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Multiple scattering of muons in beryllium

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Abstract. The defocussing and the depolarization of a high energy muon beam in a beryllium filter, often used to eliminate accompanying pions, have been studied. The quantum mechanical transport equation of Waldmann, which can also be used to describe the multiple scattering of Dirac particles, is solved with a distorted wave Born approximation. Calculations are done for both the Thomas-Fermi and the Hartree-Fock potential of the beryllium atom. It is shown that the Hartree-Fock potential leads to a less divergent beam. The depolarization of a longitudinally polarized muon beam in passage through a thin beryllium foil is also studied.

Keywords. Beryllium filter; defocussing and depolarization of a muon beam; Waldmann's transport equation; multiple scattering; Thomas-Fermi and Hartree-Fock potential of beryllium.

1. Introduction

High energy muon scattering experiments provide useful information in nuclear and subnuclear physics. The choice of muon as the projectile has some special advantages. In particular, it eliminates several problems which one encounters with an electron beam in a bubble chamber, *e.g.*, uncertainty in the scattered electron energy due to Bremsstrahlung and pair production of Bremsstrahlung photons in the hydrogen (Carroll 1973). A number of methods are now available for production of a high energy muon beam. The one followed at SLAC in some recent experiments (Ballam *et al* 1972, Carroll 1973) involved the method of pair production in a high Z target by the Bremsstrahlung radiation of a primary electron beam. The muon beam is usually contaminated with a large number of pions. One way to eliminate the hadronic component is to pass the mixed beam through a thickness of Be. This, however, leads to a defocussing of the muon beam. Moreover, if the muons are polarized initially, there is also a depolarization in passage through matter. The purpose of this paper is to study these two aspects of multiple scattering of a beam of muons in a Be foil. We will consider muons of momenta ranging from 200 MeV/c to 1·2 GeV/c for which these effects are significant and measurable. The results obtained will be useful in the design and analysis of the scattering experiments of muons of this momentum range.

The small angle multiple scattering (ms) of fast charged particles has been studied in detail by a number of workers (Moliére 1947, 1948; Bethe 1953; Scott 1963). Moliére's theory, in particular, gives qualitative agreement with the experimental results (Hanson *et al* 1951) for unpolarized electrons. Since we also intend to study here the polarization effects of relativistic muons, we shall consider instead the quantum mechanical transport equation of Waldmann (1957), which can be used to describe the ms of particles with spin. The equation can be written as

$$\hat{P}\,\frac{\partial D(\hat{P},r)}{\partial r} = n\int A(P\,P')\,D(\hat{P}'\,r)\,A^{+}(P,P')\,d\hat{P}'$$

$$+\,2\pi ni\Big[A(P,P)\,D(\hat{P},r) - D(\hat{P},r)\,A^{+}(P,P)\Big]. \qquad (1)$$

Here D is the distribution matrix for the muon beam, P' the momentum of a particle undergoing a single scattering to come out with the momentum P with $|P| = |P'|$, and $\hat{P}$ and $\hat{P}'$ are unit vectors in the direction of P and P' respectively. The number of scatterers per unit volume is given by n. The scattering amplitude $A(P,P')$ is defined as

$$\sigma(P,P')\,\rho(\mathscr{P}) = A(P,P')\,\rho'(\mathscr{P}')\,A^{+}(P,P'), \qquad (2)$$

where $\rho'(\mathscr{P}')$ and $\rho(\mathscr{P})$ are the non-relativistic spin density matrices* of the beam before and after one scattering, and $\sigma(P,P')$ is the scattering cross-section. The first term on the right hand side of eq. (1) is an obvious generalization of the scattering out term of the classical transport equation while the second term takes into account a possible spin flip even for an undeviated particle. The 2×2 matrix D may be written as

$$D(\hat{P},r) = \tfrac{1}{2}\Big[F(\hat{P},r)\,I + \pi(\hat{P},r)\,.\,\sigma\Big]. \qquad (3)$$

The function $F(\hat{P},r)$ is the analogue of the classical distribution function and describes the essential features of multiple scattering of a beam of unpolarized particles. The vector $S(\hat{P},r) = \pi(\hat{P},r)/F(\hat{P},r)$ describes the average polarization of the beam.

