Pramāna, Vol. 8, No. 5, 1977, pp. 447-456. © Printed in India.

The asteroidal belt and Kirkwood gaps—II. Kinematical theory

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MS received 2 February 1977

Abstract. We have developed a kinematical theory for the asteroidal belt and Kirkwood gaps from the point of view of stellar dynamics. We have generated the potential that would produce these gaps and have made a spectral analysis study. We have shown that these gaps could be due to spiral tubes of matter in the ecliptic plane as a consequence of differential rotation and spatial interference of density waves. We have also shown that this mechanism could account for depletion of matter from this region.

Keywords. Nonequilibrium; effective potential; spiral tubes of matter.

1. Introduction

A spatial periodicity in the radial direction of asteroidal matter distribution in a heliocentric coordinate system was first discovered by Daniel Kirkwood (1867), and the minima in this distribution are called the Kirkwood gaps, after its discoverer. This has generated a great deal of interest. The problem is essentially this: can a quantisation of orbits (or bunching of orbits and thereby of matter) be obtained in a system wherein only long range attractive gravitational forces exist? We have shown in the previous paper (Pratap 1968; this will be referred to as 1 in this paper) that an analysis of the observed data indicates the source to be due to the spatial interference of two density waves.

The following points are known observationally:

(*a*) The asteroids have a prograde orbit with the semi-major axes a (in AU) in the interval $2 < a < 3 \cdot 8$. Only asteroids brighter than $B(a, o) = 21 \cdot 2$ (*i.e.*, $> 1 \cdot 6$ km in diameter) are observed and their number is $\sim 4 \cdot 8\ (\pm\ 0 \cdot 3) \times 10^5$. The total mass of matter is $2 \cdot 4 \times 10^{24}$ grams ($0 \cdot 4 \times 10^{-3}\ M_{\oplus}$). There could be large number of smaller asteroids which fall outside the detection level.

(*b*) There is a definite tendency for the elliptical orbits to have its major axes point towards Jupiter. This implies that Jupiter has a definite influence on the motion of asteroids besides the Sun.

(*c*) Collisions are frequent, in which accretion of matter, change of shape and fragmentation are all common. Hence the collision processes are very highly involved.

(*d*) Due to collision as well as due to Poynting-Robertson effect, the elements of the orbit get changed continuously and very often they leave their stable orbits and get captured by the other members of the solar system.

The above observations give rise to the following inferences: (i) There could be many more smaller bodies which evade observation. Hence the mass estimate given above could be less than the actual mass. (ii) The control of the planets of the solar system on the asteroidal particle gives a potential field for the particle which is highly time dependent, and hence the usual conservation laws are not preserved. (iii) Again since bound states as well as fragmentation are very common, number conservation of a particular size or mass, cannot be imposed on the distribution function. This clearly shows that, the equilibrium distribution, if at all it exists, should be non-Maxwellian.

The early attempts to solve the problem were mainly based on the three body theory developed by Poincare, de Lawney, etc. (Brouwer 1963), wherein attempts were made to explain the bunching of orbits due to resonance. This theory however does not take into account the collisions. A recent study by Giffern (1973) of the commensurable motion in the asteroidal belt has shown that while gaps do exist at or very near the lower order commensurability ratios, *viz.*, 2:1, 7:3, 5:2 and 3:1, there are maxima at 3:2 and 1:1 (Hilda and Trojan groups). Collision dynamics was thoroughly investigated by Opik (1951), Wetherill (1967) and others in calculating the collision probability but no attempt was made to explain the formation of these ring structures called the jet streams. Alfven (1970) in a series of papers tried to develop the collision theory and invoked the idea of collisional focussing, *viz.*, that if an asteroidal particle outside a stream collides with a particle in the stream, then the particle would again come back to the same point of collision after an orbital motion and will thus be subjected to a large number of collisions before it becomes a member of this belt. This however is at variance with the calculations of Opik (1951) wherein he has shown that there is a probability for the particle to escape completely from colliding with other particles by changes in the orbital elements.

