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The effect of magnetic fields and boundary conditions on the Couette flow of nematics

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Abstract. The paper discusses the theory of Couette flow of a nematic liquid crystal. The apparent viscosity, orientation and velocity profiles are computed for *p*-azoxyanisole as functions of shear rate and magnetic field for symmetric and asymmetric molecular alignments at the boundaries and for different relative radii of the cylinders. For symmetric homeotropic boundary condition an azimuthal field exhibits a threshold analogous to a Freedericksz transition. An expression is also derived for the Freedericksz threshold in the hydrostatic case.

Keywords. Liquid crystal; *p*-azoxyanisole; Couette flow.

1. Introduction

The Couette flow of nematics has formed the subject of some theoretical and experimental studies in recent years. Atkin and Leslie (1970) have given the analytical solution and scaling analysis for this type of flow in the absence of magnetic fields. Their theory is in qualitative agreement with the experimental results of Porter and Johnson (1962) who found an increase in the apparent viscosity with a decrease in gap width. Currie (1970) has investigated a number of possible solutions in the presence of magnetic fields without, however, investigating apparent viscosity. Extending our previous studies (Kini and Ranganath 1975) we discuss here the simplest and most likely solutions for Couette flow and present detailed calculations for the apparent viscosity as functions of the shear rate (or equivalently the relative angular velocity of the two cylinders), of radial and azimuthal magnetic fields and also of the geometry of the viscometer, for symmetric and asymmetric boundary conditions of molecular alignment at the two cylinders.

2. Theory

The differential equations governing the director orientation and velocity fields for Couette flow in the presence of a magnetic field have been derived by Currie (1970). A more simple and direct approach for deriving these equations is given below. The nematic is assumed to be confined in the annular space between two coaxial cylinders of radii R_1 and R_2 ($R_2 > R_1$) with their common axis along z. They rotate about their common axis with constant angular velocities Ω_1 and Ω_2 respectively. We consider only planar flows and seek steady state solutions for the director and velocity fields in cylindrical polar coordinates

$$n_r = \sin\theta(r) \qquad n_\psi = \cos\theta(r) \qquad n_z = 0$$

$$v_r = 0 \qquad v_\psi = rw(r) \qquad v_z = 0. \tag{1}$$

The magnetic field components, consistent with the symmetry of the solution for the director orientation are those which depend only on r, with H_z equal to zero. In order that H_r and H_ψ may satisfy Maxwell's equations

$$\nabla \cdot H = 0 \quad \text{and} \quad \nabla \times H = 0$$

we must have

$$H = \left(\frac{A}{r}, \frac{B}{r}, 0\right), \tag{2}$$

where A and B are constants.

The Ericksen-Leslie equations (Ericksen 1962 *a*, Leslie 1968) governing the hydromechanics of nematics are

$$\rho \dot{v}_i = F_i + t_{ji,j} \tag{3 a}$$

$$\rho_1 \ddot{n}_i = G_i + g_i + \pi_{ji,j} \tag{3 b}$$

where ρ is the density, ρ_1 the moment of inertia of the director, F_i the external body force per unit volume, G_i the external director body force per unit volume t_{ji} the stress tensor, π_{ji} the director surface stress, g_i the intrinsic director body, force

$$t_{ji} = -p\delta_{ij} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + \mu_1 d_{kp} n_k n_p n_i n_j + \mu_2 n_j n_i N_i$$

$$+ \mu_3 n_i N_j + \mu_4 d_{ij} + \mu_5 n_j d_{ik} n_k + \mu_6 n_i d_{jk} n_k,$$

$$\pi_{ji} = \frac{\partial W}{\partial n_{i,j}}$$

$$g_i = -\frac{\partial W}{\partial n_i} + \lambda_1 N_i + \lambda_2 d_{ij} n_j + \gamma n_i,$$

$$N_i = \dot{n}_i - w_{ik} n_k, \qquad w_{ik} = (v_{i,k} - v_{k,i})/2.$$

$$d_{ik} = (v_{i,k} + v_{k,i})/2, \qquad \lambda_1 = \mu_2 - \mu_3, \ \lambda_2 = \mu_5 - \mu_6.$$

Here μ_1 to μ_6 are the viscosity coefficients introduced by Leslie (1968), W is the Frank elastic energy per unit volume ρ and γ are arbitrary constants arising from constraints that the fluid is incompressible and the director is of constant magnitude.

$$2W = k_{11}(\nabla \cdot n)^2 + k_{22}(n \cdot \nabla \times n)^2 + k_{33}[(n \cdot \nabla) n]^2$$

with k_{11}, k_{22}, k_{33} as the elastic constants of a nematic liquid crystal.

