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Improved unitarity bounds for inclusive reactions of particles with spin

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Abstract. A comprehensive discussion of the unitarity bounds on inclusive angular distributions, given the elastic amplitudes and the integrated inelastic cross section, is presented. The role played by the latter constraint in improving the bound is studied in detail. Calculations are made for $\pi^{\pm}p \rightarrow p$ (inclusive) at 6 GeV/c.

Keywords. Unitarity bounds; angular distribution; inelastic cross section; spin.

1. Introduction

The last few years have seen a systematic exploration of the constraints imposed on inelastic and inclusive distributions by the principle of unitarity through a knowledge of the corresponding elastic amplitudes. These constraints are in the form of bounds and, since they are derived without use of analyticity properties of the amplitudes, they are, generally, valid at all energies. The basic bound is on inclusive angular distributions. A number of recent refinements include a completely rigorous treatment of the spins of all particles involved (Divakaran and Vengurlekar 1976, referred to as I in this paper; this also gives references to previous work) and the imposition of additional constraints through extra experimental inputs (Divakaran, Kugler and Soffer 1976). The treatment of exclusive inelastic processes proceeds in a very similar fashion and has provided, when the integrated inelastic cross section is also given as an input in addition to elastic amplitudes, bounds which are almost saturated (Divakaran, Kugler and Soffer 1977, referred to as II; Vengurlekar 1977), giving rise to the hope that such bounds can provide stringent unitarity criteria for the choice of elastic particle wave amplitudes (or, equivalently, phase shifts) from out of a number of possible (both in principle and in practice) amplitude sets.

In the present paper we intend to give a comprehensive account of the unitarity bound on inclusive differential cross section when the corresponding integrated cross section is given. The most important new feature is again a strict treatment of particle spins, which is a non-trivial task. It is essential to do this because of the closeness, mentioned earlier, of the bounds with experimental data for some processes (see, especially, Vengurlekar 1977). Another purpose of this paper is to provide essential proofs and demonstrations of some of the facts which were, in earlier work, either justified on physical grounds or only proved sketchily.

The next section contains most of the derivations with some technical details relegated to appendices. In section 3, we compute these bounds for $\pi p \to p$ (inclusive) at 6 GeV/c pion laboratory momentum, for purposes of illustration.

As is natural, we draw heavily upon the earlier work cited above especially refs. I and II. The reader is requested to consult them for concepts, notation and discussions which he may feel are not described in sufficient detail here.

2. The bound

2.1. *Preliminaries and motivation*

We are interested in the process $A + B \to C$ (inclusive) with particles A, B and C having spins S_A, S_B and S_C and helicities m_A, m_B and m_C. The amplitude for the reaction (X_i being any, including multiparticle, state)

$$A + B \to C + X_i\,.$$

has the partial wave decomposition

$$A^C_{i,(m)_i} = \frac{\sqrt{s}}{2k} \sum_{j=\max\{|\lambda|,\,|\mu_i|\}}^{\infty} (2j+1)\, a^j_{i,(m)_i}\, d^j_{\lambda\mu_i}(\theta)\, e^{i(\lambda-\mu_i)\phi} \tag{1}$$

where $(m)_i$ stands for the set (m_A, m_B, m_C, m_i), m_i being the helicity of X_i, namely, the projection of S_i, the angular momentum of the set X_i in its own rest frame, $\lambda = m_A - m_B$ and $\mu_i = m_C - m_i$. For reasons explained in I, nothing is gained by considering the polarisation of the detected final particle C. The inclusive differential cross section of interest to us then is that in which A and B, having helicities m_A and m_B, produce a C of any helicity at angle θ and anything else. This quantity is given by

$$\frac{d\sigma_{m_A m_B}}{d\Omega} = \frac{1}{k^2} \sum_{m_C,\,i,\,m_i} \rho_i \left| \sum_j (2j+1)\, a^j_{i,(m)_i}\, d^j_{\lambda\mu_i}(\theta) \right|^2 , \tag{2}$$

where ρ is a phase space factor.

The problem considered and solved in I is that of finding an upper bound on this quantity as a function of the elastic partial wave helicity amplitudes $a^j_{m_A m_B, m_A m_B}$, through the use of unitarity in the form

$$\mathrm{Im}\, a^j_{m_A m_B,\, m_A m_B} = \sum_{m'_A,\, m'_B} |a^j_{m_A m_B,\, m'_A m'_B}|^2 + \sum_{i,\, m_i,\, m_C} |a^j_{i,(m)_i}|^2 \rho_i + \ldots \tag{3}$$

In writing this equation we have ignored multiplicity factors, *i.e.*, we have assumed that there are no final states with more than one C particle.* Furthermore, we have not written explicitly the contribution of those final states which do not contain a C. Obviously, how good a bound results depends on how small the fraction of such final states (not containing a C) is. A rough way of assessing this

* This can be a bad assumption in certain cases, *e.g.*, when C is a pion in πp or pp collisions. In such cases, eq. (3) has to be replaced by certain unitarity sum rules involving conserved quantities as was done first by Tiktopoulos and Treiman (1972). For details of how the energy conservation sum rule can be used for this purpose, see Vengurlekar (1976). In the practical case considered in this paper, $\pi p \to p$ (inclusive) at 6 GeV/c, the problem does not arise.

fraction is to convert eq. (3) into a statement on cross sections, by multiplying by $(2j+1)$ and summing over j, m_A and m_B:

$$\sigma^{tot} = \sigma^{el} + \sigma^{C} + \sigma' , \tag{4}$$

where the last two terms on the right are the integrated inelastic cross sections for final states containing and not containing a C respectively, all cross sections being for unpolarised particles. In the example of $\pi^- p \to p$ (inclusive) at 6 GeV/c, discussed in detail in section 3, $\sigma^{tot} \sim 30$ mb, $\sigma^{el} \sim 6$ mb and $\sigma^{p} \sim 10$ mb, so that $\sigma' \sim 14$ mb, which is certainly not negligible (more details will be found in sec. 3). To see what impact this can have on bounds let us, as in ref. I, define

$$\alpha^{j}{}_{(m)i} = [\sum_i \rho_i \mid a^{j}{}_{i,(m)i} \mid^2]^{1/2}, \tag{5}$$

$$\beta^{j} m_A m_B = [\sum_{m_i, m_C} (\alpha^{j}{}_{(m)i})^2]^{1/2} \tag{6}$$

and

$$B^{j} m_A m_B, m_A m_B = [\mathrm{Im}\, a^{j} m_A m_B, m_A m_B - \sum_{m'_A m'_B} \mid a^{j} m_A m_B, m'_A m'_B \mid^2]^{1/2} \tag{7}$$

and rewrite (3) as

$$B^{j} m_A m_B, m_A m_B \leq \beta^{j} m_A m_B . \tag{8}$$

The bound in I was derived first as a function of the β's and then rewritten in terms of the B's using certain monotonicity properties. Clearly, the relative value of σ' with respect to σ^{p} is a direct measure of how much information is lost in this replacement.