Trulsen (1971 *a, b*) was the first to adopt a kinetic approach to the problem of asteroids and wrote a collisional kinetic equation, with the collision term identical to that of Boltzmann with the difference that he has introduced an inelastic parameter β. For the elastic case ($\beta = 2$), his equation reduces to that of Boltzmann. It is known [Prigogine and Severne 1966, 1968, Pratap 1975] that a system of particles in a self-consistent gravitational field do not attain a Maxwellian distribution since the particles are in a long range field. Hence Boltzmann's kinetic equation having a Maxwellian distribution as solution in the asymptotic time limit and with a Boltzmann's H function as a consequence is not the adequate equation in the case of self-gravitating systems.

Jeffreys (1967) pointed out that the force experienced by an asteroidal particle can be a nongravitational one. The question that comes up is then: can a particle moving in a gravitational field provided by the planets such as Jupiter, Mars, etc., which are themselves in motion around the Sun experience a force which is *apparently* nongravitational in nature? It is this question that we propose to answer in this paper. We pose the question in the following manner. If a system of neutral particles moving in the combined gravitational fields of the Sun and the planets and which are highly collisional in nature, *i.e.*, they can either accrete, or deform or fragment, then under what condition can it exhibit a bunching of orbits? What should be the potential the particle should see so that the orbits are bunched? We propose to answer this question using the techniques developed by Jeans (1961) and Chandrasekhar (1960).

In the next section we propose to formulate the problem. In section 3, we shall give the most general solution of the problem and in the next section we shall discuss some of the particular cases. This approach is distinctly different from the earlier ones and hence throws fresh light on the problem and new directions to look for.

2. Equations of the problem

As we have indicated in the previous section, we shall consider a collisional system and define a one particle distribution function $f(\boldsymbol{r}, \mathbf{v}, t)$ satisfying a kinetic equation

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \nu f \tag{1}$$

wherein $\mathbf{v}$, $\boldsymbol{r}$ are actual velocity and position vectors and $v(t)$ is the collision frequency. Φ is the potential the particle would be seeing and is due to the Sun and the planets of the solar system and also the self-consistent field contributed by the rest of the particles. The latter may be negligible, but we can include that also for the sake of completeness. It may be noted that eq. (1) differs from the *BGK* eq. of plasma physics, in that (1) does not conserve the number density, or momentum or energy. We shall try to determine Φ as a function of r and t from (1). We can also write

$$\nabla^2 \Phi = \int d\mathbf{v} \, (f + F_P) \tag{2}$$

where F_P is the distribution of the planets, which is considered known. We would also like to point out that (1) does not imply a Boltzmann H theorem nor has Maxwellian as a solution in the asymptotic limit in time.

3. Solution of (1)

In solving (1), we shall follow the method of characteristics as developed by Jeans (1961) and Chandrasekhar (1960). We shall assume the existence of a one parametric family of surfaces defined as

$$\Omega \, (\bar{u} + n - \ln f) = \text{constant} \tag{3}$$

where $\bar{u}$ is an arbitrary function of r, $\mathbf{v}$, t. n is a function of time only and is defined as

$$n = \int_0^t \nu(t') \, dt'. \tag{4}$$

The assumption involved in writing (3) is the existence of a function similar to entropy as an integral of the system. As is known, while the system is in a non-equilibrium state the usual entropy is not a constant. Nevertheless we can define an entropy in a nonequilibrium state (Prigogine and Henin 1966). The fact that the system is in a nonequilibrium state is evident by the inclusion of the function of $\bar{u}$ as well as n. Furthermore we have f as a function of space and time.

