\sum_{l=0}^{n} (2l + 1)\, A_{l,\, n} + \sum_{l=n+1}^{\infty} (2l + 1)\, B_{n,\, l} \right] \tag{14}$$

$$\int_0^4 ds\, s^N (4 - s)^{M-1} \sum_{n=0}^{M} (2n + 1)\, \lambda_{M,\, n} \left[\sum_{l=0}^{n} (2l + 1)\, A'_{l,\, n} + \sum_{l=n+1}^{\infty} (2l + 1)\, B'_{n,\, l} \right]$$
$$= \int_0^4 ds\, s^M (4 - s)^{N-1} \sum_{n=0}^{N} (2n + 1)\, \lambda_{N,\, n} \left[\sum_{l=0}^{n} (2l + 1)\, A'_{l,\, n} + \sum_{l=n+1}^{\infty} (2l + 1)\, B'_{n,\, l} \right] \tag{15}$$

where,

$$A_{l,\, n} = [(2f_l^0(s) - 5f_l^2(s))\, R_{l,\, n}(b, c) + 3f_l^1(s)\, R_{n,\, l}(-b, c, o)]/3 \tag{16}$$

$$B_{l,\, n} = [(2f_l^0(s) - 5f_l^2(s))\, R_{n,\, l}(b, c) + 3f_l^1(s)\, R_{n,\, l}(-b, c, o)]/3 \tag{17}$$

$$A'_{l,\, n} = \frac{1}{3} \Big[(2f_l^0(s) - 5f_l^2(s)) \Big\{ \Big(1 - \frac{1}{c}\Big) R_{l,\, n}(c) + \frac{1}{c} R_{l,\, n}(o) - R_{l,\, n}(b) \Big\} + 6f_l^1(s) \Big\{ \Big(1 + \frac{1}{2c} + \frac{1}{(b - c)}\Big) R_{l,\, n}(c) - \Big(\frac{1}{(b - c)} + 1\Big) R_{l,\, n}(b) - \frac{1}{2c} R_{l,\, n}(o) \Big\} \Big] \tag{18}$$

$$B'_{n,l} = \frac{1}{3}\Big[(2f_l^0(s) - 5f_l^2(s)) \Big\{\Big(1-\frac{1}{c}\Big) R_{n,l}(c) + \frac{1}{c} R_{n,l}(o)$$

$$- R_{n,l}(b)\Big\} + 6f_l^1(s) \Big\{\Big(1 + \frac{1}{2c} + \frac{1}{(b-c)}\Big) R_{n,l}(c)$$

$$- \Big(\frac{1}{(b-c)} + 1\Big) R_{n,l}(b) - \frac{1}{2c} R_{n,l}(o)\Big\}\Big] \tag{19}$$

$$R_{n,l}(\pm x, \pm y, \pm \ldots) = \pm P_n(x) Q_l(x) \pm P_n(y) Q_l(y) \pm \ldots \tag{20}$$

$$b = -\frac{s+4}{4-s}\,; \quad c = \frac{4-3s}{s-4} \tag{21}$$

and $P_n(x)$ and $Q_l(x)$ are the Legendre functions of the first and second kind.

Symmetry of $G_k(s, t, u)$ makes the relations (14) and (15) trivially true when $M = N$. Thus at the lowest order, putting $M = 0$, $N = 1$, we get,

$$\int_0^4 ds \frac{(5s-8)}{6} [(2f_0^0 - 5f_0^2)(Q_0(b) + Q_0(c)) + 9f_1^1(-Q_1(b)$$

$$+ Q_1(c) - 1) + 5(2f_0^0 - 5f_0^2)(Q_2(b) + Q_2(c))]$$

$$= \int_0^4 ds \frac{4-s}{2} [(2f_0^0 - 5f_0^2)(Q_1(b) + Q_1(c))$$

$$+ 24f_1^1(-bQ_1(b) + cQ_1(c))] \tag{22}$$

$$\int_0^4 ds \frac{5s-8}{6(4-s)} \Big[(2f_0^0 - 5f_0^2)\Big\{\frac{4(2-s)}{4-3s} Q_0(c) - Q_0(b)\Big\}$$

$$+ 18f_1^1 \Big\{\frac{16-7s^2-8s}{4s(4-3s)} Q_1(c) - \frac{4+3s}{4s} Q_1(b) + \frac{s-4}{2(4-3s)}\Big\}$$

$$+ 5(2f_0^0 - 5f_0^2)\Big\{\frac{4(2-s)}{4-3s} Q_2(c) - Q_2(b)\Big\}\Big]$$

$$= \int_0^4 ds \Big[(2f_0^0 - 5f_0^2)\Big\{\frac{4(2-s)}{4-3s} Q_1(c) - \frac{s-4}{4-3s} Q_1(b)\Big\}$$

$$+ 18f_1^1 \Big\{\frac{16-7s^2-8s}{4s(4-3s)} cQ_1(c) - \frac{4+3s}{4s} bQ_1(b)\Big\}$$

$$+ 5(2f_0^0 - 5f_0^2)\Big\{\frac{4(2-s)}{4-3s} cQ_2(c) - bQ_2(b)\Big\}\Big]. \tag{23}$$

Increasing values of M and N, $(M \neq N)$, will lead to other relations among the partial waves. It is hoped that the entire subset of relations obtained from eqs (14) and (15), when suitably approximated, like (22) and (23), will help to pin down the correct form of the lower partial waves. However, the advantage of these relations is that, when necessary, they can be used to enforce crossing symmetry on partial waves of any order.

Here we have used only $G_1(s, t, u)$ and $G_2(s, t, u)$. The corresponding relations for $G_0(s, t, u)$ have been obtained by Deo and Patnaik (1971).

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