Since the magnetic field is inhomogeneous it contributes to the body force influencing the translational velocity

$$F_i = [\triangle\chi H_r n_r n_k + \chi_2 H_k] H_{k,i} \tag{4}$$

where $\triangle\chi$ is the diamagnetic susceptibility anisotropy ($= \chi_1 - \chi_2$) and χ_1, χ_2 ($\chi_1 > \chi_2$) are the principal diamagnetic susceptibilities of the nematic (Ericksen 1962 *a*). The director orienting external body force G_k is given by

$$G_k = \triangle\chi H_r n_r H_k. \tag{5}$$

The surviving components of the director surface stress π_{ji} and the intrinsic director body force g_k are

$$\pi_{rr} = k_{11}\left[\cos\theta\frac{d\theta}{dr} + \frac{\sin\theta}{r}\right] + k_{33}\left[\sin^2\theta\cos\theta\frac{d\theta}{dr} - \frac{\sin\theta\cos^2\theta}{r}\right]$$

$$\pi_{\psi\psi} = k_{11}\left[\cos\theta\frac{d\theta}{dr} + \frac{\sin\theta}{r}\right] + k_{33}\left[\frac{\sin\theta\cos^2\theta}{r} - \sin^2\theta\cos\theta\frac{d\theta}{dr}\right]$$

$$\pi_{r\psi} = k_{33}\left[\frac{\sin^2\theta\cos\theta}{r} - \sin^3\theta\frac{d\theta}{dr}\right]$$

$$\pi_{\psi r} = k_{33}\left[\sin\theta\cos^2\theta\frac{d\theta}{dr} - \frac{\cos^3\theta}{r}\right] \tag{6}$$

$$g_r = k_{33}\left[\frac{\cos\theta}{r}\frac{d\theta}{dr} - \sin\theta\left(\frac{d\theta}{dr}\right)^2\right] + \frac{(\lambda_1+\lambda_2)}{2}r\frac{dw}{dr}\cos\theta + \gamma\sin\theta$$

$$g_\psi = -k_{33}\left[\frac{\cos\theta}{r^2} - \frac{\sin\theta}{r}\frac{d\theta}{dr}\right] + \frac{(\lambda_2-\lambda_1)}{2}r\frac{dw}{dr}\sin\theta + \gamma\cos\theta.$$

$$G_r = \frac{A\triangle\chi}{r^2}(A\sin\theta + B\cos\theta)$$

$$G_\psi = \frac{B\triangle\chi}{r^2}(A\sin\theta + B\cos\theta) \tag{7}$$

are the r and ψ components of G.

The expressions for π_{jk} and g_i are not identical with those derived by Atkin and Leslie (1970). This is because these authors have used the expression for W proposed by Leslie (1968) which includes terms with coefficients k_{22} and k_{24} in addition to those containing k_{11} and k_{33}. We have used Frank's expression for W, and since the deformations involved are merely splay and bend the terms involving the twist constant k_{22} vanish from our expressions for π_{jk} and g_i. Further, Ericksen (1962 *b*) has shown that k_{24} does not play a part in the equations of equilibrium. In the present work we solve the equations of motion to get the apparent viscosity, θ profile and velocity profile, which are in no way determined by k_{24}. The absence of k_{22} and k_{24} is therefore justified as is confirmed by the fact that our differential eqs (9) and (15) are identical with (3.14) and (3.17) of Atkin and Leslie (1970) except for magnetic terms, and that (14) is identical with (3.16).

Under the assumption of steady state, eq. (3 *b*) can be written as

$$\frac{d\pi_{rr}}{dr} + \frac{(\pi_{rr} - \pi_{\psi\psi})}{r} + G_r + g_r + \rho_1 w^2\sin\theta = 0$$

$$\frac{d\pi_{r\psi}}{dr} + \frac{(\pi_{r\psi} + \pi_{\psi r})}{r} + G_\psi + g_\psi + \rho_1 w^2\cos\theta = 0. \tag{8}$$

Substituting for π_{jm}, g_k and G_i and eliminating $(\gamma + \rho_1 w^2)$ from the eqs (8) we obtain

$$2f(\theta)\left[\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right]r^2 + r^2\frac{df}{d\theta}\left[\frac{1}{r^2}+\left(\frac{d\theta}{dr}\right)^2\right]$$

$$+ r^3\frac{dw}{dr}(\lambda_1+\lambda_2\cos 2\theta)$$

$$+ 2\triangle\chi\,[\sin\theta\cos\theta\,(A^2-B^2)+AB\cos 2\theta] = 0 \tag{9}$$

where

$$f(\theta) = k_{11}\cos^2\theta + k_{33}\sin^2\theta.$$

The surviving components of the stress tensor t_{ji} and the external body force F_i are

$$t_{rr} = -p - f(\theta)\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{2r}\frac{df}{d\theta}\frac{d\theta}{dr} + r\frac{dw}{dr}g_1(\theta) \equiv -p - \hat{t}_{rr}$$

$$t_{\psi\psi} = -p - \frac{f(\theta)}{r^2} + \frac{1}{2r}\frac{df}{d\theta}\frac{d\theta}{dr} + r\frac{dw}{dr}g_2(\theta) \equiv -p - \hat{t}_{\psi\psi}$$

$$t_{r\psi} = \frac{f(\theta)}{r}\frac{d\theta}{dr} - \frac{1}{2r^2}\frac{df}{d\theta} + g(\theta)\,r\frac{dw}{dr}$$

$$t_{\psi r} = \frac{1}{r}\frac{d\theta}{dr}\left[\frac{1}{2}\frac{d^2f}{d\theta^2}+f\right] - \frac{1}{2}\frac{df}{d\theta}\left(\frac{d\theta}{dr}\right)^2 + \frac{r}{2}\frac{dw}{dr}\,[2g(\theta)+\lambda_1+\lambda_2\cos 2\theta]$$