It is to avoid this weakening of the bound that we fix the quantity $\sigma^{C}{}_{m_A m_B}$ as an additional constraint in this paper. The resulting improvement is quite marked as will be seen in section 3, and is even spectacular in the case of exclusive inelastic reactions (Divakaran, Kugler and Soffer 1975; Vengurlekar 1977).

2.2. *Derivation of the bound (schematic)*

The basic problem is to find an upper bound on $d\sigma_{m_A m_B}/d\Omega$, eq. (2), as a function of $B^{j} m_A m_B, m_A m_B$ and the integrated inclusive cross section

$$\sigma^{C} m_A m_B = 2\pi \int_{-1}^{1} \frac{d\sigma_{m_A m_B}(\theta)}{d\Omega}\, d(\cos\theta)$$

$$= \frac{4\pi}{k^2} \sum_j (2j+1) \sum_{m_i m_C} (\alpha^{j}{}_{(m)i})^2 = \frac{4\pi}{k^2} \cdot \bar{\sigma}^{C} m_A m_B \tag{9}$$

given the unitarity restrictions, eqs (5) to (8). As in ref. I, we start with the preliminary simplification

$$\frac{d\sigma_{m_A m_B}}{d\Omega} \leqslant \frac{1}{k^2} \sum_{m_i m_C} \Big[\sum_j (2j+1)\, \alpha^{j}{}_{(m)i} \mid d^{j}{}_{\lambda\mu_i}(\theta) \mid \Big]^2$$

$$= \frac{1}{k^2} f_{m_A m_B}(\theta). \tag{10}$$

We then maximise $f_{m_A m_B}(\theta)$ considering $a^j_{(m)i}$ as variational parameters, but now also subject to the constraint given by eq. (9), in addition to the unitarity constraints.

Since the formula that expresses the bound as well as those which arise in the intermediate steps are complicated (on account of spins) we shall first present a simplified version of the bound—one which entails a certain loss of information but compensates for it through increased clarity of the logic involved. We shall then quote the form the bound takes in complete generality (we repeat that the closeness of the bound to the data makes every complication that improves the bound worthwhile). We also give a derivation of the general bound in appendix A.

Writing $D^j(\theta) = \underset{m_i, m_C}{\text{Max}} \mid d^j_{\lambda\mu_i}(\theta) \mid$, it follows from (10) that

$$f_{m_A m_B}(\theta) \leqslant [f'(\theta)]^2 = [\sum_j (2j+1)\, \beta^j D^j(\theta)]^2, \qquad D^j(\theta) \geqslant 0. \tag{11}$$

We now maximise $f'(\theta)$, as a function of β^j subject to the constraints

$$\beta^j \leqslant B^j \tag{12}$$

and

$$\tilde{\sigma}^C = \sum_i (2j+1)(\beta^j)^2. \tag{13}$$

The quantities f', $\tilde{\sigma}^C$, D^j, β^j and B^j all still depend on the initial helicities m_A and m_B, which we have suppressed in the interests of easy writing—*i.e.*, we are still deriving a (non-optimal) bound on polarised angular distribution, given polarised input.

Standard variational methods tell us that at an extremum of $f'(\theta)$ the partial waves can be divided into two mutually exclusive subsets Λ and Λ' (which also depend on m_A and m_B, in addition to the angle θ), with

$$\beta^j = \frac{D^j(\theta)}{\lambda} < B^j; \qquad j \in \Lambda \tag{14}$$

and

$$\beta^j = B^j \leqslant \frac{D^j(\theta)}{\lambda}; \qquad j \in \Lambda' \tag{15}$$

where λ is determined, from the constraint (13), to be

$$\frac{1}{\lambda^2} = \frac{\tilde{\sigma}^C - \sum_{j\in\Lambda'} (2j+1)(B^j)^2}{\sum_{j\in\Lambda} (2j+1)(D^j)^2}. \tag{16}$$

Since a necessary condition for an extremum determined in this way to be a maximum is that λ should be positive, this equation determines λ fully, once the sets Λ and Λ' are known (it also guarantees the positivity of the variational β). These sets are, in turn, determined by the unitarity constraints (12) and eq. (14): Λ consists of all j for which

$$\frac{\tilde{\sigma}^C - \sum_{j\in\Lambda'} (2j+1)(B^j)^2}{\sum_{j\in\Lambda} (2j+1)(D^j)^2}\,(D^j)^2 < (B^j)^2 \tag{17}$$

The left side of this inequality itself involves Λ and Λ'; so (17) is a highly implicit criterion for the division of all partial waves into Λ and Λ'. Nevertheless, it is a fact that Λ and Λ' are *uniquely* determined—a proof will be found in appendix B.

The bound itself is then given by

$$f'(\theta) \leqslant \tilde{f}'(\theta)$$
$$= \sum_{j \in \Lambda'} (2j+1)\, D^j(\theta)\, B^j +$$
$$[\tilde{\sigma}^C - \sum_{j \in \Lambda'} (2j+1)(B^j)^2]^{1/2} \times [\sum_{j \in \Lambda} (2j+1)(D^j)^2]^{1/2}. \tag{18}$$

It is to be noted that Λ is necessarily a finite set, as follows from the constraint (13), since $\sum_{j_0}^{\infty} (2j+1)(D^j)^2$ is divergent for any finite j_0. So Λ' is an infinite set.

That (18) is indeed an upper bound is proved in appendix C by the 'direct subtraction method'.

