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Non-singular cosmologies in the conformally invariant gravitation theory

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Abstract. It is shown that in the framework of a conformally invariant gravitation theory, the singularity which is present in some anisotropic universes in general relativity is due to a wrong choice of conformal frame. Frames exist in which these models can be made singularity free.

Keywords. Conformal invariance; cosmology; gravitation; singularity.

1. Introduction

The most striking feature of the Robertson-Walker cosmological models in general relativity is the singularity which occurs in them in the finite past, if the cosmological constant Λ is below a critical positive value. At the singularity the density of matter becomes infinitely large, the scale function of the universe goes to zero and space-time itself becomes singular.

Could this singular behaviour of the Robertson-Walker models be due to the perfect homogeneity and isotropy assumed in constructing them? Or, can the irregularities which presumably exist in the real universe prevent the occurrence of the singularity? It has been shown in a series of theorems by Hawking and Penrose (1973) that if the general theory of relativity holds, and if some very general conditions like the positivity of energy are satisfied, then there is always a singularity in space-time. It follows from this that inhomogeneity and anisotropy cannot prevent the occurrence of a singularity in our past. The singularity seems to be an unavoidable feature of all cosmological models in general relativity.

One may look upon the cosmological singularity as a unique event representing the creation of the universe. Indeed many big-bang cosmologists take this view; some even consider this to be a positive merit of relativistic cosmology (p. 362, Hawking *et al.*, *op. cit.*). This view would have been readily acceptable if the cosmological singularity were the only possible singularity in general relativity. This, however, is not the case. As the various singularity theorems show, the singularity could arise in a variety of ways, such as in the gravitational collapse of a compact object. Space-time singularities therefore seem to be inherent in general relativity, the cosmological singularity being one of them. While one may still attempt to make a virtue of this inevitable property, it is generally recognised that such a point of view is contrary to the *modus operandi* in the rest of theoretical physics, where attempts are made to make theories free from various mathematical singularities.

Another way to deal with the singularity problem is to look for a new theory of gravitation which would lead to singularity free world models. It has been shown by Hoyle and Narlikar (1972, 1974) that in their conformally invariant theory of gravitation, homogeneous and isotropic cosmologies can be made singularity free. Their method, which is explained in § 2 is particularly simple because the Robertson-Walker models are conformally flat. It is therefore not obvious that it will work in the more general class of cosmological singularities. This paper is the first in a series of investigations on the removal of cosmological singularities in the conformal theory. Here we explain how this method works for a particular class of anisotropic world models.

2. Conformal gravitation

We shall briefly describe the conformally invariant gravitation theory of Hoyle and Narlikar (1974) and mention how the Einstein-de Sitter universe can be made singularity free in it.

We will work in a system of units in which the speed of light and Planck's constant are both unity, *i.e.*, $c = \hbar = 1$. In such a system of units, every physical quantity Q has a dimension which is some power L^n of the length unit L.

A conformal transformation is a change in the length unit. It transforms the Riemann line element

$$ds^2 = g_{ab}\, dx^a\, dx^b \tag{1}$$

through

$$\overset{*}{g}_{ab} = \Omega^2\, g_{ab} \tag{2}$$

to

$$ds^{*2} = \overset{*}{g}_{ab}\, dx^a\, dx^b, \tag{3}$$

and every conformally covariant physical quantity Q to

$$Q^* = \Omega^n\, Q. \tag{4}$$

The coordinates, being dimensionless quantities, remain unchanged. $\Omega(x)$ is a real non-zero function of space-time coordinates—we cannot have zero or infinity in the scaling of the length unit. A theory is said to be conformally invariant if it preserves its form under the above transformation.

In the system of units being used, mass has the dimension L^{-1}. It conformally transforms as

$$m^* = \Omega^{-1}\, m. \tag{5}$$

The mass of a particle is therefore no longer a constant in general, it is a space-time dependent function. It is necessary to discard the concept of constancy of mass in order to preserve the conformal invariance of the theory. Early attempts at constructing conformal theories broke down in the presence of matter because of the insistence upon maintaining constant masses.