$$F_r = -\frac{\chi_2}{r^3}(A^2+B^2) - \frac{\triangle\chi}{r^3}(A\sin\theta + B\cos\theta)^2$$

$$F_\psi = \frac{\triangle\chi}{r^3}[\sin\theta\cos\theta\,(A^2-B^2)+AB\cos 2\theta] \tag{10}$$

where

$$2g(\theta) = 2\mu_1\sin^2\theta\cos^2\theta + \mu_4 + (\mu_5-\mu_2)\sin^2\theta + (\mu_6+\mu_3)\cos^2\theta$$

$$2g_1(\theta) = 2\mu_1\sin^3\theta\cos\theta + (\mu_2+\mu_3+\mu_5+\mu_6)\sin\theta\cos\theta$$

$$2g_2(\theta) = 2\mu_1\sin\theta\cos^3\theta + (\mu_5+\mu_6-\mu_2-\mu_3)\sin\theta\cos\theta.$$

Similarly in the steady state, eqs (3 *a*) become

$$\frac{d\hat{t}_{rr}}{dr} + \frac{(\hat{t}_{rr}-\hat{t}_{\psi\psi})}{r} + F_r - \frac{\partial p}{\partial r} = -\rho r w^2$$

$$\frac{dt_{r\psi}}{dr} + \frac{1}{r}(t_{r\psi}+t_{\psi r}) + F_\psi - \frac{\partial p}{\partial\psi} = 0$$

$$\frac{\partial p}{\partial z} = 0\,. \tag{11}$$

A separation of variables yields

$$p = E\psi + L(r)$$

where E is a constant and

$$\frac{dL}{dr} = \rho r w^2 + \frac{d\hat{t}_{rr}}{dr} + \frac{(\hat{t}_{rr} - \hat{t}_{\psi\psi})}{r} + F_r \,.$$

For a single valued solution we put $E = 0$ and hence eqs (11) reduce to

$$p = L(r)$$

and

$$\frac{dt_{r\psi}}{dr} + \frac{t_{r\psi} + t_{\psi r}}{r} + F_\psi = 0. \tag{12}$$

Substituting for t_{jk} and F_m in (12),

$$2f(\theta)\left[\frac{d^2\theta}{dr^2} + \frac{1}{r}\,\frac{d\theta}{dr}\right]r^2 + \frac{df}{d\theta}\left[\frac{1}{r^2} + \left(\frac{d\theta}{dr}\right)^2\right]r^2 + r^3\,\frac{dw}{dr}\,(\lambda_1 + \lambda_2\cos 2\theta)$$
$$+ 2\triangle\chi\,[\sin\theta\cos\theta\,(A^2 - B^2) + AB\cos 2\theta]$$
$$+ 2r^3\frac{d}{dr}\left[r\frac{dw}{dr}\,g(\theta)\right] + 4r^3 g(\theta)\,\frac{dw}{dr} = 0. \tag{13}$$

Using eqs (9) and (12) we obtain

$$\frac{d}{dr}\left[r\frac{dw}{dr}g(\theta)\right] + 2\,\frac{dw}{dr}\,g(\theta) = 0$$

which integrates to yield

$$r^3\frac{dw}{dr}\,g(\theta) = c = \text{a constant} \tag{14}$$

$\tau_{ji} = e_{ipq}\, n_p\, \pi_{jq}$ is the director contribution to the couple stress. The total couple per unit length of a cylinder of radius r is

$$\tau' = 2\pi\,(r^2\, t_{r\psi} + r\tau_{rz}) = 2\pi c = \text{a constant.}$$

Hence c is a measure of the couple per unit length on either cylinder. Using eqs (14) and (9),

$$f(\theta)\left[\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr}\right]r^2 + r^2\frac{df}{d\theta}\left[\frac{1}{r^2} + \left(\frac{d\theta}{dr}\right)^2\right]$$
$$+ 2\triangle\chi\,[\sin\theta\cos\theta\,(A^2 - B^2) + AB\cos 2\theta]$$
$$+ \frac{c}{g(\theta)}(\lambda_1 + \lambda_2\cos 2\theta) = 0. \tag{15}$$

We assume boundary conditions for θ and w in the form

$$\theta(R_1) = \theta_1; \qquad \theta(R_2) = \theta_2$$
$$w(R_1) = \Omega_1; \qquad w(R_2) = \Omega_2. \tag{16}$$

For a certain value of c we solve eq. (15) to obtain θ as a function of r, compatible with (16). Then we integrate eq. (14) to obtain the relative angular velocity, $\triangle \Omega$, between the two cylinders

$$\triangle \Omega = \Omega_2 - \Omega_1 = c \int_{R_1}^{R_2} \frac{dr}{r^3 g(\theta)}$$

and calculate the apparent viscosity

$$\eta = \frac{c(R_2^2 - R_1^2)}{2(\Omega_2 - \Omega_1) R_1^2 R_2^2}.$$

It is of interest to note that the apparent viscosity does not depend directly on the angular velocity of either cylinder but on their relative angular velocity.