2.3. *Exact treatment of spins*

We go back now to the quantity $f_{m_A m_B}(\theta)$. The variational problem is now more complicated and the solution is given in appendix A. Again there is only one set of $a^j{}_{(m)i}$ as a candidate for a maximum ($\lambda_{m_A m_B} > 0$) which therefore gives an upper bound on $f_{m_A m_B}(\theta)$. This bound is

$$f_{m_A m_B}(\theta) \leqslant \sum_{m_i, m_C} \triangle^2{}_{(m)i} \tag{19}$$

where $\triangle_{(m)i}$ are solutions of the non-linear equations

$$\triangle_{(m)i} = \triangle_{(m)i}\, \Phi_{(m)i}, \tag{20}$$

$$\Phi_{(m)i} = \sum_{j \in \Lambda_{m_A m_B}} \frac{(2j+1)\,[d^j{}_{\lambda\mu_i}(\theta)]^2\, [\tilde{\sigma}^C{}_{m_A m_B} - \sum_{j' \in \Lambda'_{m_A m_B}} (2j'+1)(B^{j'}{}_{m_A m_B, m_A m_B})^2]^{1/2}}{[\sum_{j' \in \Lambda_{m_A m_B}} (2j'+1) \sum_{m_i' m_C'} \triangle^2{}_{(m')i}\, [d^{j'}{}_{\lambda\mu'_i}(\theta)]^2]^{1/2}}$$
$$+ \sum_{j \in \Lambda'_{m_A m_B}} \frac{(2j+1)\,[d^j{}_{\lambda\mu_i}(\theta)]^2\, B^j{}_{m_A m_B, m_A m_B}}{[\sum_{m_i' m_C'} \triangle^2{}_{(m')i}\, [d^j{}_{\lambda\mu'_i}(\theta)]^2]^{1/2}}. \tag{21}$$

The set $\Lambda_{m_A m_B}$ consists of all j for which

$$\frac{\sigma^C{}_{m_A m_B} - \sum_{j' \in \Lambda'_{m_A m_B}} (2j'+1)(B^{j'}{}_{m_A m_B, m_A m_B})^2}{\sum_{j' \in \Lambda_{m_A m_B}} (2j'+1)\,[\delta^{j'}{}_{m_A m_B}(\theta)]^2}\, [\delta^j{}_{m_A m_B}(\theta)]^2 < (B^j{}_{m_A m_B, m_A m_B})^2, \tag{22}$$

$$[\delta^j{}_{m_A m_B}(\theta)]^2 \equiv \sum_{m_i, m_C} \triangle^2{}_{(m)i}\, [d^j{}_{\lambda\mu_i}(\theta)]^2, \tag{23}$$

all other j's belong to $A'_{m_A m_B}$. The proof that for a given m_A, m_B and θ these sets are uniquely determined by (22) proceeds as before (appendix B).

The equations which determine the bound have, not surprisingly, some points of similarity with the corresponding equations of ref. I. In particular, if $A_{m_A m_B}$ were empty, the expression for $\Phi_{(m)i}$ reduces to the corresponding expressions, eq. (15) of ref. I. As in that situation, the set of all possible values of m_i also decomposes into two subsets $S(\theta, m_C)$ and $T(\theta, m_C)$ for fixed values of m_A, m_B and θ (we drop the subscripts m_A and m_B to conform with the notation of I):

$$\triangle_{(m)i} = 0, \qquad \Phi_{(m)i} < 1 \text{ for } m_i \in S\ (\theta, m_C), \tag{24}$$

$$\Phi_{(m)i} \neq 0, \qquad \Phi_{(m)i} = 1 \text{ for } m_i \in T(\theta, m_C), \tag{25}$$

where, in the equation for $\Phi_{(m)i}$, the sums over m'_i in the denominators are to be carried out over the set $T(\theta, m'_C)$. Once the sets S and T are known at a given angle, as well as the sets A and A', eq. (25) is a set of $(2S_C + 1)\, t_C$ equations for the same number of unknown (and non-vanishing) $\triangle$'s [t_C is the number of helicities m_i which belong to $T(\theta, m_C)$ at a given angle]. These equations become especially simple at $\theta=0$. The only non-vanishing $\triangle$ is then $\triangle_{m_A, m_B, m_C, m_B + m_C - m_A}$ and eq. (21) then becomes

$$\Big\{ \Big[\sum_{j \in A_{m_A m_B}} (2j+1)\Big]^{1/2} \Big[\tilde{\sigma}^C_{m_A m_B} - \sum_{j \in A'_{m_A m_B}} (2j+1)\,(B^j_{m_A m_B,\, m_A m_B})^2\Big]^{1/2}$$

$$+ \sum_{j \in A'_{m_A m_B}} (2j+1)\, B^j_{m_A m_B,\, m_A m_B} \Big\} \cdot \Big[\sum_{m_C} \triangle^2_{(m)}\,(\theta = 0)\Big]^{-1/2} = 1,$$

$$(m) \equiv (m_A, m_B, m_C). \tag{26}$$

This is trivially solved for the bound (knowing A and A') which in fact is the same as the bound (at $\theta = 0$) given by the simplified treatment of sec. 2.2.

For a general θ, determining the sets S and T is very complicated. For small nonzero θ, however, we can make use of the properties of the d^j functions to find, as in ref. 1, the sets S and T from the conditions of eqs (24) and (25) and compute the bound. We find that for small θ, we have only to solve a small number of equations $\Phi = 1$.

2.4. *A lower bound on* $\sigma^C_{m_A m_B}$

It is a fact that the bound we have derived is a monotonically increasing function of $\tilde{\sigma}^C_{m_A m_B}$. Even for the simplified bound of sec. 2.2 this is not entirely trivial to show because, even though the *explicit* dependence on $\tilde{\sigma}^C$ has this property, the sets A and A' also depend on $\tilde{\sigma}^C$, see (17). A proof of this fact is given in appendix D.

This has the consequence that our bound can be considered to be also a *lower* bound on $\tilde{\sigma}^C$ given the (experimental) inclusive angular distribution and the elastic amplitudes (ref. II). For the best such bound, we shall of course need the entire angular distribution as input, but partial knowledge of the angular distribution also gives a (not necessarily the best) bound. Examples of such a bound are given in ref. II and in Vengurlekar (1977) (for the exclusive case).