If we now substitute (3) in (1) we get a defining equation for $\bar{u}$ as

$$\frac{\partial \bar{u}}{\partial t}+(\mathbf{v}\,\nabla)\,\bar{u}-\nabla\Phi\cdot\frac{\partial \bar{u}}{\partial \mathbf{v}}=0. \tag{5}$$

Eq. (5) is a homogeneous differential eq. and is free from the collision frequency ν defined in (1). We now impose an integral condition on (5), *viz.*, that $\bar{u}$ is an arbitrary function of the Schwartzchild ellipsoidal velocity distribution, *i.e.*,

$$\bar{u}=\bar{u}\,(\psi) \tag{6}$$

where

$$\psi=\sum_{i,j} a_{ij}\,(v_i-u_i)\,(v_j-u_j)+\sigma. \tag{7}$$

In the above, the coefficients of the ellipsoid as well as the mean velocity $\boldsymbol{u}$ and the function σ are all functions of space and time but independent of the velocity $\mathbf{v}$. This would imply that Φ, $\boldsymbol{u}$, a_{ij} and σ are all unknown functions to be determined. We imposed the condition (6) because, we imposed a similar condition in the case of interstellar plasma and *predicted* that the magnetic surfaces, *i.e.*, surfaces on which magnetic lines of force as well as current lines lie, are helicoids, with its axis along the spiral axis (Pratap 1968). This was later on verified by Mathewson and Nicolls (1968). We now assume that this distribution could generally be true for matter in a gravitating system.

We reduce the problem to a two dimensional one, assuming that the particles are all confined to the ecliptic. In this approximation (7) can be written in a cartesian coordinate system as

$$\psi=a\,(v_1-u_1)^2+b\,(v_2-u_2)^2+2h\,(v_1-u_1)\,(v_2-u_2)+\sigma \tag{8}$$

where a, b, h, and u_1, u_2, σ are all functions of x, y, and t. We can, following Chandrasekhar (1960), write the explicit expressions for a, b, h, u_1, u_2 and σ as

$$a=\phi^2+\kappa y^2;\qquad b=\phi^2+\kappa x^2;\qquad h=-\kappa xy \tag{9}$$

$$u=\frac{\phi'}{\phi}\,\varpi-\frac{\beta\times\varpi}{\phi^2+\kappa\varpi^2} \tag{10}$$

and

$$\sigma=-\frac{\beta^2\varpi^2}{\phi^2+\kappa\varpi^2}+2\Phi_1\,(\varpi^2/\phi^2)+\text{constant} \tag{11}$$

where $\varpi=(x^2+y^2)^{1/2}$ is the radial distance in a cylindrical coordinate system, and κ and β are absolute constants and ϕ is an arbitrary function of time. We have taken β in the z direction and Φ_1 is an arbitrary function of the arguments specified in (11). We can also write the potential the particle would be seeing as

$$\Phi=-\frac{\phi''}{\phi}\,\varpi^2+\phi^{-2}\,\Phi_1\,(\varpi^2/\phi^2)\,. \tag{12}$$

The average velocity may be compared with that obtained by Trulsen [1971—eq. (34)] and we find that Trulsen's expression can be obtained from (10), if we choose $\kappa=0$ and ϕ a constant, independent of time. This incidently proves that the mean velocity one obtains does not critically depend on the form of the collision integral. The mean velocity $\boldsymbol{u}$ given above consists of a differential velocity term besides the rotational one, which one should expect for any gravitational system. The limitation of this approach lies in the fact that we have the solution in terms of characteristic functions and that we have only surfaces in the configuration space in which the solutions lie; and that again is the strong point in this formulation for one can build models by giving explicit forms for these functions. We shall discuss one such solution in the next section.