Substitution of the expression (3) in eq. (1) leads to a pair of linked equations for F and π. If the small Mott polarization of the muon in a single scattering is neglected, the two equations are completely decoupled and formal solutions of the eq. (1) can be obtained. Mühlschlegel and Koppe (1958) studied this equation in the first Born approximation. One of us (Mukherjee 1967) followed up with a distorted wave approximation (Mukherjee and Majumdar 1965) after reducing the Dirac equation with a generalized form of Sommerfeld-Maue approximation. The calculations gave good agreement with the experimental results (Hanson *et al* 1951) for electrons of intermediate energies, $\epsilon \simeq 15$ MeV. The present calculations will follow the method developed earlier (Mukherjee 1967). However, to make use of the formal solutions obtained there, one has to calcu-

* We shall neglect in our calculations the small Mott polarization. This permits us to use the non-relativistic spin density matrices. The validity of this approximation is studied by Mendlowitz and Case.

late the single scattering amplitude $A(P, P')$ for an appropriate potential. It will be useful to consider in this context both the Thomas-Fermi and the Hartree-Fock potential of the beryllium atom. This will be done in the next section.

2. The screening angle in a Be foil

In Moliére's theory of small angle ms, the entire dependence of the distribution function on the atomic potential is taken into account by a single parameter, the screening angle, θ_a. In our method, there is one more potential-dependent parameter λ, which for a high energy muon in a low-Z material like Be, can be taken to be equal to 1. It is, therefore, sufficient to study the dependence of θ_a on the potential. We first consider the Thomas-Fermi (TF) potential and make use of Moliére's representation, *viz.*,

$$V(r) = \frac{a}{r} \sum_{i=1}^{3} a_i \, e^{-b_i r/r_0}, \tag{4}$$

where $a = \alpha Z$, r_0 is the TF radius, and the constants are given by

$$a_1 = 0{\cdot}10 \qquad a_2 = 0{\cdot}55 \qquad a_3 = 0{\cdot}35$$
$$b_1 = 6{\cdot}0 \qquad b_2 = 1{\cdot}2 \qquad b_3 = 0{\cdot}3. \tag{5}$$

The expression agrees with the exact TF function within $0{\cdot}002$ for $0 \leqslant r/r_0 \leqslant 6$. The expression for the screening angle for the potential (5) can be written, following Mukherjee (1967) as:

$$\lambda \ln \theta_a = \lambda \ln \theta_0 + e^{\pi a(\epsilon/p)} \left\{ \sum_j a_j^2 \ln b_j \, e^{-2a \tan^{-1} 2 \cdot \theta_j} \right.$$
$$- 2 \operatorname{Re} \sum_{j<k} a_j a_k \left(\frac{\theta_j^2 + 2i\theta_j}{\theta_k^2 - 2i\theta_k} \right)^{ia(\epsilon/p)} \left[\frac{1}{2} - \frac{b_j^2 \ln b_j - b_k^2 \ln b_k}{b_j^2 - b_k^2} \right.$$
$$\left.\left. + ia \frac{\epsilon}{p} \left(\frac{b_k^2}{b_j^2 - b_k^2} \ln^2 \frac{b_j}{b_k} - \tfrac{1}{2} L_2 \left(\frac{b_j^2 - b_k^2}{b_j^2} \right) \right) \right] \right\}, \tag{6}$$

where

$$\lambda = e^{\pi a(\epsilon/p)} \operatorname{Re} \sum_{j,k} a_j a_k \left(\frac{\theta_j^2 + 2i\theta_j}{\theta_k^2 - 2i\theta_k} \right)^{ia(\epsilon/p)}$$

ϵ and p are the energy and momentum of the muon, $\theta_j = b_j \theta_0$, $\theta_0 = 1/pr_0$, and $L_2(x)$ is Euler's dilogarithm function. The values of θ_a for the TF potential for muons of momenta ranging from 200 MeV/c to $1{\cdot}2$ GeV/c are shown in figure 1. The TF potential is not likely to be very accurate for a small-Z atom, like Be. We have, therefore, considered also the Hartree-Fock potential calculated for Be (Hartree and Hartree 1935) including the exchange term (Fleischmann 1960). The following expression gives a good fit to the HF potential considered by Fleischmann;