3. Results

Computations have been carried out for *p*-azoxyanisole (PAA). The elastic and viscosity coefficients have been assumed to be the same as those used by Tseng *et al* (1972; see table 1 of Kini and Ranganath 1975) and the anisotropy of diamagnetic susceptibility $\triangle \chi$ has been taken to be $0{\cdot}136 \times 10^{-6}$ cgs (Gasparoux and Prost 1971). The equations have been solved by the orthogonal collocation method used by Finlayson 1972; Tseng *et al* 1972; see also Villadsen and Stewart 1967. We have chosen 16 collocation points corresponding to the sixteen zeros of the Legendre polynomial P_{16}, with double precision arithmetic. Computations have been repeated with 24 collocation points for cases involving large deformations. The values of η obtained from the two calculations are found to agree within 1%. We have chosen $R_1 = 1$ cm and $R_2 = 1{\cdot}005$ cm for studying the variation of η with shear rate and magnetic fields. We have considered two boundary conditions.

$\underline{\theta_1 = \pi/2 = \theta_2}$

At low shear rates η remains almost constant at the Miesowicz value $\eta_A = (\mu_4 + \mu_5 - \mu_2)/2 = 0{\cdot}092$ poise for PAA (Miesowicz 1936, 1946). As the shear rate increases η decreases and approaches the Miesowicz value $\eta_{\parallel} = (\mu_3 + \mu_4 + \mu_6)/2 = 0{\cdot}024$ poise for PAA, corresponding to molecules being oriented along the flow. When a radial field $H_r (= A/r)$ is applied there is a stabilizing effect and η decreases at first more slowly with the increase of shear rate and finally approaches $\eta_{\parallel}$ at large shear rates.

In general η decreases in the presence of an azimuthal field $H_\psi (= B/r)$ (figure 1 *a*). At low shear rates η remains almost constant at η_A, until B attains a value

$$B_c \approx \frac{\pi R_1}{(R_2 - R_1)} \left[\frac{k_{33}}{\triangle \chi}\right]^{\frac{1}{2}}$$

(see appendix), which in the static case corresponds to the Freedericksz threshold below which there is no deformation. Above B_c there is a fairly rapid change of η with field and for large B, η approaches $\eta_{\parallel}$ (figure 2 *a*). At low shear rates

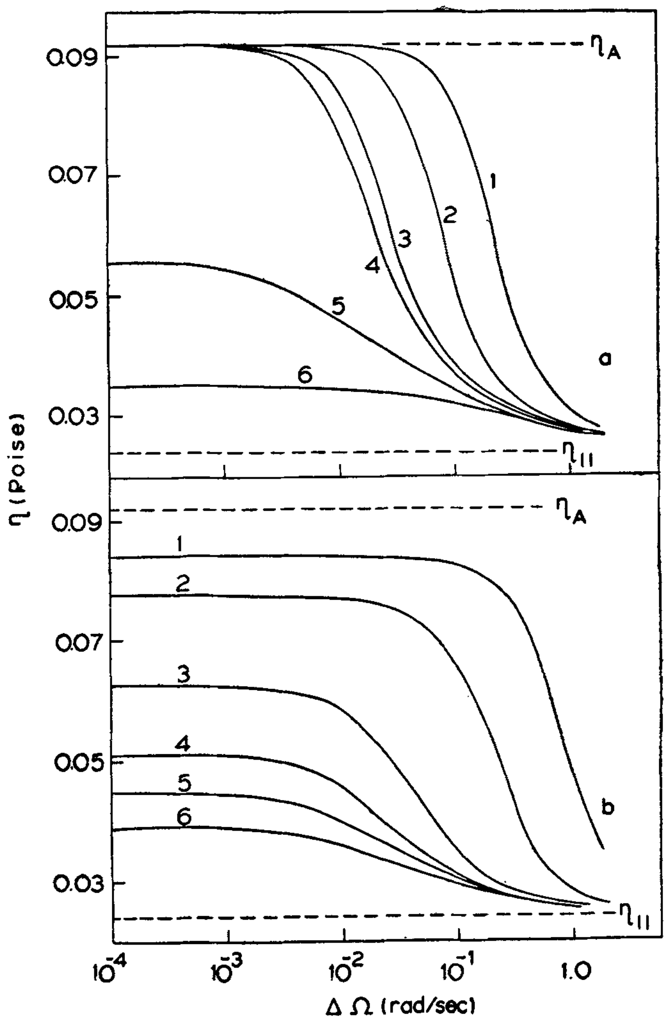

Figure 1. Variation of apparent viscosity with shear rate and magnetic field for $R_2/R_1 = 1\cdot005$.
(*a*) $\theta_1 = \pi/2 = \theta_2$;
Radial magnetic field A = (1) 5000, (2) 2500, (3) 0 gauss cm;
Azimuthal magnetic field B = (4) 1000, (5) 2000, (6) 3000 gauss cm;
(*b*) $\theta_1 = 0$; $\theta_2 = \pi/2$
A = (1) 10000, (2) 5000, (3) 2000, (4) 1000, (5) 0 gauss cm, (6) B = 1000 gauss cm.

the value of η depends on whether B is greater or less than B_c but at high shear rates it approaches $\eta_{\parallel}$ regardless of the magnitude of B. At large shear rates η goes up from about $\eta_{\parallel}$ and attains η_A as A increases from a low to a high value (figure 2 *b*).