3. Evaluation of the bound and application to a concrete example

3.1. *General remarks*

As in ref. I, we shall assume that all elastic helicity amplitudes which we need for the computation of the bound are exponentials in t near the forward direction so that the partial wave amplitudes are Gaussians in $(j+\frac{1}{2})$. This is an excellent approximation at energies which are not too low, certainly for the case for which we compute the bounds here, namely $\pi^{\pm} p \to p$ (inclusive) at 6 GeV/c pion momentum. We shall therefore use the form $B^j \sim e^{-(j+\frac{1}{2})^2}$ in the following discussion. There is, however, no difficulty of principle in working with arbitrary function or numerical inputs for the B^j, as can arise, *e.g.*, from low energy phase shift analysis (*see* Vengurlekar 1977).

We shall terminate all j sums at a finite j_{max}. The errors introduced by this truncation can be made as small as we wish by choosing j_{max} sufficiently large—see the appendix of ref. I for a detailed discussion of this point.

3.2. *Solution of the equations for the bound*

In spite of the complicated appearance of the determining equations, (20)–(22), the bound is very easily evaluated at $\theta = 0$. Since this is also going to be the basis for the iterative evaluation of the bound at arbitrary angles, we describe the forward case with some care first.

At $\theta = 0$, $\delta^j m_A m_B$ becomes independent of j, $(= \sum_{m_C} \triangle^2_{(m)})$ and so cancels from inequality (22), which becomes

$$\frac{\tilde{\sigma}^C m_A m_B - \sum_{j \in \Lambda' m_A m_B} (2j'+1)(B^{j'} m_A m_B, m_A m_B)^2}{\sum_{j' \in \Lambda m_A m_B} (2j'+1)} < (B^j m_A m_B, m_A m_B)^2, \qquad (j \in \Lambda m_A m_B). \quad (27)$$

In other words, there is a unique (since the left side is a constant and the right side a Gaussian in $j+\frac{1}{2}$) J such that Λ $(\theta = 0)$ consists of all partial waves up to $J-1$ and $\Lambda'(\theta = 0)$, of all partial waves $J, J+1, \ldots j_{max}$. J is then given (not insisting on pedantic accuracy) by

$$\frac{1}{J^2}\left[\tilde{\sigma}^C m_A m_B - \sum_{j=J}^{j_{max}} (2j+1)(B^j m_A m_B, m_A m_B)^2\right] = (B^J m_A m_B, m_A m_B)^2 . \quad (28)$$

As a function of J, the left side of this equation is negative for small values of J (because $\tilde{\sigma}^C m_A m_B < (4\pi/k^2)^{-1} \overset{tot}{\sigma} m_A m_B \equiv \overset{tot}{\tilde{\sigma}} m_A m_B$), positive for large values and decreases asymptotically (for j_{max} large enough) like J^{-2}. The right side is of course a positive Gaussian. So J is necessarily finite. Having determined J, the forward bound is easily calculated from eqs (26) and (28):

$$\sum_{m_C} \triangle^2_{(m)} (\theta = 0) = [J^2 B^J m_A m_B, m_A m_B + \sum_{j=J}^{j_{max}} (2j+1) B^j m_A m_B, m_A m_B]^2. \quad (29)$$

It may be noted that J depends on m_A and m_B. We also remark that removing the $\tilde{\sigma}^C$ constraint is equivalent to replacing $\tilde{\sigma}^C m_A m_B$ by $\overset{tot}{\tilde{\sigma}} m_A m_B - \overset{el}{\tilde{\sigma}} m_A m_B$. Equation (28) then has the solution $J = 0$ and we recover, from eq. (29), the bound of ref. I.

The value of $\sum_{m_C} \triangle^2_{(m)}$ $(\theta \neq 0)$ is now taken as a first approximation to the bound at a small value of $\theta \neq 0$. Dividing both sides of inequality (22) by $[\delta^j{}_{m_A m_B}(\theta)]^2$, and using this first approximant, the left side is again independent of j, while the right side is fully computable. This will determine the first approximants to the sets $\Lambda(\theta)$ and $\Lambda'(\theta)$, which are fed back into eq. (21) to determine the second approximant to $\sum_{m_C} \triangle^2_{(m)}(\theta)$ and so on. Convergence is very rapid.

This procedure works for values of $\theta < \theta_1$, the first transition angle, at which $\triangle_{(m)i}$ is non-vanishing for more than one value of m_i. The calculation of the bound then proceeds in the way described in ref. I, the sets being determind in successive approximations as indicated above.

3.3. *Application to* $\pi^\pm p \to p$ *(inclusive)*

We have already discussed in ref. I the salient points of the parametrisation of the πp partial wave amplitudes at 6 GeV/c, following the amplitude analyses of Halzen and Michael (1971). The new input required here is the value of $\bar{\sigma}^p{}_{m_p}$.

For spin $\frac{1}{2}$–spin 0 scattering, with the detected particle spin summed, the value of σ^p for both initial helicities is the same:

$$\sigma^p = \sigma_+{}^p = \sum_{m=+,-} \sigma^p_{m,+} = \sigma_-{}^p \tag{30}$$

in the notation of ref. I, on account of parity (and rotation) invariance—σ^p is in fact the *unpolarised* cross section for the production of one final proton. This is calculable from a knowledge of σ^{tot}, σ^{el} and $\langle n \rangle$, the average proton multiplicity. Assuming that all final states contain either one proton or none (an excellent approximation for the case under consideration), we must have

$$\langle n \rangle \sigma^{tot} = \sigma^{el} + \sigma^p \tag{31}$$

which determines σ^p. Of course, if a complete knowledge of the cross sections for individual channels is available, that provides a direct determination of the value of σ^p. Thus, for example, Honecker *et al* (1969) have presented partial cross sections for various topologies and reaction channels in 16 GeV/c $\pi^- p$ interactions. In particular, an estimate of σ^p can be obtained from their tables. This, using eq. (31), leads to $\langle n_p \rangle \sim 0{\cdot}55$ at 16 GeV/c. In the absence of similar tabulated information at 6 GeV/c, we make the reasonable assumption that the value of $\langle n_p \rangle$ does not change a great deal as p_{lab} is increased from 6 GeV/c to 16 GeV/c. Taking $\langle n_p \rangle$ to be between 0·5 and 0·6, we obtain from eq. (31) the estimate $\sigma^p \sim$ 9 to 12 mb for 6 GeV/c $\pi^- p$ interactions. This compares well with the value obtained from the curves given by Honecker *et al* (1969) showing variation of the dominant proton producing cross sections in $\pi^- p$ interactions as a function of the momentum of incoming pion. For $\pi^+ p$ also we assume $\langle n_p \rangle$ to lie between 0·5 and 0·6, giving $\sigma^p \sim 7$ to 10 mb at 6 GeV/c.