4. A model

In this section, we shall construct a specific model by giving a particular form for these functions. We shall choose the function (3) as

$$\bar{u} + n - \ln f = \text{constant}. \tag{13}$$

The relation (13) would give the one particle distribution function as

$$f = Ce^{n+\bar{u}} \tag{14}$$

where C is a constant. The explicit form of f depends on the explicit form of $\bar{u}$ and n. Again from (6), we have

$$\bar{u} = -\psi \tag{15}$$

where ψ is given by (8) with all the parameters defined in (9)–(12). Substituting (15) in (14) and integrating over the velocity variables, we get the density distribution as

$$\rho(r, t) = \frac{\text{const}}{\phi(\phi^2 + \kappa\varpi^2)^{1/2}} \exp\left[n + \frac{\beta^2\varpi^2}{\phi^2 + \kappa\varpi^2} - 2\phi\phi''\varpi^2 - 2\phi^2\Phi\right] \tag{16}$$

Φ being the effective potential as given by (12). If we choose the arbitrary functions in such a way that the exponential function in the above takes the foim of a trigonometric function, then this is identical to the eq. foi the density as given by Lin and Shu [1964—eq. (6) on p. 648].

Some of the general features of the density distribution could be investigated. The function is a maximal in space at $\omega = (\varpi/\phi)$ given by

$$\kappa(1 + \kappa\omega^2)^{-1} - 2\beta^2(1 + \kappa\omega^2)^{-2} + 4\phi^3\phi'' + 4\phi^2\frac{d\Phi}{d\omega^2} = 0 \tag{17}$$

and this maximal is a maxima or minima depending on whether

$$\frac{d^2\rho}{d\omega^2} = (1 + \kappa\omega^2)^{-1/2}\left[-\frac{\kappa(1 - \kappa\omega^2)}{(1 + \kappa\omega^2)^2} + \frac{2\beta^2(1 - 3\kappa\omega^2)}{(1 + \kappa\omega^2)^3} - 4\phi^3\phi'' - 2\phi^2\frac{d^2\Phi}{d(\omega^2)^2}\right] \lessgtr 0 \tag{18}$$

subject to (17). One can obtain various conditions depending on the form of ϕ and Φ.

Case I. ϕ = constant and Φ = constant.

Eq. (17) yields a solution as

$$\omega^2 = \frac{1}{\kappa}\left(\frac{2\beta^2}{\kappa} - 1\right) \tag{19}$$

and substituting this in (18), we get

$$\frac{d^2\rho}{d\omega^2} = (\kappa/2\beta^2)^{1/2}\,[(\kappa/2\beta^2)\,2\kappa\,(1-2\beta^2/\kappa)]. \tag{20}$$

Since (19) is the radial distance, it has to be positive definite, and this implies that $2\beta^2/\kappa > 1$. This condition makes (20) negative definite provided κ and β are both real and positive. Thus the density will have a maximum at the distance ω given by (19). If on the other hand κ is negative, then ω in (19) is positive definite, but the function (16) becomes imaginary. Since ρ is a density function, which is real and positive definite, the negative κ is inadmissible.

Case II. ϕ = a function of time, but $d\Phi/d\omega^2 = 0$.

In this case, we are retaining the time dependence of ϕ thereby including a time scale, but take the potential to be constant. The relation (17) then reduces to

$$\kappa\,(1+\kappa\omega^2)^{-1} - 2\beta^2\,(1+\kappa\omega^2)^{-2} + 4\phi^3\phi'' = 0. \tag{21}$$

The solution of (21) which is physically relevant can be written as

$$1+\kappa\omega^2 = -\frac{\kappa}{8\phi^3\phi''} + \left[\left(\frac{\kappa}{8\phi^3\phi''}\right)^2 + \frac{\beta^2}{2\phi^3\phi''}\right]^{1/2} \tag{22}$$

and the relation (18) after some manipulations is given by

$$\frac{d^2\rho}{d\omega^2} = -\frac{4\kappa\omega^2}{(1+\kappa\omega^2)^{1/2}}\,[2\phi^3\phi'' + \beta^2/(1+\kappa\omega^2)] \tag{23}$$

(22) should be positive definite and the right hand side should be greater than unity. This would make (23) negative definite and hence will give a maxima. Eq. (23) can also alternate sign depending on the function ϕ and ϕ''. If ϕ is a harmonic function in time, then (22) can be positive definite, while (23) can be either be positive or negative. We thus get a series of maxima and minima depending on the time dependent function ϕ. Thus we do get a temporal periodicity in the density distribution in a time dependent problem which was absent in case I wherein we considered a time independent situation.

