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Korteweg - de Vries solitons with different co-ordinate stretchings

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Abstract. The differences between the soliton solutions of the K–dV equation for a homogeneous, collisionless plasma, consisting of cold ions and isothermal electrons arising due to the two different sets of stretched co-ordinates have been discussed. In particular, the differences between the amplitudes and the widths of the solitons and their variations with the soliton velocity have been indicated. Further, the experimental implications of these differences and also of the two sets of stretched co-ordinates have been discussed.

Keywords. Solitary wave; soliton; stretched co-ordinates; homogeneous plasma; amplitude; Mach number.

1. Introduction

The ion acoustic solitary waves in a homogeneous collisionless plasma consisting of cold ions and isothermal electrons have been studied by a number of workers (Zabusky and Kruskal 1965; Gardner *et al* 1967; Hirota 1971; Jeffry and Kakutani 1972). The Korteweg–de Vries equation which describes their propagation has for its independent variables certain stretched co-ordinates and is derived from the basic set of fluid equations using reductive perturbation method (Washimi and Taniuti 1966). There exist, however, two different sets of stretched co-ordinates in the literature which both give the same K–dV equation in the two sets for the homogeneous medium.

It has been stated (Asano 1974) that while one set is appropriate to study the propagation in spatially inhomogeneous media, the other set is appropriate for the study in temporally varying (spatially homogeneous) media. The precise role of these two stretchings is, however, still not entirely clear. In particular, if we have a spatially homogeneous medium which is also time independent, we may ask whether the solitary waves governed by the K–dV equations resulting from the two different stretchings have the same or different propagation characteristics in the (x, t) space. This question becomes specially important when one needs to compare the results of an experiment on soliton propagation in a homogeneous medium with theory. One will have to examine carefully which of the two solitons described by the two K–dV equations (corresponding to the two stretchings) is to be identified with the experimentally observed soliton.

It is somewhat surprising that this simple question has not, to our knowledge, been explicitly considered. In fact, in this paper, we find that the two different solitons do indeed have different propagation characteristics in the (x, t) space and correspond from an experimental point of view to two different modes of launching the solitons.

2. The K–dV equation in two different stretched co-ordinates

The basic set of equations required to derive the K–dV equations for the weakly nonlinear ion-acoustic waves in a homogeneous, collisionless plasma consisting of cold ions and isothermal electrons is,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nV) = 0 \tag{1}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{\partial \phi}{\partial x} = 0 \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2} - e^{\phi} + n = 0 \tag{3}$$

where n is the ion density, V is the ion fluid velocity, ϕ is the electrostatic potential, and x and t are space and time co-ordinates, all the quantities being suitably normalized with respect to plasma parameters, plasma density, ion acoustic velocity, a characteristic potential (KT_e/e), electron Debye length, ion plasma period respectively (*see*, for instance Davidson 1972).

The two sets of stretched co-ordinates discussed in the literature (Washimi and Taniuti 1966; Davidson 1972) are

$$\left.\begin{aligned} \xi &= \epsilon^{1/2}(x - t) \\ \tau &= \epsilon^{3/2}\, t \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} \xi &= \epsilon^{1/2}(x - t) \\ \eta &= \epsilon^{3/2}\, x \end{aligned}\right\}. \tag{5}$$

If we carry out the reductive perturbation analysis for eqs (1)–(3) using the co-ordinates (4) and (5), we get the following equations respectively (Davidson 1972; Washimi and Taniuti 1966).

$$\frac{\partial \phi_1}{\partial \tau} + \phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi_1}{\partial \xi^3} = 0 \tag{6}$$

$$\frac{\partial \phi_1}{\partial \eta} + \phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi_1}{\partial \xi^3} = 0\ . \tag{7}$$

These two equations are identical in form and differ only in the interchange of τ and η variables.

3. Solution of the K-dV equations for the two sets of stretched co-ordinates

We know that both eqs (6) and (7) admit soliton solutions with a constant velocity u in (x, t) space. To afford a proper comparison of the propagation characteristics of the two solitons described by the two eqs (6) and (7) we must obtain soliton solutions corresponding to the same soliton velocity u in the (x, t) space in both cases. We thus look for solutions for ϕ_1, which depend on the variables x and t through a variable $Z = \epsilon^{\frac{1}{2}} (x - ut)$. The factor $\epsilon^{\frac{1}{2}}$ ensures the large width of the solitons.

Using the definitions of ξ and τ, and ξ and η we find

$$Z = (\xi - a\tau); \quad a = (u - 1)/\epsilon \tag{8}$$

corresponding to the stretching (4) and eq. (6), and

$$Z = (u\xi - a\eta); \quad a = (u - 1)/\epsilon \tag{9}$$

corresponding to the stretching (5) and eq. (7).