The conformally invariant action functional for the theory is

$$S = - \sum_a \int m_a \, da, \tag{6}$$

where m_a, the mass of the a^{th} particle is given by

$$m_a(X) = \sum_{b \neq a} \int \lambda^2 G(X, B) \, db. \tag{7}$$

B is a typical point on the trajectory of the b^{th} particle. λ is a coupling constant and $G(X, B)$ is a symmetric scalar propagator which satisfies

$$\left(\Box_X + \frac{1}{6} R(X) \right) G(X, B) = \frac{1}{\sqrt{-g(X)}} \delta_4(X, B), \tag{8}$$

where $\delta_4(X, B)$ is the four-dimensional delta function and $R(X)$ is the Riemann scalar curvature at the point X. Equation (7) shows that the mass of every particle arises from interactions with all other particles in the universe. This is therefore a Machian view of the nature of mass. Field equations and particle trajectories are obtained from (6) as usual.

In the presence of a very large number of particles, it can be shown that the mass of every particle can be approximated by the same function $m(x)$. If now a conformal transformation is made with

$$\Omega(x) \propto m(x), \tag{9}$$

particle masses become constant and the field equations reduce to

$$R_{ab} - \tfrac{1}{2} g_{ab} R = - 8\pi G \, T_{ab}, \tag{10}$$

which are just the Einstein equations.

It is possible that particle masses vanish in some regions of space-time. In such a region the conformal function $\Omega(x)$ of eq. (9) will be zero, the approximation to Einstein's theory will not be valid and the full conformal theory will have to be used.

The Robertson-Walker line element for the Einstein-de Sitter model in the usual notation is

$$ds^2 = dt^2 - S^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \tag{11}$$

where the solution of the relativistic field equations gives

$$S(t) \propto t^{2/3}, \rho \propto t^{-2}. \tag{12}$$

Here ρ is the density of matter in the form of dust. Clearly there is a singularity at $t = 0$.

Introducing a new time coordinate τ by

$$\tau = \int_0^t \frac{dt'}{s(t')}$$

reduces (11) to

$$ds^2 = Q^2(\tau) [d\tau^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)], \tag{13}$$

where

$$Q(\tau) = S(t).$$

Choosing the conformal function to be

$$\Omega(\tau) = [Q(\tau)]^{-1} \tag{14}$$

we can make a transition to Minkowski space-time:

$$ds^2 = d\tau^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

In this frame particle masses are given by

$$m^*(\tau) \propto \tau^2, \tag{15}$$

and it can be shown that the universe contains a uniform static distribution of particles in flat space-time; it is singularity free. It might seem as though we have merely replaced the singularity in the geometry by a zero in the mass function (15) at $\tau = 0$. But in the conformal theory, instead of starting with the Einstein-de Sitter universe $\tau > 0$ we start with the whole of Minkowski space-time. Particle world lines stretch unbroken from $\tau = -\infty$ to $\tau = \infty$. Particle masses vanish at $\tau = 0$ because of the —ve contributions which arise in the region $\tau < 0$. We can pass to the usual conformal frame using the function

$$\Omega \propto \tau^2$$

at all times except $\tau = 0$. At this instant the vanishing of the conformal function will produce a singularity if the transformation is made. The situations in the usual theory and the conformal theory are shown in figures 1 and 2.

3. Anisotropic universes

Having seen how the singularity in homogeneous isotropic cosmologies can be dealt with in the conformal theory, we will now go to more general homogeneous anisotropic universes.

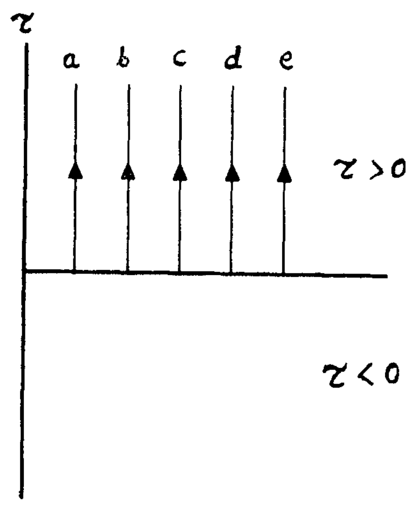

Figure 1. *a*, *b*, *c*, *d*, *e* are particle world-lines, confined to the region $\tau > 0$ of Minkowski space-time.