Case III. In the general case, the result depends on the functional form of Φ. We shall discuss this case in the next section, wherein we shall consider in detail the potential function Φ.

The following points may be noted: The density is an explicit function of time through $n(t)$. n has been introduced through the collision frequency as defined by the integral relation (4). Since the collision frequency $\nu(t)$ is time dependent, n defines a time scale through the integral relation. The arbitrary function ϕ is also a constant of integration and hence define another time scale. Hence the problem involves essentially two time scales.

5. Kirkwood gaps

We have in I shown that Kirkwood gaps are due to the interference of two density waves in a circular disc of matter around the Sun in the ecliptic plane. This disc may be rotating and probably with a differential rotation as given by the mean velocity **u** (10). We shall now determine the potential which a particle in the asteroidal belt would be seeing. We can do this by inverting eq. (16) and write

$$\Phi = \frac{1}{2\phi^2}[n(t) + \beta^2 \varpi^2/(\phi^2 + \kappa\varpi^2) - 2\phi\phi''\varpi^2$$
$$- \ln\phi\,(\phi^2 + \kappa\varpi^2)^{1/2} - \ln\rho\,(\varpi^2)]. \quad (24)$$

The most dominant term in ϖ in (24) is the third term as ϖ increases. We have considered the case when ϕ = constant and have plotted in figure 1 the function

$$\Phi = [\beta^2/\kappa\,(1 + \kappa\omega^2) + \ln(1 + \kappa\omega^2)^{1/2} + \ln\rho\,(\varpi)] \quad (25)$$

using the observed distribution for $\rho\,(\varpi)$ as given in I.

One can see clearly that the gaps occur in two distinct domains and the orbits between $2\,\text{AU} < \varpi < 3{\cdot}2$ AU are more stable than the orbits occurring in the outer region. This implies that while the two bunches of particles are correlated the particles from the outer rings when fall into one of the inner rings, they become more stable, while the probability of a particle from the inner ring escaping to the outer one is very low. Furthermore particles from the outer rings can escape faster, thereby matter gets depleted from the system. This feature is amply reflected in the auto-correlation function plotted in I. As we have already mentioned, if the rings are formed by a spiral tube, then the particles are more stable as they are in the inner spiral, but as they come out, become less stable and could finally escape from the system. Thus there could be a critical region along the length of the spiral (at about 3·2 AU from the centre) wherein the stability changes from more stable to less stable situations. This could probably explain the leakage of matter from the system and this could happen due to collision.

In figure 2 we have plotted the Fourier vectors of the curve in figure 1. We find a great deal of difference between the corresponding plot for the density given in I. The most dominant vector is indeed $n = 1$. The vector however is in the third quadrant with a phase of 230° instead of in the first quadrant with phase of