We now assume the solution $\phi_1 = \phi_1 (Z)$ which, using eq. (8) becomes

$$\phi_1 (Z) = \phi_1 (\xi - a\tau).$$

It can be easily shown that the stationary solution of (6) subject to the boundary conditions

$$\phi_1, \frac{d\phi_1}{dZ}, \frac{d^2\phi_1}{dZ^2} \to 0 \text{ as } |Z| \to \infty \tag{10}$$

is given by

$$\begin{aligned} \phi_1 &= 3a \cdot \sec h^2 \left[\sqrt{\frac{a}{2}} \; (\xi - a\tau) \right] \\ &= \frac{3(u-1)}{\epsilon} \cdot \sec h^2 \left[\sqrt{\frac{u-1}{2}} \; (x - ut) \right] \end{aligned} \tag{11}$$

Similarly, the solution of eq. (7) in the form $\phi_1 (Z) = \phi_1 (u\xi - a\eta)$ subject to the same boundary conditions (10) is given by

$$\begin{aligned} \phi_1 &= \frac{3a}{u} \sec h^2 \left[\sqrt{\frac{a}{2u^3}} \; (u\xi - a\eta) \right] \\ &= \frac{3(u-1)}{\epsilon u} \cdot \sec h^2 \left[\sqrt{\frac{u-1}{2u^3}} \; (x - ut) \right] . \end{aligned} \tag{12}$$

4. Comparison of the two solitons

Comparing the two solutions (11) and (12), we obviously note that while the two solitons propagate with the same velocity u, the dependence on u of their widths and amplitudes is quite different. The amplitude and width functions are compared below for the two cases in figures 1 and 2 which show considerable differences between the two solitons, particularly for Mach numbers of the order of 1·4. However, since the upper limit on the amplitude of an ion-acoustic soliton is of the order of 1·3, and the corresponding limit on the Mach number is of the order 1·5 (Leontovich 1966), the comparison beyond the value $u \approx 1 \cdot 5$ is not relevant.

There is another important difference between eqs (6) and (7) and their solutions which does not show up in the stationary form of their solutions (11) and (12). Here the identity of the x and t variables is somewhat lost because of

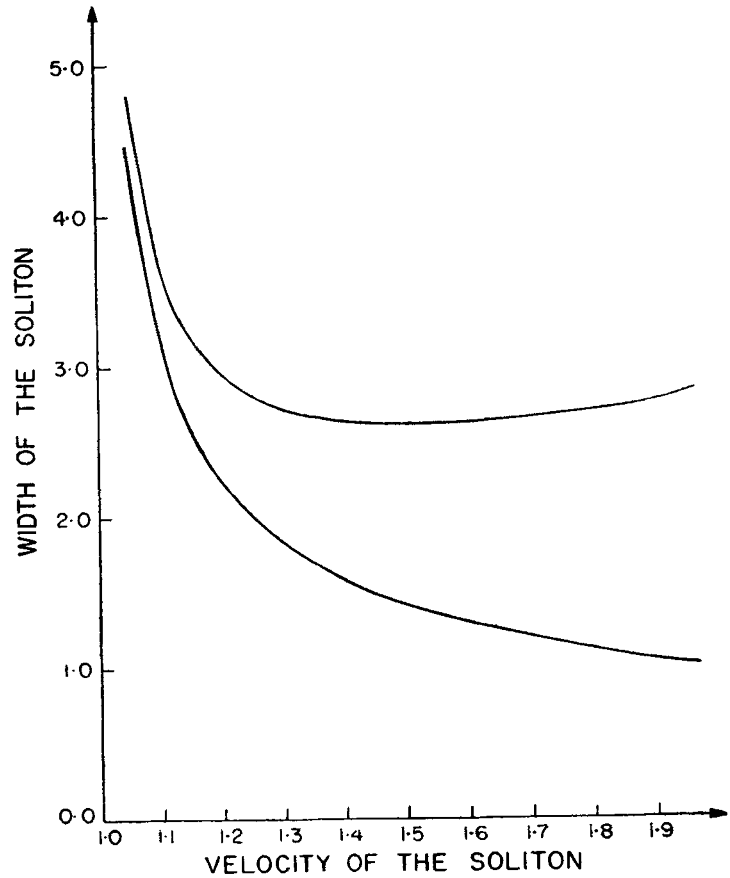

Figure 1. Comparison of the widths of the two solitons given by eqs (11) and (12). The upper curve is for the soliton given by eq. (12) and the lower one is for that of eq. (11).

the solitons in the soliton frame being stationary. The real difference between eqs (6) and (7) appears when we transform them to the x and t variables. From eqs (4) and (5), the inverse transformations are

$$\left.\begin{aligned} x &= \epsilon^{-1/2}\,\xi + \epsilon^{-3/2}\,\tau \\ t &= \epsilon^{-3/2}\,\tau \end{aligned}\right\} \tag{13}$$

and

$$\left.\begin{aligned} x &= \epsilon^{-3/2}\,\eta \\ t &= \epsilon^{-3/2}\,\eta - \epsilon^{-1/2}\,\xi \end{aligned}\right\} \tag{14}$$