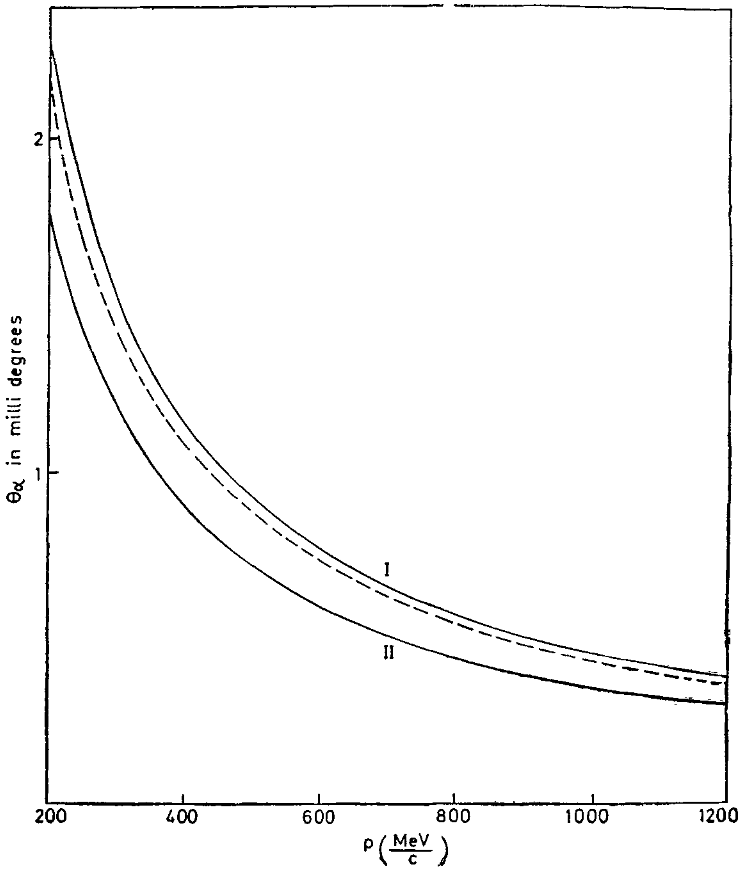

Figure 1. The screening angles for different momenta of the incident muon. The curves I and II correspond to the HF potential and the TF potential respectively of the Be atom. The dashed curve is obtained from Moliére's interpolation formula (9).

$$V(r) = \frac{4Z_p(r)}{r} = \frac{4a}{r}\left[a_1 e^{-b_1 r/r_0} + a_2\left(\frac{r}{r_0}\right) e^{-b_2 r/r_0}\right], \tag{7}$$

with

$$a_1 = 1\cdot 00 \qquad b_1 = 1\cdot 09\frac{r_0}{a_0}$$

$$a_2 = -1\cdot 54\frac{r_0}{a_0} \qquad b_2 = 3\cdot 28\frac{r_0}{a_0}, \tag{8}$$

where a_0 is the Bohr radius. We have compared in table 1 the values of Z_p obtained from (7) with those obtained by Fleischmann (1960). The calculation for the screening angle is repeated with this potential following Mukherjee (1967, 1968) and the screening angle θ_a for various energies of the muon shown in figure 1, are obtained. It may be useful to compare with the interpolation expression for the screening angle obtained by Moliére, for the TF poential,

$$\theta_a^2 = \theta_0^2 (1\cdot 13 + 3\cdot 76\,\beta^2) \tag{9}$$

where $\beta = Ze^2/\hbar v$, v being the velocity of the projectile. We have shown in figure 1 the screening angles obtained from this formula for various momenta of the muons. Those are larger than the screening angles obtained by us for the TF potential but consistently lower than those for the HF potential.

The knowledge of the screening angle is sufficient to determine the distribution function $F(\chi, t)$, giving the probability per unit solid angle of a total deflection by an angle χ from the incident direction after traversing a thickness t. We first define the characteristic angle θ_c by the relation

$$\theta_c = \sqrt{4\pi N}\,\frac{\epsilon a}{p^2}, \tag{10}$$

where N gives the number of scatterers per unit area of the Be foil. In (10), a factor $\sqrt{(1 + 1/Z)}$, which one includes in the case of an electron to account for the incoherent electron-electron scattering, has been omitted, since the correction cannot be taken into account easily in this case. The angles θ_c and θ_a determine a parameter B from the relation

$$B - \ln B = \ln \frac{e\lambda\theta_c^2}{\gamma^2\theta_a^2}, \tag{11}$$

where $\ln\ \gamma$ is Euler's constant. Considering a change of scale given by

$$\phi = \chi/\theta_c\sqrt{B\lambda}, \tag{12}$$

it is possible to express the reduced distribution function $f(\phi, t)$ exactly in the Moliére's form, *viz.*,

$$f(\phi, t) = \frac{1}{2\pi}\int_0^\infty \eta\, d\eta\, J_0(\phi\eta) \exp\left\{-\frac{\eta^2}{4} + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4}\right\}. \tag{13}$$

