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Tachyon trajectories in Schwarzschild's space–time

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Abstract. The motion of a tachyon in the empty Schwarzschild solution outside a mass m is discussed. It is shown that a tachyon falling radially inwards never reaches the space-time singularity at the origin. Instead, it is bounced back at a point inside the Schwarzschild radius. The causal and non-causal aspects of such a bounce are considered. It is shown that a tachyon dropped from a radial co-ordinate $< 2 \cdot 56\, m$ always arrives before it went in whereas a tachyon dropped from a radial co-ordinate $> 3 \cdot 27\, m$ always arrives later than its starting time. The more general case of a tachyon with a finite angular momentum is also analyzed. The possible astrophysical consequences of the presence of tachyons near condensed or collapsing objects and black holes are qualitatively discussed.

Keywords. General relativity; tachyons; black holes.

1. Introduction

Recently several authors (Narlikar and Sudarshan 1976; Davies 1975; Barashenkov 1974) have considered the involvement of tachyons in astrophysical and cosmological phenomena. One motivation behind such considerations has been the search for tachyons which have so far remained undetected in laboratory experiments. High energy astrophysics provides settings much more dramatic than could ever be achieved in a terrestrial laboratory and it is likely that tachyons may show up in astronomical events.

One such setting is the instant of the big-bang-if the universe did have a big-bang origin a finite time ago. If tachyons were created at the big-bang, their survival time depends on their initial energy. Narlikar and Sudarshan (1976) showed that the primordial tachyon, in the absence of any interaction with matter except through gravitation, encounters a time barrier. The epoch of this barrier depends on the initial tach-on energy as well as on the type of big-bang cosmological model. Beyond this epoch the tachyon wavefunction decays very rapidly with a time scale of the order of

$$10^{-8}\left(\frac{H}{H_m}\right)^{1/3}\left(\frac{m_e}{M}\right)^{2/3} \text{sec.} \tag{1}$$

Here $m_e =$ mass of the electron, $M =$ mass parameter of the tachyon, $H =$ Hubble constant at present epoch, $H_m =$ Hubble constant at the epoch of annihilation. The survival criterion can be used to determine the limits on m_e/M (Narlikar and Sudarshan, *op. cit.*).

The above example illustrates the peculiar effects which might arise if tachyons exist in the universe in a significant quantity. In this paper we investigate another such effect which also arises when tachyons are considered in a strong gravitational field. Highly collapsed objects or black holes have been increasingly invoked in the discussions of astrophysical phenomena. Can tachyons play a significant role in such cases ? Does the time barrier encountered in the cosmological problem appear when tachyons move near black holes ? We begin by setting out the geometrical description of the space-time associated with a Schwarzschild black hole and then consider the various aspects of tachyon propagation.

2. The Schwarzschild space-time

In the familiar Schwarzschild co-ordinate the line element is given by

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \qquad (2)$$

Here $r = 0$ is the singular source of gravitation, having the mass m. (The units are so chosen that the speed of light c and the gravitational constant G are unity). The so-called Schwarzschild radius is $r = 2m$. We will first assume that the space $r > 0$ is empty so that (2) applies throughout. Later we shall consider what happens if the gravitating object occupies a region $r < r_b$ so that (2) holds for $r > r_b$.

As is well-known, the line-element (2) is unsatisfactory in that it only describes 'half' of the entire physical space-time. A more satisfactory co-ordinate system is that of Kruskal (1960) and Szekeres (1960) which is given by the transformation

$$u = \begin{cases} \left(\frac{r}{2m} - 1\right)^{1/2} \exp\left(\frac{r}{4m}\right) \cosh\left(\frac{t}{4m}\right), & \text{for } r > 2m, \\ \left(1 - \frac{r}{2m}\right)^{1/2} \exp\left(\frac{r}{4m}\right) \sinh\left(\frac{t}{4m}\right), & \text{for } r < 2m, \end{cases}$$

$$v = \begin{cases} \left(\frac{r}{2m} - 1\right)^{1/2} \exp\left(\frac{r}{4m}\right) \sinh\left(\frac{t}{4m}\right), & \text{for } r > 2m, \\ \left(1 - \frac{r}{2m}\right)^{1/2} \exp\left(\frac{r}{4m}\right) \cosh\left(\frac{t}{4m}\right), & \text{for } r < 2m. \end{cases} \qquad (3)$$