At any given shear rate η decreases when the ratio R_2/R_1 is increased (figure 3 *a*). This is in qualitative agreement with the results of Porter and Johnson (1962) who found an increase in the apparent viscosity for a decrease in gap width. Atkin and Leslie (1970) have predicted that the extremum of θ, for $\theta_1 = \theta_2 = \pi/2$ at any shear rate should occur at a point $r_m = [R_1R_2]^{\frac{1}{2}}$. When $(R_1 + R_2)/2 \gg$

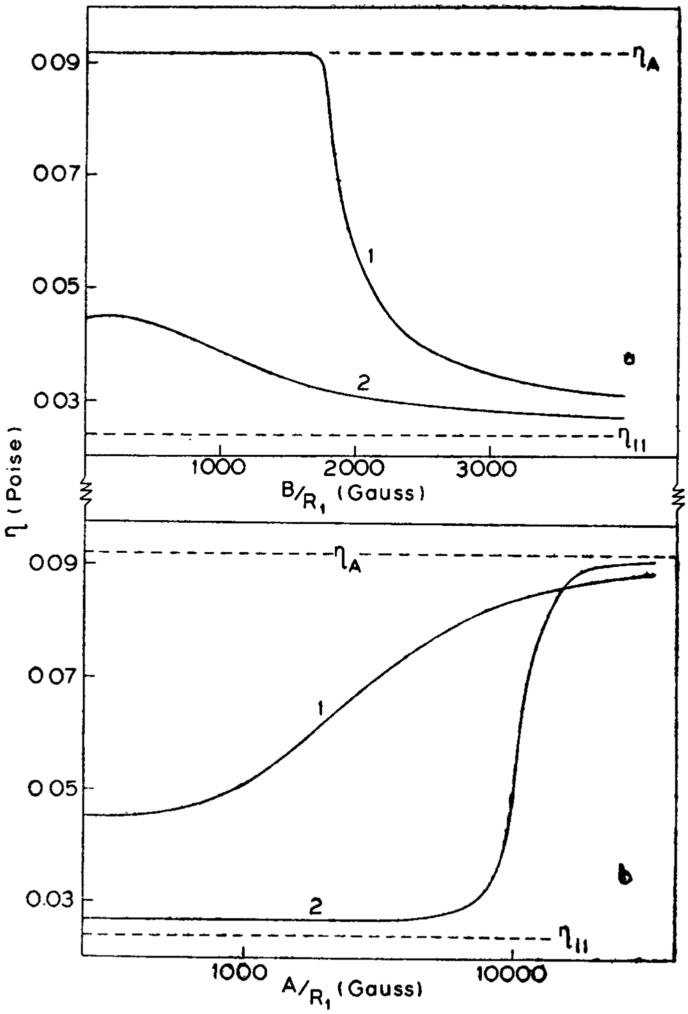

Figure 2. Variation of apparent viscosity with magnetic field for $R_2/R_1 = 1{\cdot}005$. (*a*) Azimuthal field $c = 10^{-3}$ dyne (1) $\theta_1 = \pi/2 = \theta_2$, (2) $\theta_1 = 0$; $\theta_2 = \pi/2$; (*b*) Radial field, (1) $\theta_1 = 0$; $\theta_2 = \pi/2$, $c = 10^{-3}$ dyne (2) $\theta_1 = \theta_2 = \pi/2$; $c = 10{\cdot}0$ dyne.

$(R_2 - R_1)$, as is true in the present case with $R_1 = 1$, $R_2 = 1{\cdot}005$ cm and $r_m \approx (R_1 + R_2)/2$, the orientation profile is almost symmetric. But the asymmetry in the orientation profile becomes perceptible as the ratio R_2/R_1 increases (figure 3 *b*).

$\theta_1 = 0$; $\theta_2 = \pi/2$

The variation of η with shear rate, magnetic field and geometry in this configuration can also be explained in a similar manner. Two points may be emphasized (*a*) the initial value of η at low shear rates for $R_2/R_1 = 1{\cdot}005$ cm and in the absence of magnetic fields is $0{\cdot}5\,\eta_A$, about half of what it is for $\theta_1 = \pi/2 = \theta_2$ because a deformation is present even in the absence of flow. As expected this initial value increases with increasing A, attaining η_A at large values of A; and (*b*) there is no threshold field for H.

