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Differential equation approach for the energy average of the scattering function

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Abstract. An exact differential equation is given to evaluate the energy average of the scattering function. The advantage of the differential equation as compared to the earlier methods based on series expansion is that one has to evaluate only single sums over the complex poles of the S-matrix. Using Wigner's semicircle law for the distribution of the real parts of the poles of the scattering matrix, the earlier expression for the energy average of the scattering function is rederived.

Keywords. Average cross-section; differential equation technique; transmission coefficient.

1. Introduction

The resonance nuclear reactions have been extensively studied (Vogt 1959, 1968; Mekjian 1973) for quite some times now. As was first pointed out by Bohr, they arise due to the formation and the subsequent decay of the compound nucleus. The other phenomena which were observed later and which have made the experimental and theoretical study of the resonance reactions interesting are the cross-section fluctuation (Ericson 1963), doorway states (Mekjian 1973), etc. An important quantity in this field is the energy averaging (Moldauer 1967, 1969; Nazakat Ullah 1968) of the resonance cross-sections. Agassi and Weidenmüller (Agassi and Weidenmüller 1975) have recently developed a new technique based on the shell-model approach to study this problem. The earlier methods (Moldauer 1967, 1969; Nazakat Ullah 1968) evaluate the energy averages of the scattering function by making series expansions and then resumming certain terms of this series and finally showing that the terms which are left out are either small or cancel each other because of alternating sign. The purpose of the present work is to show that a differential equation technique can be developed to evaluate the energy averages of the resonance cross-sections. As we shall show later, the differential equation technique is quite simple and has the advantage that only single sums over the complex poles of the S-matrix have to be evaluated to arrive at the final expression for the average cross-section.

In section 2 we establish the differential equation for the energy average of the scattering function, and solve it in section 3. Section 4 contains the concluding remarks.

2. Formulation

Let us write the pole resonance form of the scattering function $S(E)$ as

$$S(E) = \prod_{\mu} (E - Z_\mu^*)/(E - Z_\mu), \tag{1}$$

where $Z_\mu = \epsilon_\mu - i/2\ \Gamma_\mu$ and where we have omitted a phase factor which multiplies the product over resonances μ.

Using the Lorentzian energy resolution function having a half-width $I/2$ and following Feshbach, Kerman and Lemmer (Feshbach, Kerman and Lemmer 1967), the energy average of $S(E)$ can be written as

$$\langle S(E_0) \rangle = \prod_{\mu} \left(E_0 + i\frac{I}{2} - Z_\mu^* \Big/ E_0 + i\frac{I}{2} - Z_\mu \right). \tag{2}$$

We now establish the differential equation to evaluate the product in expression (2). For simplicity we write f for $\langle S(E_0) \rangle$. Replacing each Γ_μ by $\lambda\Gamma_\mu$ where λ is a real parameter, we can rewrite expression (2) as

$$f(\lambda) = \prod_{\mu} \left((E_0 - \epsilon_\mu) + \frac{i}{2}(I - \lambda\Gamma_\mu) \Big/ (E_0 - \epsilon_\mu) + \frac{i}{2}(I + \lambda\Gamma_\mu) \right). \tag{3}$$

From expression (2) and (3) it is obvious that if $\lambda = 0$, $f(\lambda = 0) = 1$ and if $\lambda = 1$, $f(\lambda = 1)$ becomes the desired energy average $\langle S(E_0) \rangle$ which we are interested in. Since the product in expression (3) is convergent (Titchmarsh 1961) we can write the following differential equation for $f(\lambda)$ by differentiating expression (3) with respect to λ

$$\frac{df}{d\lambda} = \left\{ -\frac{i}{2} \sum_{\mu} \left(\Gamma_\mu \Big/ (E_0 - \epsilon_\mu) + \frac{i}{2}(I - \lambda\Gamma_\mu) \right) \right.$$
$$\left. -\frac{i}{2} \sum_{\mu} \left(\Gamma_\mu \Big/ (E_0 - \epsilon_\mu) + \frac{i}{2}(I + \lambda\Gamma_\mu) \right\} f \right), \tag{4}$$

with the boundary condition $f(0) = 1$. This is the desired differential equation.