The resulting line-element is

$$ds^2 = \frac{32m^3}{r} \exp\left(-\frac{r}{2m}\right) (dv^2 - du^2) - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \qquad (4)$$

with r determined from the implicit relation

$$\left(\frac{r}{2m} - 1\right) \exp\left(\frac{r}{2m}\right) = u^2 - v^2. \qquad (5)$$

The space-time diagram in Kruskal co-ordinates is shown in figure 1. The part of the diagram given by

$$u + v > 0 \qquad (6)$$

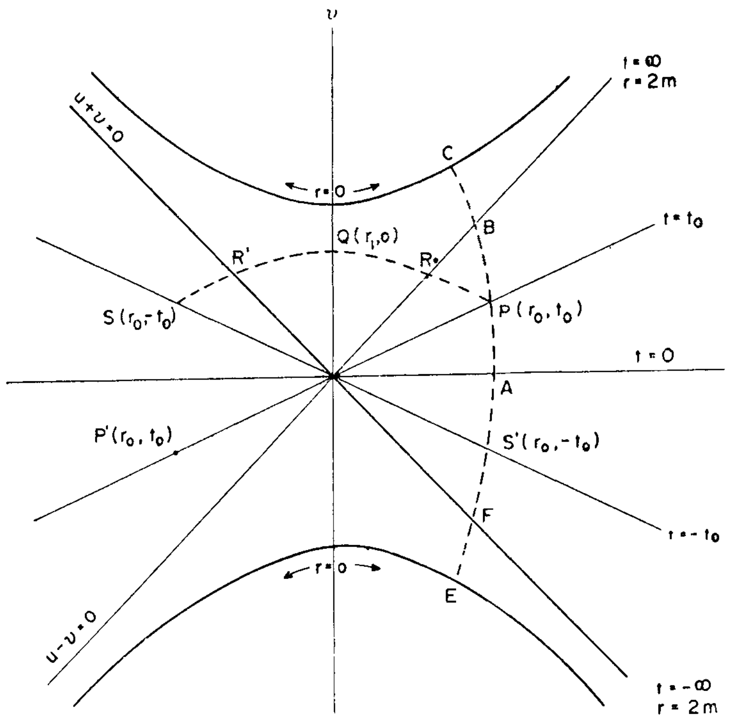

Figure 1. The Kruskal-Szekeres diagram with the trajectories EFABC and PRQR′S of a material particle and a tachyon respectively. The material particle may emerge from the $r = 0$ singularity (white hole) and fall back into the $r = 0$ singularity (black hole). The tachyon bounces at Q. It will arrive at $r = r_0$ earlier than its departure if, as shown in the diagram, $t_0 > 0$.

is the one normally used by (2). Thus, as shown in figure 1, the material particle radially falling into the singularity at $r = 0$ is described by the stretch ABC of the dotted line shown. Along AB, the t-coordinate steadily increases from 0 to $+\infty$. Along BC it decreases from $+\infty$ to a finite value. However, the proper time of the particle continuously increases from A to C and is finite throughout its motion. In the same diagram the stretch EFA denotes the motion of a particle ejected from $r = 0$ and reaching the point A. However, part of this stretch, EF, goes into the 'other half' of space-time $u + v < 0$. The phenomena of white holes refer to this second type of trajectories while the collapse into a black hole is described by the first type of trajectory. Clearly the entire Kruskal-Szekeres diagram is necessary to describe the phenomena of black and white holes.

For tachyon trajectories also the entire space-time diagram will be required. Since this means that in general two points in space-time have the same (r, t) co-ordinates, we will identify such points. For example, the points P and P′ in figure 1 will both be identified (provided their angular co-ordinates are the same) with the point $r = r_0$, $t = t_0$ in the line-element (2). Identification of this type has been considered by other authors also. For example, Israel (1966) has made a strong case for this type of identification, even though this implies no global past-future distinction. Honig *et al* (1974) who have discussed space-like geodesics in the extended Schwarzschild manifold have also advocated this type of identification from symmetry arguments.