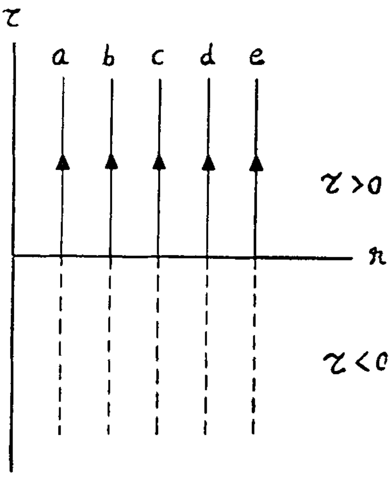

Figure 2. Particle world-lines a, b, c, d, e occupy both the regions, $\tau > 0$ and $\tau < 0$ of Minkowski space-time.

The Robertson-Walker models are all conformally flat, and it is possible to straightaway transform to the singularity free Minkowski space. But conformal flatness is not a property shared by all world models and a different and more general approach becomes necessary. We will show that in the case of the anisotropic universes considered here, a conformal frame exists in which space-time is singularity free in the sense that (i) the Riemann tensor in a locally flat coordinate system, (ii) the coordinate invariant $R_{abcd}\, R^{abcd}$ and (iii) the comoving (scalar) density of matter are all finite everywhere. It will be seen that the conformal function becomes singular and the particle masses vanish at this instant. As in § 2 we will then attribute the singularity, which occurs in the Einstein conformal frame, to this breakdown of the conformal function rather than to any peculiarity in the physics.

The simplest non-empty, spatially homogeneous, anisotropic universes are the Bianchi type I spaces. These have an abelian isometry group (Ryan and Shepley 1975). We will first consider a model belonging to the class which contains a pressure-free perfect fluid, and we will assume for simplicity that the cosmological constant $\Lambda = 0$. There exist comoving coordinates (t, x, y, z) such that the line element for this universe takes the form (Hawking and Ellis 1973, p. 143)

$$ds^2 = dt^2 - X^2(t)\, dx^2 - Y^2(t)\, dy^2 - Z^2(t)\, dz^2. \tag{16}$$

Substitution of this line element in the field equations leads to

$$X(t) = S \cdot \left(\frac{t^{2/3}}{S}\right)^{2\sin\alpha}$$

$$Y(t) = S \cdot \left(\frac{t^{2/3}}{S}\right)^{2\sin(\alpha + 2\pi/3)},$$

$$Z(t) = S \cdot \left(\frac{t^{2/3}}{S}\right)^{2\sin(\alpha + 4\pi/3)}, \tag{17}$$

where $S(t)$ is defined by

$$S^3(t) = XYZ = \frac{9}{2} Mt(t + \Sigma). \tag{18}$$

Here Σ (> 0) is a constant determining the magnitude of anisotropy and α $(-\pi/6 < \alpha < \pi/2)$ is a constant determining the direction in which the most rapid expansion takes place. The conservation law

$$(\rho u^k)_{;k} = 0$$

shows that the comoving matter density ρ is given by

$$\rho = \frac{3M}{4\pi S^3},$$

where M is a constant.

Hereafter we will consider only the special case $\alpha = \pi/2$. The expansion rates $\dot{X}/X$, $\dot{Y}/Y$, $\dot{Z}/Z$ in the x-, y- and z-directions are then given by

$$\frac{\dot{X}}{X} = \frac{2(t + 3\Sigma/2)}{3t(t + \Sigma)}, \tag{19}$$

$$\frac{\dot{Y}}{Y} = \frac{\dot{Z}}{Z} = \frac{2}{3(t + \Sigma)};$$

the average expansion rate is given by

$$\frac{\dot{S}}{S} = \frac{2(t + \Sigma/2)}{3t(t + \Sigma)}.$$

The different expansion rates in different directions show that the universe is anisotropic. Evaluating the vorticity and shear tensors (Ellis 1971), it can be shown that there is shear but no rotation. The singularity which appears in this model is of the 'pancake' type: at $t = 0$, X is zero but Y and Z are non-zero; matter expands monotonically in all directions, starting from an indefinitely high rate in the x-direction, but with finite rates in the y- and z-directions.

We now introduce a locally flat coordinate system at an arbitrary point (t_0, x_0, y_0, z_0) through the successive coordinate transformations (Landau and Lifshitz 1971)

$$x^a \rightarrow x'^a = P_0^{(k)} x^k,$$

$$x'^a \rightarrow x''^a = x'^a + \frac{1}{2} (\Gamma'^a_{bc})_0 x'^b x'^c,$$

where

$$P_0^{(0)} = 1,\ P_0^{(1)} = X(t_0),\ P_0^{(2)} = Y(t_0),\ P_0^{(3)} = Z(t_0).$$

In the new coordinate system

$$(g''_{ab})_0 = \eta_{ab} = \mathrm{diag}(1, -1, -1, -1), \quad (\Gamma''^a_{bc})_0 = 0,$$

This is the physically relevant coordinate system for a fundamental observer comoving with the matter; measurements are always made in a locally flat coordinate system.