Using this information, we have computed for small angles the bounds on $(d\sigma/d\Omega)$ for $\pi^\pm p \to p$ (inclusive) at 6 GeV/c (figures 1 and 2). Curves 2 and 3 of figure 1 (figure 2) are the bounds, eq. (18), for $\pi^- p$ ($\pi^+ p$) with $\langle n_p \rangle$ taken to be 0·6 and 0·5 respectively. The expected improvement over these by treating spins more efficiently is evident from curves 2′ and 3′ of figure 1 (figure 2) which are the bounds, eq. (19), for $\pi^- p$ ($\pi^+ p$) again with $\langle n_p \rangle = 0·6$ and 0·5 respectively. To emphasize the role of the additional input of σ^p in improving the results of ref. I,

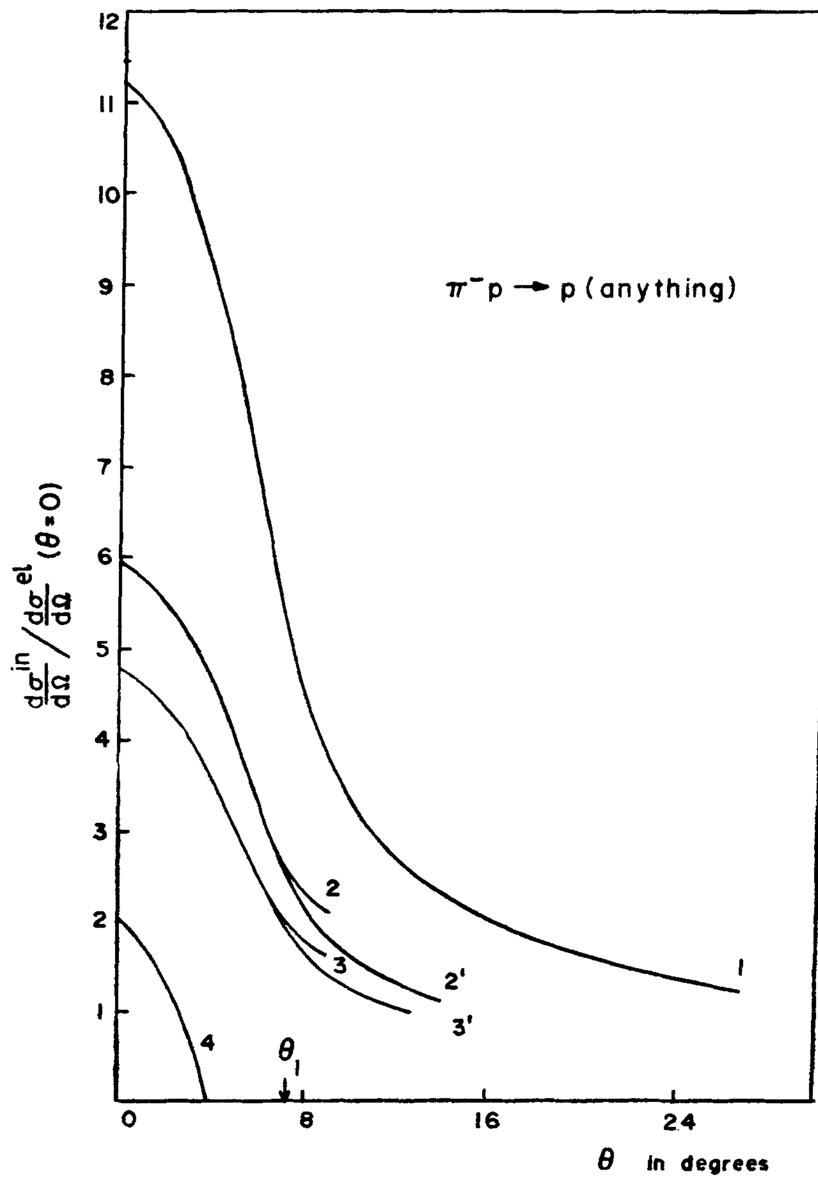

Figure 1. Bounds on the differential inclusive cross section ($d\sigma/d\Omega$) for $\pi^- p \rightarrow p$ (inclusive) at incident pion momentum 6 GeV/c. Curves 2 and 2′ (3 and 3′) are the bounds, eqs (18) and (19) respectively, computed for $\langle n_p \rangle = 0·6$ ($\langle n_p \rangle = 0·5$). For comparison, the upper (curve 1) and the lower (curve 4) bounds of ref. I are also shown.