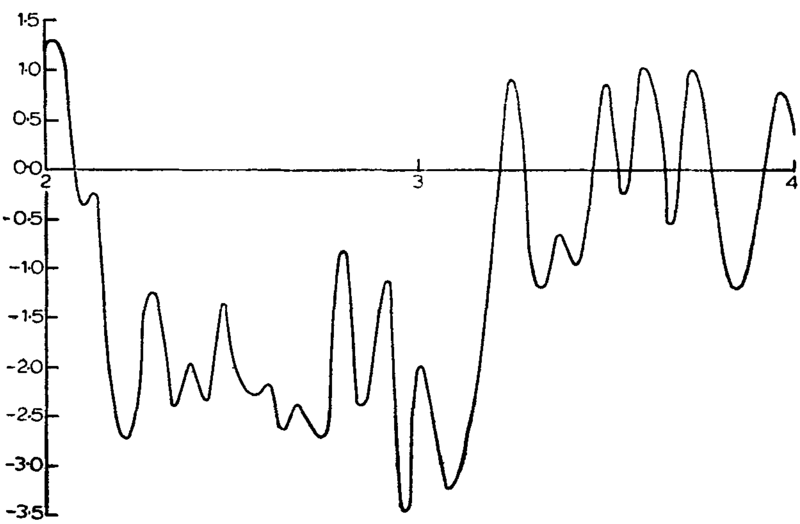

Figure 1. A plot of the potential function given by eq. (25). The x-axis gives the radial distance $\omega\,(= \bar{\omega}/\phi)$, and the y-axis the amplitude. The potential wells correspond to the maxima of the density distribution function. The region between 2·1 AU and 3·2 AU is more stable as compared to the other region.

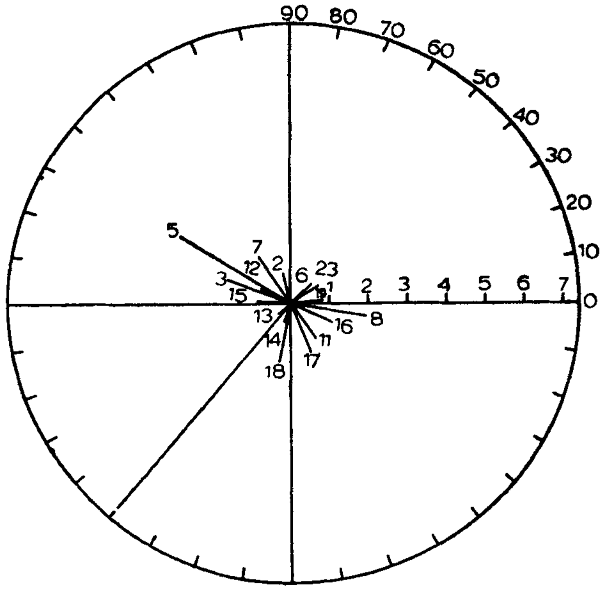

Figure 2. A polar plot of the Fourier components of Φ as given in figure 1. The numbers against each vector represent the value of n appearing in the Fourier series.

50°. There is thus a phase shift by π for this vector between ρ and Φ. Furthermore the vacant region in the Fouiier space are now $50° < \phi < 100°$ and $180° < \phi < 220°$. While there is a general turning of the circular pattern, the vectors are more or less distributed evenly in the two segments. This is primarily because, we have other terms in Φ besides $\ln \rho$ and these terms also have a deciding influence.

One can also see that the two domains we have mentioned here reflects the nature of the potential curve given in figure 3.

In figure 3 we have plotted the power spectrum (PS) based on the MEM. One can see that the maximum frequencies observed here correspond to those observed in the case of ρ as well. Here again we find that the power is concentrated in the lower modes and show the predominance of the non-Marcovian feature. In table 1 we give the frequencies in the degree of dominance and the corresponding power.

We have also given the corresponding wavelengths and the number waves in the interval. We did not do an MEM computation based on the Burg method which would have given a much sharpei frequency estimate.

In the above formulation, we have inverted the density distiibution function given by (16) to obtain Φ. On the other hand if we use the Poisson's equation given in (2) we can then write

$$\nabla^2 \Phi = \frac{4\pi}{\phi(\phi^2 + \kappa\tilde{\varpi}^2)^{1/2}} \exp\left[n + \frac{\beta^2\tilde{\varpi}^2}{\phi^2 + \kappa\tilde{\varpi}^2} - 2\phi\phi''\tilde{\varpi}^2 - 2\phi^2\Phi\right]. \quad (26)$$

Thus we come to the conclusion that these gaps are due to two systems of density waves travelling in the matter disc in the ecliptic. Eq. (26) may be considered as defining eq. for the potential function Φ and we clearly see that on the LHS