The independent non-vanishing components of the Riemann tensor at (t_0, x_0, y_0, z_0) in the new coordinate system are (dropping all primes)

$$R^0_{101} = \frac{\ddot{X}}{X}(t_0) = \frac{-2(t_0 + 3\Sigma)}{9t_0(t_0+\Sigma)^2},$$

$$R^0_{202} = R^0_{303} = \frac{\ddot{Y}}{Y}(t_0) = \frac{-2}{9(t_0+\Sigma)^2}, \tag{20}$$

$$R^1_{212} = R^1_{313} = \frac{\dot{X}\dot{Y}}{XY}(t_0) = \frac{4(t_0 + 3\Sigma/2)}{9t_0(t_0+\Sigma)^2},$$

$$R^2_{323} = \frac{\dot{Y}\dot{Z}}{YZ}(t_0) = \frac{4}{9(t_0+\Sigma)^2};$$

the coordinate invariant $R_{abcd}R^{abcd}$ is

$$R_{abcd}R^{abcd} = 4\left\{\left(\frac{\ddot{X}}{X}\right)^2 + \left(\frac{\ddot{Y}}{Y}\right)^2 + \left(\frac{\ddot{Z}}{Z}\right)^2 + \left(\frac{\dot{X}\dot{Y}}{XY}\right)^2 + \left(\frac{\dot{X}\dot{Z}}{XZ}\right)^2 + \left(\frac{\dot{Y}\dot{Z}}{YZ}\right)^2\right\}_{t=t_0} \tag{21}$$

It is seen from (20) that some of the components of the Riemann tensor diverge as t_0 go to zero. Equation (21) shows that this is a coordinate independent effect; the matter density too becomes infinitely large at $t_0 = 0$.

We will now transform to a new conformal frame using (2), taking Ω to be a function of the time t alone. The line element (16) then transforms to

$$ds^{*2} = \Omega^2 dt^2 - \Omega^2 X^2 dx^2 - \Omega^2 Y^2 dy^2 - \Omega^2 Z^2 dz^2. \tag{22}$$

Introducing a new time coordinate

$$\tau = \int^t \Omega(t')\, dt' \tag{23}$$

and defining

$$A(\tau) = \Omega(t) X(t), \qquad B(\tau) = \Omega(t) Y(t), \quad C(\tau) = \Omega(t) Z(t), \tag{24}$$

this line element can be written in the form

$$ds^{*2} = d\tau^2 - A^2(\tau)\, dx^2 - B^2(\tau)\, dy^2 - C^2(\tau)\, dz^2. \tag{25}$$

The conformal theory, we are considering here, reduces to Einstein's theory in the frame in which the particle masses are constant in the approximation of a very large number of particles, an approximation which is valid in cosmology. (16) is a solution of the Einstein equations. It therefore follows that (25), which is obtained from (16) by a conformal and a coordinate transformation, is a solution of the conformal theory in the new frame. Moreover, A, B and C appear in the new line element in the same form as X, Y and Z do in the old one. It therefore follows that in the new conformal frame, geometric objects like the Riemann tensor contain A, B, C and their derivatives, in the same way as geometric objects in the old conformal frame contained X, Y, Z and their derivatives. Using this the independent non-vanishing components of the Riemann tensor at (t_0, x_0, y_3, z_0) in a locally flat coordinate system in the new conformal frame are