3. Rederivation of the expression for $\langle S(E_0) \rangle$

We would next like to integrate the differential equation (4) from $\lambda = 0$ to $\lambda = 1$. Let us recall that the width I of the Lorentzian has to be large enough so that it contains an appreciable number of resonances (Feshbach, Kerman and Lemmer 1967), that is $I \gg \Gamma_\mu$ and since λ lies between 0 and 1, we get the following expression for $S(E_0)$ by integrating the differential equation (4),

$$\langle S(E_0) \rangle = \exp\left(-i \sum_{\mu} \left\{ \Gamma_\mu \Big/ (E_0 - \epsilon_\mu) + \frac{i}{2} I \right\} \right). \tag{5}$$

Therefore, we see that the advantage of the differential-equation technique as compared to the series expansion methods lies in the fact that no expansions, resummations and no estimations for the terms which are thrown out are to be made. It is for the first time that it is shown that the condition $I \gg \Gamma_\mu$ is all what is needed to rewrite $\langle S(E_0) \rangle$ in the form given by expression (5).

To proceed further we replace the summation over the resonances μ by an integration and replace the widths Γ_μ by their average value $\langle \Gamma_\mu \rangle$. The quantity $\langle S(E_0) \rangle$ can then be written as

$$\langle S(E_0)\rangle = \exp\left(-i\langle\Gamma_\mu\rangle \int d\epsilon_\mu \left\{\rho(\epsilon_\mu)\Big/(E_0-\epsilon_\mu)+\frac{i}{2}I\right\}\right), \tag{6}$$

where $\rho(\epsilon_\mu)$ is the density of ϵ_μ. We take the density $\rho(\epsilon_\mu)$ to be given by Wigner's semi-circle law (Wigner 1957) as

$$\rho(\epsilon_\mu) = \frac{\pi}{2ND^2}\sqrt{\frac{4N^2D^2}{\pi^2}-\epsilon_\mu^2}, \tag{7}$$

where N is the total number of resonances, D their average spacing and $\langle\epsilon_\mu\rangle = E_0$ is taken to be zero.

Putting this distribution in expression (6) and carrying out the integration we get

$$\langle S\rangle = \exp\left[-\frac{\pi\langle\Gamma_\mu\rangle}{D}\left\{\left(1+\frac{\pi^2I^2}{16N^2D^2}\right)^{\frac{1}{2}}-\frac{\pi I}{4ND}\right\}\right]. \tag{8}$$

As usual in the theory of average cross-sections we now let the averaging energy interval $\triangle E = ND$ much larger than I. This finally gives us

$$\langle S\rangle = \exp\left(-\frac{\pi\langle\Gamma_\mu\rangle}{D}\right), \tag{9}$$

which was earlier derived using series expansion.

4. Concluding remarks

As we had remarked earlier the problem of energy averaging of the pole-resonance form of the scattering matrix is an important theoretical problem in the study of the average resonance cross-sections. We have shown that an exact differential equation can be established to evaluate such quantities. Since the parameter λ lies between zero and unity the sums on the right hand side of equation (4) can be evaluated in a straightforward fashion. Because of this it is obvious that the differential equation technique is much more elegant than the earlier series expansions in handling the problem of energy averaging. Using Wigner's semi-circle law for the distribution of ϵ_μ we have rederived the earlier expression for the average of the scattering function $S(E)$.

References

Agassi D and Weidenmüller H 1975 *Phys. Lett.* **56** 309
Ericson T E 1963 *Ann. Phys.* (*N.Y.*) **23** 390
Feshbach H, Kerman A K and Lemmer R H 1967 *Ann. Phys.* (*N.Y.*) **41** 230
Mekjian A 1973 *Nucleon-Nucleus Collisions and Intermediate Structure in Advances in Nuclear Physics* eds. M Baranger and E Vogt (New York: Plenum Press) Vol **8** p. 1
Moldauer P A 1967 *Phys. Rev. Lett.* **19** 1047
Moldauer P A 1969 *Phys. Rev.* **177** 1841
Nazakat Ullah 1968 *Phys. Rev. Lett.* **20** 1510
Titchmarsh E C 1961 *Theory of functions* (London: Oxford University Press)
Vogt E 1959 *Resonance Reactions, Theoretical in Nuclear Reactions* eds. P M Endt and M Demur. (Amsterdam: N. H. Pub. Co.) Vol. I, Chapter V
Vogt E 1968 *The Statistical Theory of Nuclear Reactions in Advances in Nuclear Physics* eds. M Baranger and E Vogt (New York: Plenum Press) Vol. 1, p. 21
Wigner E P 1957 *Statistical Properties of Real Symmetric Matrices with Many Dimensions*, Canadian Mathematical Congress Proceedings (Toronto: Univ. of Toronto Press), p. 174