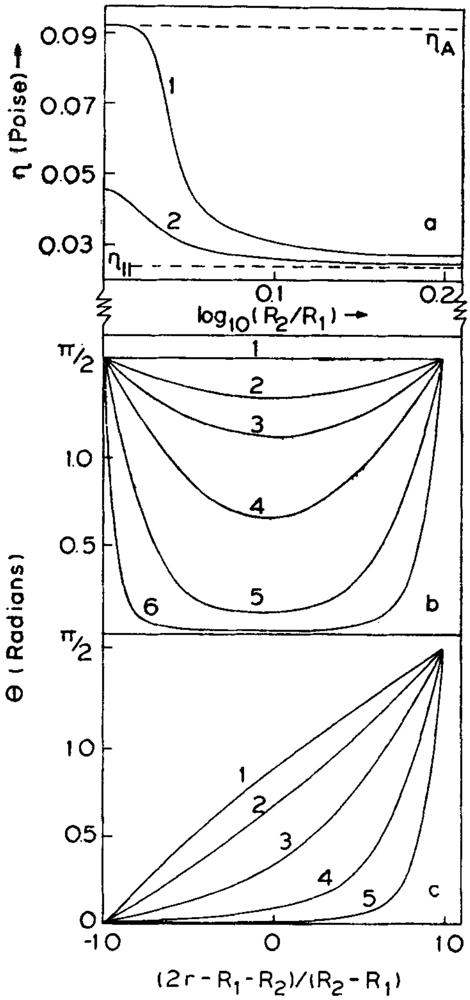

Figure 3. Variation of apparent viscosity and orientation profile with geometry.
(*a*) Variation of apparent viscosity with $\log_{10}(R_2/R_1)$; $c = 10^{-3}$ dyne, (1) $\theta_1 = \pi/2 = \theta_2$, (2) $\theta_1 = 0$; $\theta_2 = \pi/2$;
(*b*) Orientation profile $\theta_1 = \pi/2 = \theta_2$, $c = 10^{-3}$ dyne, $R_2/R_1 =$ (1) 1·005, (2) 1·05, (3) 1·071, (4) 1·1, (5) 1·25, (6) 2·0;
(*c*) Orientation profile $\theta_1 = 0$; $\theta_2 = \pi/2$, $c = 10^{-3}$ dyne, $R_2/R_1 =$ (1) 1·005, (2) 1·05, (3) 1·1, (4) 1·25, (5) 2·0.

The corresponding orientation and velocity profiles for the two types of boundary conditions are given in figures 3, 4 and 5.

It is found that there is a certain similarity between the results for Couette flow with $R_1 = 1$ cm, $R = 1.005$ cm and those obtained for shear flow (Kini 1976) for a $50\,\mu$ sample; this is as it should be since for very large values of r the two results should coincide.

We have not treated the case in which $\theta_1 = \theta_2 = 0$ with molecules aligned along the flow direction. Atkin and Leslie (1970) predict that at high shear rates all molecules will be aligned at an angle θ_0 given by $\cos 2\theta_0 = -\lambda_1/\lambda_2$. With the parameters used for PAA, $\theta_0 = 0$ so that there will be no change in η with shear rate and the behaviour will be Newtonian.