The integral has to be evaluated numerically to obtain $F(\chi, t)$, keeping (12) in mind. The plot of $F(\chi, t)$ against χ is a modified Gaussian distribution with a tail which describes essentially the single scattering phenomena through large angles. The defocussing of the beam can be studied quantitatively by calculating the $1/e$-th width of the emergent beam. For example, consider a muon beam of momentum 200 MeV/c passing through a Be foil of thickness 0·256 cm. which gives $\theta_c = 0\cdot 002$ radians and $B = 10$. The $1/e$-th width of the emergent beam with HF potential is 0·006 rad. Thus even if the beam is cut-off on emergence at its $1/e$-th width, the defocussing will lead to a large beam width when the target is placed at a distance of few meters.

Two features of our results should be pointed out:

(i) As is clear from figure 1, the HF potential gives a larger screening angle than what is obtained with the TF potential. This implies that the HF potential will lead to a lower value for the $1/e$-th width. In the case of electrons, Mukherjee (1968) has shown that the HF potential gives a better agreement with the experimental results, as one would normally expect for a Be target. The TF potential gives too large a value for the atomic potential of Be, as is clear from table 1,

Table 1. Atomic potential of beryllium

r/a_0	Hartree-Fock potential $Z_p(r)$ as obtained from (7)	$Z_p(r)$ Fleischmann	Thomas-Fermi potential $Z_p(r)$ Moliére's representation
0·00	1·000	1·000	1·000
0·04	0·904	0·909	0·921
0·08	0·822	0·830	0·856
0·25	0·593	0·596	0·662
0·45	0·455	0·450	0·517
0·70	0·358	0·353	0·396
0·80	0·329	0·326	0·359
0·90	0·303	0·302	0·327
1·00	0·279	0·280	0·300
1·20	0·235	0·240	0·254
1·40	0·196	0·204	0·220
1·60	0·162	0·170	0·190
1·80	0·133	0·140	0·166
2·00	0·109	0·114	0·147
2·20	0·088	0·092	0·131
2·60	0·058	0·058	0·105
3·00	0·038	0·037	0·086
3·20	0·030	0·029	0·078
3·60	0·020	0·019	0·064
4·00	0·013	0·012	0·052

(ii) We have considered above the example of a thin foil. For effective screening of the pions, one usually requires a fairly thick Be filter. An increase in the thickness t will have two effects. Since θ_0 and hence B will increase, the ms distribution will change its shape. The second and the more important effect will be a change in the scale of χ to ϕ, given by (12), leading to a greater inherent divergence of the beam. With a thick filter, one would, therefore, require a larger number of focussing systems.

3. Polarization effects

A muon is longitudinally polarized when it comes from a pion decay and this provides an easy method of obtaining a polarized muon beam. Let us assume that $\mathcal{P}_0$ is the initial polarization of the beam. Even though we have neglected Mott polarization in our calculation, particles coming out in a given direction have actually traversed different paths and hence their polarization vectors may be pointing in different directions. This will lead to a small depolarization of the beam. If τ is the degree of polarization, given by $\tau = |\pi|/F$, it has already been shown (Scott 1963, Mukherjee 1967) that $\tau = |\mathcal{P}_0|$ to order S^2 where

$$S = \frac{(\epsilon - 1)^2}{\epsilon^2} B\theta_0^2. \tag{14}$$

At high energies, $\epsilon \gg 1$, B is almost independent of energy, and hence $S \sim 1/\epsilon^2$ It is useful to study also the rotation of the polarization vector of the beam in passage through matter. To be specific, let us consider a thin foil of thickness 0·8 mm and a polarized muon beam of momentum 200 MeV/c. The solutions of the transport eq. (1) with $\mathcal{P}_0$ in the z-direction, can be written as