$$
\begin{aligned}
R^{*0}_{101} &= \frac{\ddot{A}}{A}(\tau_0) = \frac{1}{\Omega^2(t_0)}\left(\frac{\ddot{X}}{X} + \frac{\dot{X}\dot{\Omega}}{X\Omega} + \frac{\ddot{\Omega}}{\Omega} - \frac{\dot{\Omega}^2}{\Omega^2}\right)_{t=t_0},\\
R^{*0}_{202} &= R^{*0}_{303} = \frac{\ddot{B}}{B}(\tau_0) = \frac{1}{\Omega^2(t_0)}\left(\frac{\ddot{Y}}{Y} + \frac{\dot{Y}\dot{\Omega}}{Y\Omega} + \frac{\ddot{\Omega}}{\Omega} - \frac{\dot{\Omega}^2}{\Omega^2}\right)_{t=t_0},\\
R^{*1}_{212} &= R^{*1}_{313} = \frac{\dot{A}\dot{B}}{AB}(\tau_0) = \frac{1}{\Omega^2(t_0)}\left(\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{\Omega}}{X\Omega} + \frac{\dot{Y}\dot{\Omega}}{Y\Omega} - \frac{\dot{\Omega}^2}{\Omega^2}\right)_{t=t_0},\\
R^{*2}_{323} &= \frac{\dot{B}\dot{C}}{BC}(\tau_0) = \frac{1}{\Omega^2(t_0)}\left(\frac{\dot{Z}\dot{Y}}{ZY} + \frac{\dot{Y}\dot{\Omega}}{Y\Omega} + \frac{\dot{Z}\dot{\Omega}}{Z\Omega} - \frac{\dot{\Omega}^2}{\Omega^2}\right)_{t=t_0}
\end{aligned}
\tag{26}
$$

and the coordinate invariant $R^{*}_{abcd}\, R^{*abcd}$ is

$$
\begin{aligned}
R^{*}_{abcd}\, R^{*abcd} = 4\Bigg\{&\left(\frac{\ddot{A}}{A}\right)^2 + \left(\frac{\ddot{B}}{B}\right)^2 + \left(\frac{\ddot{C}}{C}\right)^2\\
&+ \left(\frac{\dot{A}\dot{B}}{AB}\right)^2 + \left(\frac{\dot{A}\dot{C}}{AC}\right)^2 + \left(\frac{\dot{B}\dot{C}}{BC}\right)^2\Bigg\}_{t=t_0},
\end{aligned}
\tag{27}
$$

where t_0 is the t-time corresponding to $\tau = \tau_0$.

We are free to make any choice for the conformal function $\Omega(t)$. Choosing

$$\Omega(t) = t + \frac{1}{t}, \tag{28}$$

we get from (26)

$$\tau = \frac{t^2}{2} + \ln t, \tag{29}$$

where the constant of integration has been given the value zero by appropriately resetting our clocks. The Riemann tensor components (26) can now be obtained as explicit functions of t_0 or τ_0 using eqs (17), (19) and (28). These expressions will not be reproduced here as they are rather unwieldy. It will suffice to say that

$$R^{*0}_{101} = R^{*1}_{212} = R^{*1}_{313} = 0,$$

$$R^{*0}_{202} = R^{*0}_{303} = R^{*2}_{323} = 1,$$

and

$$R^{*}_{abcd}\, R^{abcd} = 12$$

at $t_0 = 0$, *i.e.*, $\tau_0 = -\infty$, while all the components vanish in the infinite future.

The mass m^*, matter density ρ^* and particle number density ν^* in the new frame are given by

$$m^* = \frac{m}{\Omega} = \frac{mt}{t^2+1},$$

$$\rho^* = \frac{\rho}{\Omega^4} = \frac{\text{constant} \cdot t^3}{(t+\Sigma)(t^2+1)^4}, \tag{30}$$

and

$$\nu^* = \frac{\nu}{\Omega^3} = \frac{\text{constant} \cdot t^2}{(t+\Sigma)(t^2+1)^3},$$

where m is the constant particle mass, ρ the matter density and ν the number density in the Einstein frame. m^*, ρ^* and ν^* vanish at $t_0 = 0$, *i.e.*, $\tau_0 = -\infty$, and in the infinite future. Particle number density vanishes in the infinite past and future because the particles are infinitely dispersed and not because of destruction of particles. Particle creation and destruction are not contemplated in the present model—the baryon number is conserved.