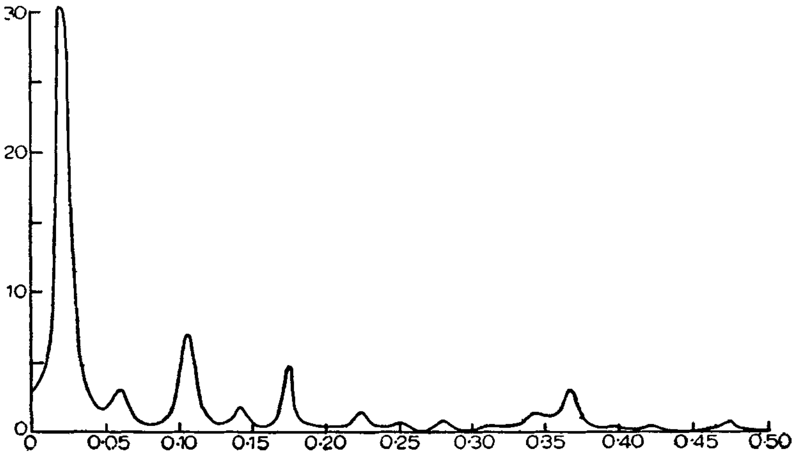

Figure 3. A PS plot based on the MEM giving the dominant frequencies. The major frequencies are the same as that of ρ in figure 1. However there is a phase difference of π. The x-axis gives the frequency and the y-axis the power.

Table 1. Table gives the frequency and the corresponding power. The third column gives the absolute value of the wave vector, and the fourth, the wave length. The fifth column gives the number of waves in the interval. These are the numbers against the vectors in figure 2.

Frequency	Power	κ	λ	n
0·020	33	0·5	2 AU	1
0·105	6·9	2·626	0·381 AU	5
0·175	4·6	4·375	0·23 AU	9
0·060	3·0	1·5	0·67 AU	3
0·367	2·9	9·175	0·108 AU	18

we have a Laplacian on Φ while on the RHS the function appears in the exponent. This is precisely what one gets in the nonlinear Vlasov equation when we integrate the eq. by the method of characteristics. It is also significant, that the collision frequency $n(t)$ also appears in the exponent, which will modify the potential function as a function of time. Thus we see explicitly the role of the collision frequency in the definition of the effective potential.

6. Conclusions

The main questions we have raised in this paper and tried to answer are the following.

(1) Starting with the observed matter distribution, we generated the effective potential function, and have shown that the potential function is indeed an explicit as well as implicit function of time. (2) We have shown that the phenomena of gaps is indeed due to spatial interference of density waves. (3) We have shown

that the PS of the potential function also shows the non-stationarity and hence the phenomena is either quasilinear or nonlinear in space, and furthermore is non-Marcovian. Hence the macroscopic description essentially is given by a non-Marcovian differential equation and not by the usual hydrodynamic equations which are Marcovian. (4) This analysis also shows that a three body approach to this problem is not sufficiently adequate to explain the feature and that any single particle picture will leave many of the essential features unexplained. The role played by the collective effects are very important, and this manifests itself in the small but significant cumulative effects which build up in time. (5) The non-Marcovian effect is also exhibited in the appearance of more than one time scale in the problem.

The analysis is weak in some respects. Since this has been carried out mainly by determining the characteristic functions, there is a great deal of arbitrariness in the case of any one specific problem. This however can be considered as a strong point as well. We can generate a great variety of models using this solution by giving particular forms for these functions. It may be pointed out that the choice made in the model is not unique.

Acknowledgements

I would like to thank W H Jeffereys for initiating me into this fascinating problem while I was at the University of Texas at Austin in 1968. I am deeply indebted to Suhasini Ravipati for many useful discussions I had during the process of generating the potential, and the statistical analysis of the result. I am also thankful to Ranna A Patel and Varsha Prabhavalkar for helping me with computer programmes.

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