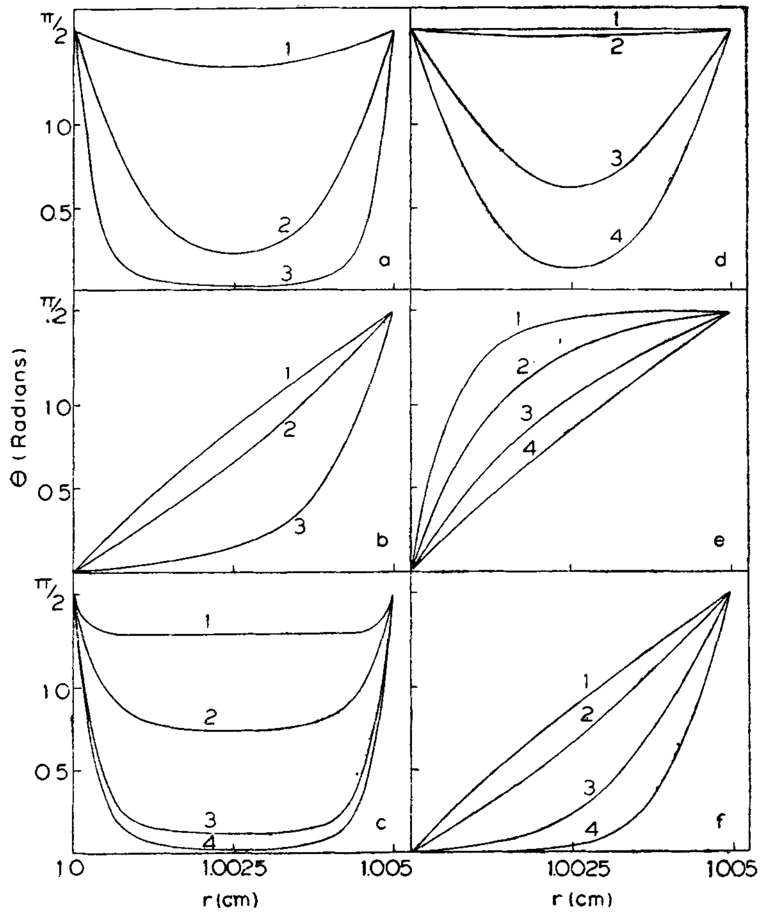

Figure 4. Orientation profiles for various magnetic fields and shear rates for $R_2/R_1 = 1{\cdot}005$.
(*a*) $H = 0$, $\theta_1 = \theta_2 = \pi/2$, $c =$ (1) 0·1, (2) 1·0, (3) 10·0 dynes;
(*b*) $H = 0$, $\theta_1 = 0$; $\theta_2 = \pi/2$, $c =$ (1) 0·01, (2) 0·1, (3) 1·0 dyne;
(*c*) $c = 10{\cdot}0$ dyne, $\theta_1 = \pi/2 = \theta_2$, $A =$ (1) 15000, (2) 10000, (3) 5000, (4) 0 gauss cm;
(*d*) $c = 10^{-3}$ dyne, $\theta_1 = \pi/2 = \theta_2$, $B =$ (1) 0, (2) 1700, (3) 2000, (4) 3000 gauss cm;
(*e*) $c = 10^{-3}$ dyne, $\theta_1 = 0$; $\theta_2 = \pi/2$, $A =$ (1) 4000, (2) 2000, (3) 1000, (4) 0 gauss cm;
(*f*) $c = 10^{-3}$ dyne, $\theta_1 = 0$; $\theta_2 = \pi/2$, $B =$ (1) 0, (2) 1000, (3) 2000, (4) 3500 gauss cm·

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Appendix

Leslie (1970)* has considered a nematic confined in the annular space of two coaxial cylinders of radii R_1 and R_2 ($R_2 > R_1$), in the presence of a radial magnetic field

* Thanks are due to a referee for bringing this work to the author's notice.

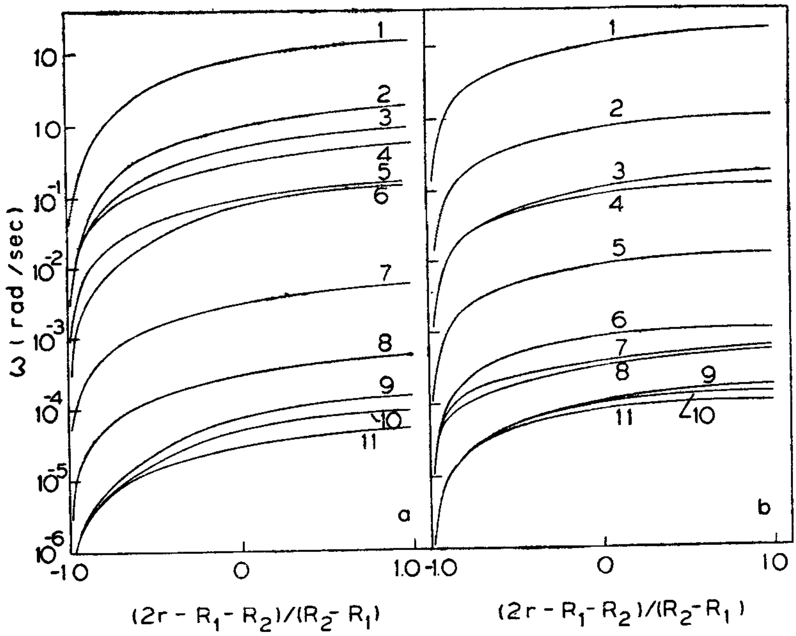

Figure 5. Velocity profiles for various magnetic fields and shear rates.
(*a*) $\theta_1 = \pi/2 = \theta_2$, (1) $c = 10^{-3}$ dyne, $H = 0$, $R_2/R_1 = 1\cdot25$;
$c = 10\cdot0$ dyne, $R_2/R_1 = 1\cdot005$, $A =$ (2) 0, (3) 10000, (4) 15000 gauss cm; (5) $H = 0$, $c = 10^{-3}$ dyne, $R_2/R_1 = 1\cdot083$; $H = 0$, $R_2/R_1 = 1\cdot005$, $c =$ (6) 1·0, (7) 0·1, (8) 10^{-2} dyne; $c = 10^{-3}$ dyne, $R_2/R_1 = 1\cdot005$, $B =$ (9) 3000, (10) 2000, (11) 0 gauss cm.
(*b*) $\theta_1 = 0$, $\theta_2 = \pi/2$;
$c = 10^{-3}$ dyne, $H = 0$, $R_2/R_1 =$ (1) 1·25, (2) 1·1, (3) $c = 1\cdot0$ dyne, $H = 0$, $R_2/R_1 = 1\cdot005$; (4) $c = 10^{-3}$ dyne, $H = 0$, $R_2/R_1 = 1\cdot05$; $R_2/R_1 = 1\cdot005$, $H = 0$, $c =$ (5) 0·1, (6) 0·01 dyne; $c = 10^{-2}$ dyne, $R_2/R_1 = 1\cdot005$, $A =$ (7) 5000, (8) 10000 gauss cm;
$c = 10^{-3}$ dyne, $R_2/R_1 = 1\cdot005$, $B =$ (9) 3000, (10) 1500, (11) 0 gauss cm.

$$= \frac{H}{r}, \qquad H_\psi = 0, \qquad H_z = 0.$$

Seeking a solution for the director in the form $n_r = \sin\theta(r)$, $n_\psi = \cos\theta(r)$, $n_z = 0$ he finds that for boundary conditions $\theta(R_1) = 0 = \theta(R_2)$ there exists a critical value of H_c given by

$$H_c = \left[\left\{\frac{k_{11}\pi^2}{\left[\ln\left(\frac{R_2}{R_1}\right)\right]^2} + k_{11} - k_{33}\right\}\frac{1}{\Delta\chi}\right]^{\frac{1}{2}}$$

below which there is no deformation. For this case if the value of R_2/R_1 exceeds a value R_c, given by

$$\frac{k_{11}\pi^2}{[\ln R_c]^2} = (k_{33} - k_{11}),$$

H_c can become imaginary. This is a direct consequence of k_{11} being smaller than k_{33} for all known nematics.