The scale factors and expansion rates in the three directions are

$$A(\tau) = \left(\frac{9M}{2}\right)^{-1/3} \frac{t^2+1}{(t+\Sigma)^{1/3}},$$

$$B(\tau) = C(\tau) = \left(\frac{9M}{2}\right)^{2/3} \frac{(t^2+1)(t+\Sigma)^{2/3}}{t},$$

$$\frac{\dot{A}}{A}(\tau) = \frac{5t^3 + 6\Sigma t^2 - t}{3(t^2+1)^2(t+\Sigma)},$$

$$\frac{\dot{B}}{B}(\tau) = \frac{\dot{C}}{C}(\tau) = \frac{5t^3 + 3\Sigma t^2 - t - 3\Sigma}{3(t^2+1)^2(t+\Sigma)}.$$

We therefore see the following picture of the Binachi type I, zero pressure, $\alpha = \pi/2$ universe in the new conformal frame: the universe originates in the infinite past with *zero* particle masses, *zero* matter density and *finite* Riemann curvature. The scale factor in the x-direction is finite, in the other two infinite. At the origin there is contraction in the y- and z-directions, but there is neither contraction nor expansion in the x-direction. As time progresses a contraction builds up in the the x-direction as well. At still later times, the contraction along the three axes changes to expansion. This expansion continues for ever; in the infinite future particle masses and matter and number densities again go to zero. In between these phases of contraction, there is an epoch of maximum but *finite* matter density.

Transformation to a proper conformal frame therefore relegates the origin of the universe, which occurs in the finite past in the Einstein frame, to the infinite past. In this frame matter density and curvature are always finite, even at the origin at $\tau = -\infty$. There is no singularity in the universe. The singularity of the line element (16) was due to bad choice of the conformal frame and not because of a breakdown of physics.

We will now consider the anisotropic Kasner universe. This is an empty universe with the metric (Landau and Lifshitz 1971, p. 362)

$$ds^2 = dt^2 - t^{2P_1} dx^2 - t^{2P_2} dy^2 - t^{2P_3} d_z^2, \tag{31}$$

where P_1, P_2, P_3 are any three numbers satisfying

$$P_1 + P_2 + P_3 = P_1^2 + P_2^2 + P_3^2 = 1.$$

If we arrange these three numbers in the order $P_1 < P_2 < P_3$, their values will lie in the intervals

$$-\tfrac{1}{3} \leqslant P_1 \leqslant 0, \quad 0 \leqslant P_2 \leqslant \tfrac{2}{3}. \quad \tfrac{2}{3} \leqslant P_3 \leqslant 1. \tag{32}$$

The Kasner model is empty and therefore anti-Machian. But we are interested in the behaviour of cosmological models near the singularity, and in this regime, matter can be introduced into the metric of the empty universe, neglecting its back reaction on the gravitational field. The evolution of matter density is then determined simply by its equation of motion in the given field, and is given by

$$\rho \sim t^{-2\,(1-P_3)}\,.$$

The matter density and Riemann curvature diveege at $t = 0$.

We conformally transform the metric (31) with

$$\Omega\,(t) = t + \frac{1}{t}\,,$$

and set up a locally flat coordinate system at an arbitrary point (t_0, x_0, y_0, z_0). It then follows, as in the case of the Bianchi type I universe, that in the new frame the curvature tensor and the invariant $R_{abcd}\,R^{abcd}$ are always finite. The matter density in the new frame,

$$\rho^* \sim \frac{t^{2+2P_3}}{(t^2+1)^4},$$

is also finite at all times because of (32). The singularity is therefore removed and the origin shifted to the infinite past.

Thus in the anisotropic models discussed above, the singularity is again identified with the $m = 0$ hyper-surface in the conformally invariant theory. In the Friedmann case, this hyper-surface is in the finite past (both in t and τ scales) and this property was used by Hoyle (1975) in giving an interesting interpretation of the microwave background. In Hoyle's interpretation the background arises from the thermalization of the starlight generated in the $\tau < 0$ half of the universe. For such an explanation to work in the anisotropic models discussed above the space-time will have to be extended beyond the $m = 0$ surface at $\tau = \pm\infty$.

5. Conclusion

It has been shown in the past that the singularity in the Robertson-Walker models could be removed in the framework of a conformally invariant gravitation theory. It could have been the case that such a result was obtained because of the special properties of these models and that it was not extendable to more general cosmologies. It would indeed have been very ironical had this turned out to be true, because there was a time when it was believed that it was the highly symmetric character of the Robertson-Walker models which produced the singularity!

We have now shown that the singularity which appears in some more general anisotropic universes can also be removed, in the sense that a conformal frame exists in which the Riemann curvature and matter density are always finite. This result leads us to believe that every cosmological model can be made singularity free in the conformally invariant theory. We expect to investigate this conjecture in the future, under a more general set of assumptions.

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