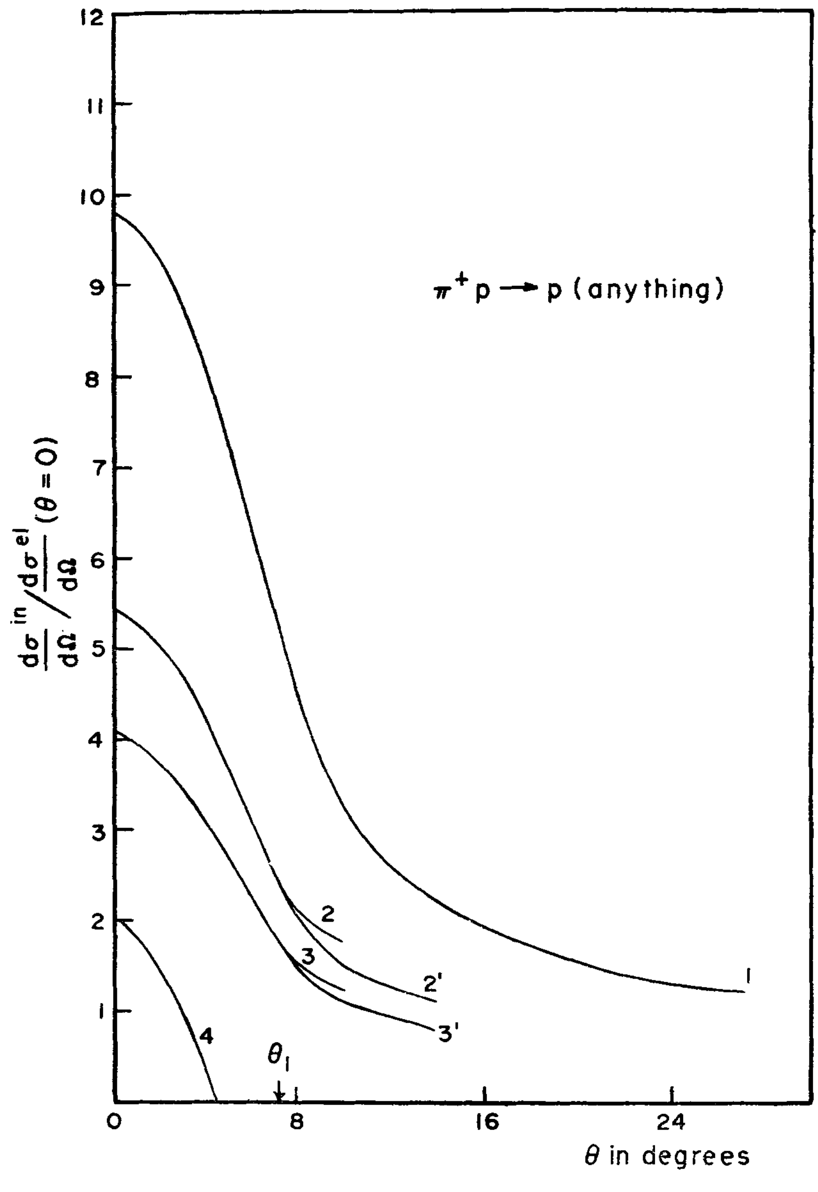

Figure 2. Bounds on the differential inclusive cross section $(d\sigma/d\Omega)$ for $\pi^+ p \to p$ (inclusive) at incident pion momentum 6 Gev/c. Curves 2 and 2′ (3 and 3′) are the bounds, eqs (18) and(19) respectively, computed for $\langle n_p \rangle = 0{\cdot}6$ ($\langle n_p \rangle = 0{\cdot}5$). For comparison, the upper (curve 1) and the lower (curve 4) bounds of ref. I are also shown.

we have reproduced from ref. I the upper (curve 1) and lower (curve 4) unitarity bounds for $\pi^- p$ (figure 1) and $\pi^+ p$ (figure 2). A substantial improvement is evident from these figures.

The bounds can also be expressed in a form corresponding to a given polarisation of the proton rather than to a given helicity. This proceeds parallel to the treatment in I and so we do not describe it here.

4. Conclusions

In practical terms, the improvement in the bound, brought about by fixing the value of σ^C, is clear from the results of the last section. We have already seen that the bound increases monotonically with σ^C for a fixed set $\{B^j\}$ of elastic amplitudes, so that the best bounds will result for processes with the smallest value for σ^C-indeed, as $\sigma^C \to 0$, the bound also vanishes. This follows from (20), (21) and (22) and the positivity of $\beta^j{}_{m_A m_B}$. When $\tilde{\sigma}^C{}_{m_A m_B} = 0$, $\Lambda'_{m_A m_B}$ must be empty; otherwise the left side of (22) will be negative (this is consistent with the fact that then (22) itself tells us that all j belong to $\Lambda_{m_A m_B}$). Consequently from (21), $\Phi_{(m)_i}$ vanishes identically—the first term, because the factor $[\ ]^{1/2}$ in the numerator is zero and the second because the sum is over an empty set. Then from (20), we have $\triangle_{(m)_i} = 0$.

In inclusive processes, however, it is difficult for σ^C to be very small, for obvious reasons. In this respect, the case of exclusive processes with small cross sections should be of interest. At the same time, it is of course true that σ^C cannot be very small without the differential cross section itself being small everywhere. Also the partial wave unitarity, eq. (8) is a stronger constraint for inclusive processes than for exclusive ones. The chance of the bounds being close to the data depends on the balancing between these factors. It also depends on the way $(d\sigma/d\Omega)$ behaves as a function of θ. Thus, for example, the bounds [which are decreasing functions of θ [or $|t|$] will be closer to the data if $(d\sigma/d\Omega)$ shows a broad peak for near-forward angles. This is borne out by the results of Vengurlekar (1977) in the case of $\pi^- p \to \pi^0 n$. One reason for applying the method to exclusive reactions is the easy availability of the necessary input as well as data on $(d\sigma/d\Omega)$ for comparison. The interest in this is enhanced by the fact that our bounds depend not only on $\tilde{\sigma}^C$ but also on the elastic amplitudes which are not well determined even in principle. Since the bounds of ref. II and of Vengurlekar (1977) show near-saturation, it is reasonable to hope that bounds of this kind can provide a relatively simple method of discriminating between partial wave amplitudes or, equivalently, phase shifts all of which correspond to the same measurable *elastic* quantities.