Here we consider a similar problem, but with the boundary condition $\theta(R_1) = \theta(R_2) = \pi/2$ for an azimuthal magnetic field

$$H_r = 0, \qquad H_\psi = \frac{B}{r}, \qquad H_z = 0.$$

We assume that the director lies in the $r\psi$ plane and seek a solution in the form $\boldsymbol{n} = (\sin\theta(r), \cos\theta(r), 0)$ with boundary conditions $\theta(R_1) = \pi/2$ and $\theta(R_2) = \pi/2$ in the presence of a magnetic field $\boldsymbol{H} = (0, B/r, 0)$.

Equations (3 *b*) simplify to

$$2r^2 f(\theta)\frac{d^2\theta}{dr^2} + r^2\frac{df}{d\theta}\left(\frac{d\theta}{dr}\right)^2 + 2rf\left(\frac{d\theta}{dr}\right) + \frac{df}{d\theta}$$

$$-2\triangle\chi B^2\sin\theta\cos\theta = 0.$$

Multiplying by $\frac{d\theta}{dr}$ and integrating we obtain

$$f(\theta) + r^2 f(\theta)\left(\frac{d\theta}{dr}\right)^2 - \triangle\chi H^2\sin^2\theta = C' \tag{17}$$

where C' is a constant.

An analysis similar to the one given by Atkin and Leslie (1970) shows that $\frac{d\theta}{dr} = 0$ and $\theta = \theta_m$ an extremum at $r = r_m = [R_1R_2]^{\frac{1}{2}}$. From (17) we get $C' = f(\theta_m) - \triangle\chi B^2\sin^2\theta_m$. Substituting in (17) and integrating we find

$$\int_{R_1}^{r_m}\frac{dr}{r} = \log\frac{r_m}{R_1} = \int_{\pi/2}^{\theta_m}\frac{[f(\phi)]^{\frac{1}{2}}\,d\phi}{[\triangle\chi B^2(\sin^2\phi - \sin^2\theta_m) + f(\theta_m) - f(\phi)]^{\frac{1}{2}}}. \tag{18}$$

Using the Legendre transformation

$$\cos\phi = \cos\theta_m\sin\lambda$$

in (18) we obtain

$$\log\frac{r_m}{R_1} = -\int_0^{\pi/2}\left[\frac{k_{33} + (k_{11} - k_{33})\sin^2\lambda\cos^2\theta_m}{\{\triangle\chi B^2 + k_{11} - k_{33}\}\{1 - \sin^2\lambda\cos^2\theta_m\}}\right]^{\frac{1}{2}} d\lambda.$$

Taking limits as $\theta_m \to \pi/2$ we obtain a critical value B_c of B given by

$$B_c = \left[\frac{1}{\triangle\chi}\left\{\frac{k_{33}\pi^2}{\left(\log\frac{R_1}{R_2}\right)^2} + k_{33} - k_{11}\right\}\right]^{\frac{1}{2}}.$$

It is of interest to note that since $k_{33} \geqslant k_{11}$ for nematics the expression for B_c can never become either zero or imaginary for reasonable values of R_1 and R_2 (*i.e.* $R_1 \neq 0$, $R_2 \neq 0$, $R_2 \geqslant R_1$).

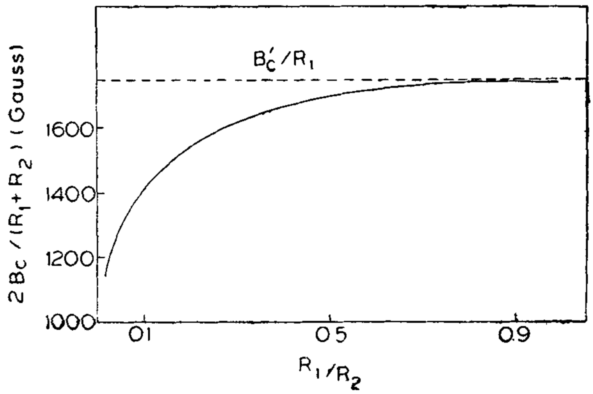

Figure 6. Variation of $2B_c/(R_1 + R_2)$ as a function of R_1/R_2 for PAA, with $R_2 - R_1 = 50\,\mu$.

f R_1 and R_2 are large compared to $d = R_2 - R_1$ we have

$$\frac{B_c'}{R_1} \approx \frac{\pi}{d}\left[\frac{k_{33}}{\Delta\chi}\right]^{\frac{1}{2}}.$$

For PAA

$$\frac{B_c'}{R_1} \approx 1750.$$

Figure 6 shows a plot of $\dfrac{2B_c}{(R_1 + R_2)}$ as a function of $\dfrac{R_1}{R_2}$ for a constant gap width of $50\,\mu$.

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