Using (13) and (14), we transform eqs (6) and (7) to the (x, t) variables and obtain respectively

$$\frac{\partial\phi_1}{\partial t} + \left[\frac{\partial\phi_1}{\partial x} + \epsilon\phi_1\,\frac{\partial\phi_1}{\partial x}\right] + \frac{1}{2}\,\frac{\partial^3\phi_1}{\partial x^3} = 0 \tag{15}$$

$$\frac{\partial\phi_1}{\partial x} + \left[\frac{\partial\phi_1}{\partial t} - \epsilon\,\phi_1\,\frac{\partial\phi_1}{\partial t}\right] - \frac{1}{2}\,\frac{\partial^3\phi_1}{\partial t^3} = 0 \tag{16}$$

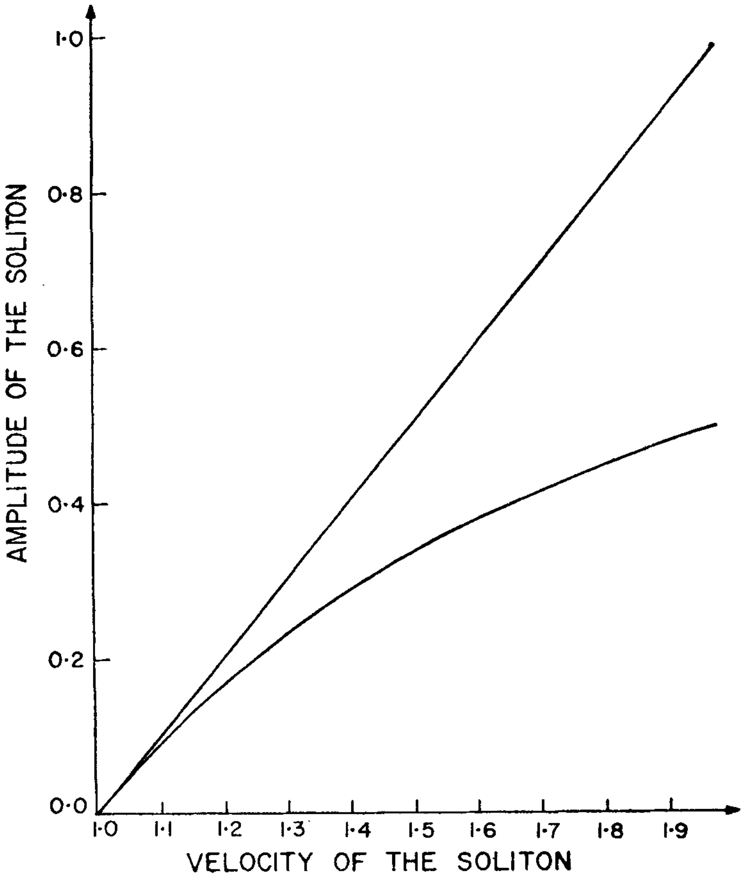

Figure 2. Comparison of the amplitudes of the two solitons given by eqs (11) and (12). The straight line is for the soliton given by (11) and the curved line for that of (12).

A comparison between eqs (15) and (16) shows, except for some signs, the interchange of the role of the x and t variables between the two equations. It is clear that these two equations will give entirely different time evolution of some initial pulse.

Equation (15) involves a term with a first order time derivative of ϕ_1. As an initial value problem then, one would need to specify as an initial value the values of ϕ_1 at all points of space. If we, therefore, have a spatial pulse of ϕ_1, the soliton that will result should correspond to eqs (4), (11) and (15). On the other hand, eq. (16) involves a third order derivative in time, and to solve this as an initial value problem, one would need to specify the function ϕ_1, its first and second time derivatives at some initial time. Such a situation would correspond to a temporal pulse and the soliton that results should correspond to eqs (5), (12) and (16).

From an experimental point of view, it is much simpler to have a temporal pulse than a spatial one. Thus the solitons obtained experimentally from a temporal pulse (for instance John and Saxena 1976) should be compared for their propagation characteristics with the solitons that are derived from eqs (5), (12) and (16).

Detailed experiments have not been carried out to check the propagation characteristics of the above soliton (that is, the variation of the width and amplitude with the velocity u). It would be interesting to do so. Furthermore, it would also be interesting to study the spatial pulse soliton and check its propagation characteristics against our theoretical results.

5. Conclusions

We have shown that even though the K-dV equation in the stretched co-ordinates for a homogeneous, collisionless plasma consisting of cold ions and isothermal electrons is the same for the two different sets of stretched co-ordinates, the two solitons do differ considerably in their propagation characteristics in the (x, t) space. The differences are quite considerable for Mach numbers of the order of 1·4. Further, we have discussed the experimental implications of the two sets of stretched co-ordinates and show that they correspond to two different ways of launching the solitons in the actual experiments.

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