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Appendix A

The problem is to maximise $f_{m_A m_B}(\theta)$, eq. (10), given $B^j{}_{m_A m_B}$, $\tilde{\sigma}^C{}_{m_A m_B}$ [eq. (9)]. and the unitarity restrictions of (8). At an extremum, the variational solutions for $\{\alpha^j{}_{(m)_i}\}$ are found, by standard methods, to satisfy the equations

$$\triangle_{(m)_i} \, | d^j{}_{\lambda\mu_i}(\theta) | - (\lambda_{m_A m_B} + \mu^j{}_{m_A m_B}) \, \alpha^j{}_{(m)_i} = 0 \tag{A.1}$$

where $\triangle_{(m)_i}$ is the following function of $\alpha^j{}_{(m)_i}$:

$$\triangle_{(m)_i} = \sum_j (2j+1) \, | d^j{}_{\lambda\mu_i}(\theta) | \, \alpha^j{}_{(m)_i} \tag{A.2}$$

and λ and μ^j are Lagrange multipliers, to be determined from the constraints. Because of the inequality constraints of unitarity, the partial waves again decompose into two sets, Λ and Λ' [see, *e.g.*, Einhorn and Blankenbecler (1971) for an account of the theory]:

(1) $j \in \Lambda_{m_A m_B}$ when $\beta^j m_A m_B < B^j m_A m_B m_A m_B$. In this case $\mu^j m_A m_B = 0$ and eq. (A.1) has the formal solution

$$a^j_{(m)i} = \lambda^{-1}_{m_A m_B} \mid d^j{}_{\lambda\mu_i}(\theta) \mid \triangle_{(m)i}. \tag{A.3}$$

(2) $j \in \Lambda' m_A m_B$ when $a^j_{(m)i}$ are such that $\beta^j m_A m_B = B^j m_A m_B, m_A m_B$, *i.e.*,

$$(\lambda m_A m_B + \mu^j m_A m_B)^{-2} \sum_{m_i m_C} \triangle^2_{(m)i} [d^j{}_{\lambda\mu_i}(\theta)]^2 = (B^j m_A m_B\, m_A m_B)^2$$

so that

$$a^j_{(m)i} = \frac{\triangle_{(m)i} \mid d^j{}_{\lambda\mu_i}(\theta) \mid B^j m_A m_B}{[\sum\limits_{m'_i, m'_C} \triangle^2_{(m')i} [d^j{}_{\lambda\mu'_i}(\theta)]^2]^{1/2}}, \tag{A.4}$$

$$\mu'_i = m'_C - m'_i, \qquad (m')_i = (m_A, m_B, m'_C, m'_i).$$

The (positive) quantity $\lambda_{m_A m_B}$ is obtained by solving

$$\tilde{\sigma}^C m_A m_B = \lambda^{-2}_{m_A m_B} \sum_{j \in \Lambda_{m_A m_B}} (2j+1) \sum_{m_i, m_C} \triangle^2_{(m)i} [d^j{}_{\lambda\mu_i}(\theta)]^2$$
$$+ \sum_{j \in \Lambda'_{m_A m_B}} (2j+1)(B^j m_A m_B, m_A m_B)^2 \quad , \tag{A.5}$$

leading to eq. (21) of the text.

Computing $(\beta^j m_A m_B)^2$ from (A.3) and (A.4), we are also led to the criterion, inequality (22) of the text, for the division of $\{j\}$ into Λ and Λ'. A necessary condition for the solutions of eqs (20) and (21) to be a *maximum* is that for all m_i which have $\triangle_{(m)i} = 0$, $\Phi_{(m)} < 1$. This can be easily proved in the same way as done in Divakaran *et al* (1973) for their bound. Hence we omit the proof here.

Appendix B

We first show that the division of $\{j\}$ into Λ and Λ' is unique for the simplified discussion of section 2.2.

Suppose that there are two partitions (Λ_1, Λ'_1) and (Λ_2, Λ'_2) of $\{j\}$, both satisfying the criterion (17) of the text:

$$\frac{\tilde{\sigma}^C - B(\Lambda'_i)}{D(\Lambda_i)} (D^j)^2 < (B^j)^2, \quad j \in \Lambda_i, \tag{B.1}$$

$$\frac{\tilde{\sigma}^C - B(\Lambda'_i)}{D(\Lambda_i)} (D^j)^2 \geqslant (B^j)^2, \quad j \in \Lambda'_i, \tag{B.2}$$

$$i = 1, 2,$$

where we have used the abbreviations $B(\Lambda_i) = \sum_{j \in \Lambda_i} (2j+1) B_j^2$, etc. To show that $\Lambda_1 = \Lambda_2$ (and so $\Lambda'_1 = \Lambda'_2$), it is sufficient to show that $U \equiv \Lambda_1 \cap \Lambda'_2$ and $V \equiv \Lambda_2 \cap \Lambda'_1$ are *both* empty, for then $j \in \Lambda \Rightarrow j \notin \Lambda'_2 \Rightarrow j \in \Lambda_2$ and *vice versa*.

We show first that at least one of the sets U and V must be empty.

Proof.—Let $j \in U$. Then $j \in A_1$ and so

$$\frac{\tilde{\sigma}^C - B(A'_1)}{D(A_1)} (D^j)^2 < (B^j)^2 \qquad \text{for all } j \in U.$$

Multiplying by $(2j+1)$ and summing over all $j \in U$, we have

$$\frac{\tilde{\sigma}^C - B(A'_1)}{D(A_1)} D(U) < B(U). \tag{B.3}$$

But since $j \in A_2'$ also, we can use (B.2) to conclude

$$\frac{\tilde{\sigma}^C - B(A'_2)}{U(A_2)} D(U) \geqslant B(U). \tag{B.4}$$

It follows that

$$\frac{\tilde{\sigma}^C - B(A'_1)}{D(A_1)} < \frac{\tilde{\sigma}^C - B(A'_2)}{D(A_2)} \tag{B.5}$$

since U is non-empty.

In an exactly similar way, the assumption that V is non-empty leads to the precisely opposite conclusion. Q.E.D.

Let us next assume therefore that V is empty. A_1 and A_2 have the decomposition (+ denotes the union of disjoint sets)

$$A_1 = A_2 + U \tag{B.6}$$

$$A'_2 = A_1' + U. \tag{B.7}$$

Proof.—Since V is empty, $j \in A_2 \Rightarrow j \notin A'_1 \Rightarrow j \in A_1$. Thus A_2 is a subset of A_1 and so $A_1 \cap A_2 = A_2$. (B.6) and (B.7) now follow from the decomposition

$$A_1 = A_1 \cap A_2 + A_1 \cap A'_2$$

$$A_1 + A_2 = A'_1 + A'_2 (= \{j\}). \quad \text{Q.E.D.}$$

Finally (B.6) and (B.7) imply that U is also empty.

Proof.—Suppose, on the contrary, that U is not empty. Then (B.5) holds. Now eliminate A_2 and A'_2 from (B.5) using (B.6) and (B.7). The result is

$$\frac{\tilde{\sigma}^C - B(A'_1)}{D(A_1)} D(U) > B(U) \tag{B.8}$$

which contradicts (B.3).