$$F(\phi, \psi; t) = \frac{1}{2\pi} I_1, \tag{15}$$

$$\pi_x (\phi, \psi; t) \pm i\pi_y (\phi, \psi; t) = e^{\pm i\psi} \frac{|\mathscr{P}_0|}{2\pi} I_2, \tag{16}$$

$$\pi_z (\phi, \psi; t) = \frac{|\mathscr{P}_0|}{2\pi} I_3, \tag{17}$$

where

$$I_1 = \int \eta \, d\eta \exp\left[\Omega - \Omega_0 - \frac{S}{4}\left(\Omega'' + \frac{1}{\eta}\Omega'\right)\right] J_0 (\phi\eta), \tag{18}$$

$$I_2 = \int \eta \, d\eta \exp\left[\Omega - \Omega_0 + \frac{S}{4}\Omega''\right]\left(-\frac{\sqrt{S}}{\omega}\sinh \omega\Omega'\right) J_1 (\phi\eta), \tag{19}$$

$$I_3 = \int \eta \, d\eta \exp\left[\Omega - \Omega_0 + \frac{S}{4}\Omega''\right]\left(\cosh \omega\Omega' - \frac{S}{4\omega\eta}\sinh \omega\Omega'\right) J_0 (\phi\eta), \tag{20}$$

with

$$\omega = \sqrt{S + S^2/16\,\eta^2} \tag{21}$$

$$\Omega - \Omega_0 = -\frac{\eta^2}{4} + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4}, \tag{22}$$

the primes on Ω indicating differentiations with reference to η. The angle of rotation of the polarization vector can be obtained from the relation,

$$\sin \theta_L = \frac{|\pi_z + i\pi_y|}{F}. \tag{23}$$

The integrals (18–20) have been evaluated numerically, with a cut-off at $\eta = 2e^{B/2}$. We have shown in table 2 the values of θ_L for various χ in a 0·8 mm Be foil. It

Table 2. Angle of rotation of the polarization vector of a muon beam of momentum 200 MeV/c in a 0·8 mm Be foil.

ϕ	$100\,\chi$ (degrees)	$100\,\theta_L$ (degrees)
0	0	0
0·4	4·17	7·82
0·8	8·34	11·74
1·2	12·51	23·49
1·6	16·69	31·31
2·0	20·86	39·15
3·0	31·29	58·72

may be pointed out that the particles are themselves coming out at an angle χ, and hence $|\theta_L - \chi|$ gives the rotation of the average polarization vector relative to the momentum. It is interesting to note that the average polarization vector rotates by an angle larger than the angle of total deflection.

4. Summary

The following summarizes the main results of this paper. The ms distribution function of a muon beam in a Be foil has been studied considering both HF and TF potentials. The screening angles θ_a for various momenta of the muon beam are shown in figure 1. To apply these results to a scattering experiment, one has to calculate the critical angle θ_c (which depends on the number of scatterers per unit area and the energy of the incident beam) and the screening angle θ_a. The two angles then determine B, the only parameter that enters in the expression for the reduced distribution function $f(\phi, t)$. This calculation is based on the small angle approximation, *viz.*, replacement of cos χ by 1, sin χ by χ and integrating over χ from 0 to ∞, instead of from 0 to π, anticipating that the integrand anyway will fall off sufficiently rapidly. The accuracy of this approximation has been checked by Lewis (1950), Bethe (1953), and Scott (1963), and it is considered to be sufficient for the purpose.

We have made a distorted wave approximation to calculate the single scattering solution. The method has been discussed in detail by Mukherjee (1965, 1967) and the approximation essentially consists in neglecting the quadratic and higher terms in $\alpha Z(\epsilon/P)$, an approximation which should be fairly accurate for the beryllium target. Our results could be improved also by considering the modification in the atomic potential due to the crystalline structure of Be. Fleischmann (1960) has already made a qualitative estimate of this correction for electron scattering. In the second part of the paper, we have studied the depolarization of a muon beam which is initially longitudinally polarized. The angle of rotation of the polarization vector for various angles of total deflection of a muon beam of 200 MeV/c momentum have been tabulated to indicate the order of the effect. If the thickness of the foil is increased, the depolarization of the beam will increase as is evident from relation (14). Our results will not differ significantly even if the small Mott polarization on each single scattering is taken into account.

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