Thus *both* U and V are empty, which is what we set out to show.

In the general case (section 2.3), the inequality (22) has exactly the same structure as (B.1), with $\delta^j{}_{m_A m_B}$ replacing D^j, and the proof goes through as above.

Appendix C*

That (18) indeed gives an upper bound can be easily proved by the 'direct subtraction method' as follows. Consider

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* The proof closely follows the technique employed by Singh and Roy (1970).

$$\begin{aligned}\triangle &= \bar{f}'(\theta) - f'(\theta) \\ &= \sum_{\Lambda} (2j+1)\beta_j^0 \; D^j + \sum_{\Lambda'} (2j+1) B^j D^j \\ &\quad - \sum_{\Lambda+\Lambda'} (2j+1)\beta^j D^j \end{aligned} \tag{C.1}$$

where we write β_j^0 for D^j/λ [see eqs (14), (15)], such that $\beta_j^0 < B^j$ for $j \in \Lambda$ and $\beta_j^0 > B^j > \beta^j$ for $j \in \Lambda'$. Now since $D^j = \beta_j^0 \lambda$, all j, we get

$$\begin{aligned}\triangle &= \Big[\sum_{\Lambda} (2j+1)(\beta_j^0)^2 + \sum_{\Lambda'} (2j+1) B^j \beta_j^0 - \sum_{\Lambda+\Lambda'} (2j+1)\beta^j \beta_j^0 \Big] \lambda \\ &= \Big[\sum_{\Lambda} (2j+1) [(\beta_j^0)^2 - (\beta^j \beta_j^0)] + \sum_{\Lambda'} (2j+1)\beta_j^0 (B^j - \beta^j) \Big] \lambda \,. \end{aligned} \tag{C.2}$$

Using

$$(\beta_j^0)^2 - \beta^j \beta_j^0 = \frac{(\beta_j^0 - \beta^j)^2 - [(\beta^j)^2 - (\beta_j^0)^2]}{2} \tag{C.3}$$

we have

$$\begin{aligned}\triangle = \Big\{ &\tfrac{1}{2} \sum_{\Lambda} (2j+1)(\beta_j^0 - \beta^j)^2 - \tfrac{1}{2} \sum_{\Lambda} (2j+1) [(\beta^j)^2 - (\beta_j^0)^2] \\ &+ \sum_{\Lambda'} (2j+1)\beta_j^0 [B^j - \beta^j] \Big\} \lambda \end{aligned} \tag{C.4}$$

Now from the constraint equation

$$\tilde{\sigma} = \sum_{\Lambda+\Lambda'} (2j+1)(\beta^j)^2 = \sum_{\Lambda} (2j+1)(\beta_j^0)^2 + \sum_{\Lambda'} (2j+1)(B^j)^2, \tag{C.5}$$

we obtain

$$\sum_{\Lambda} (2j+1) [(\beta^j)^2 - (\beta_j^0)^2] = \sum_{\Lambda'} (2j+1) [(B^j)^2 - (\beta^j)^2]. \tag{C.6}$$

Hence

$$\begin{aligned}\triangle = \Big\{ &\tfrac{1}{2} \sum_{\Lambda} (2j+1)(\beta_j^0 - \beta^j)^2 + \sum_{\Lambda'} (2j+1)(B^j - \beta^j)\beta_j^0 \\ &- \tfrac{1}{2} \sum_{\Lambda'} (2j+1) [(B^j)^2 - (\beta^j)^2] \Big\} \lambda. \end{aligned} \tag{C.7}$$

Consider

$$\Big[(B^j - \beta^j)\beta_j^0 - \frac{[(B^j)^2 - (\beta^j)^2]}{2} \Big]$$

$$= \tfrac{1}{2} (B^j - \beta^j) [(\beta_j^0 - B^j) + (\beta_j^0 - \beta^j)]. \tag{C.8}$$

Since $\beta_j^0 > B^j > \beta^j$, $j \in \Lambda'$, and since $\lambda > 0$, we have from eqs (C.7) and (C.8)

$\triangle \geqslant 0$ Q.E.D.

Note that $\triangle = 0$ only if $\beta^j = \beta_j^0$, $j \in \Lambda$ and $\beta^j = B^j$, $j \in \Lambda'$.

Appendix D

We shall once again use the simpler bound of section 2.2 to illustrate the proof of the monotonicity of $f^{(0)}$, the bound, in $\tilde{\sigma}^C$.

As we have mentioned in the text, the difficulty in the proof arises from the dependence of Λ and Λ' on $\tilde{\sigma}^C$. The way to overcome this problem is through the recognition that Λ and Λ' are sets of integers.* Thus, as $\tilde{\sigma}^C$ increases smoothly, Λ and Λ' change only at those values $\tilde{\sigma}_n{}^C$ of $\tilde{\sigma}^C$ where an infinitesimal increase $\delta\tilde{\sigma}$ in it will cause (some) one partial wave to shift from Λ to Λ', *i.e.*,

$$\frac{\tilde{\sigma}_n{}^C - \delta\tilde{\sigma} - B(\Lambda')}{D(\Lambda)}(D^{j_0})^2 < (B^{j_0})^2$$

and

$$\frac{\tilde{\sigma}_n{}^C + \delta\tilde{\sigma} - B(\Lambda')}{D(\Lambda)}(D^{j_0})^2 \geqslant (B^{j_0})^2$$

for some j_0 and for $\delta\tilde{\sigma}$ arbitrarily small and positive.

Whenever $\tilde{\sigma}^C$ increases *without* passing through any $\tilde{\sigma}_n{}^C$, the bound (18) clearly increases. Thus, the curve of $f^{(0)}$ as a function of $\tilde{\sigma}^C$ will at worst consist of pieces, each of which shows a continuous rise—the points of possible discontinuity being at $\tilde{\sigma}_n{}^C$. To prove the desired result, it is now sufficient to show that the bound $f^{(0)}$ is a single-valued function of $\tilde{\sigma}^C$ for *all* $\tilde{\sigma}^C$—it is not even necessary, for this purpose, for $f^{(0)}$ to be continuous in $\tilde{\sigma}^C$. But this follows trivially from our proof of uniqueness of Λ and Λ'.

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