3. Tachyon trajectories

In the flat space-time of special relativity, if $\boldsymbol{v}$ is the Newtonian 3-velocity of a tachyon, its energy momentum vector $(\boldsymbol{P}, E)$ has the components

$$\boldsymbol{P} = \frac{M_0 \boldsymbol{v}}{\sqrt{v^2 - 1}}, \qquad E = \frac{M_0}{\sqrt{v^2 - 1}} \tag{7}$$

where M_0 is the mass parameter of the tachyon. (This corresponds to an imaginary mass iM_0.)

In general relativity the tachyon, under no external forces, follows a space-like geodesic:

$$\frac{d^2 x^i}{d\sigma^2} + \Gamma^i_{kl} \frac{dx^k}{d\sigma} \frac{dx^l}{d\sigma} = 0 \tag{8}$$

where $x^i(\sigma)$ are the co-ordinates on the tachyon world-line at an affine parameter σ, and Γ^i_{kl} are the Christoffel symbols. We can choose $d\sigma = ids$ and write a first integral of (8) as

$$d\sigma^2 = -ds^2 = -g_{ik}\, dx^i\, dx^k \tag{9}$$

where g_{ik} is the metric tensor. Notice that for a tachyon ds is imaginary and $d\sigma$ is real. We now apply these results to the Schwarzschild line element (2).

3.1 *Radial trajectories*

Consider first the case of a tachyon at $r = r_0$, $t = t_0$ and moving radially inward along a geodesic $\theta =$ constant, $\phi =$ constant. The time component of (8) has the first integral

$$\left(1 - \frac{2m}{r}\right) \frac{dt}{d\sigma} = \Gamma \,(\text{constant}). \tag{10}$$

At large distance from the central object $r \gg 2\,m$, the space-time is almost flat and we may identify the constant Γ with E/M_0. (9) then gives

$$\left(\frac{dr}{dt}\right)^2 = \frac{1}{\Gamma^2}\left(1 - \frac{2m}{r}\right)^2 \left(1 + \Gamma^2 - \frac{2m}{r}\right). \tag{11}$$

This tells us that the tachyon *cannot* reach the singular point $r = 0$; it must rebound at

$$r = \frac{2m}{1 + \Gamma^2} = r_1 \,(\text{say}). \tag{12}$$

This result was earlier obtained by Raychaudhuri (1974). It is worth noting that although at $r = r_1$, dr/dt vanishes and the tachyon is apparently 'at rest', no paradox is involved. Since $r_1 < 2\,m$ a radial displacement dr at $r = r_1$ is *time-like* while a temporal displacement dt is *space-like*. The tachyon trajectory is described by the curve PRQR′S in figure 1. Q is the point of bounce while R and R′ represent the events when the tachyon crosses the Schwarzschild radius, once inward and once outward. The symmetry of the trajectory means that the point S has co-ordinates $(r_0, -t_0)$. There is no loss of generality in assuming that the bounce occurs at $t = 0$ as shown in figure 1. Since the line-element (2) is static, we can always arrange this by a suitable time translation. We now relate t_0 to r_0 and Γ and ask the following question: "under what circumstances the tachyon can return to r_0 before it started?" This is equivalent to the condition under which

$$t_0 > 0. \tag{13}$$

For $r > 2m$ the integral of (11) gives for inward motion

$$t - t_0 = \int_r^{r_0} \frac{\Gamma\, dr}{\left(1 - \frac{2m}{r}\right)\left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2}} = 2m\,\{F(r, \Gamma) - F(r_0, \Gamma)\}, \tag{14}$$

where

$$F(r, \Gamma) = \ln \frac{\left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2} + \Gamma}{\left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2} - \Gamma} - \frac{\Gamma\,(2\Gamma^2 + 3)}{2\,(1 + \Gamma^2)^{3/2}}$$

$$\times \ln \cdot \frac{(1 + \Gamma^2)^{1/2} + \left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2}}{(1 + \Gamma^2)^{1/2} - \left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2}}$$

$$- \frac{\Gamma}{(1 + \Gamma^2)} \frac{r}{2m} \left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2}. \tag{15}$$

As $r \to 2m+$, $F(r, \Gamma) \to +\infty$. This corresponds to the point P on the trajectory. For $r_1 < r < 2m$ we similarly have for inward motion

$$t = 2m\ G(r, \Gamma) \tag{16}$$

where

$$G(r, \Gamma) = \ln \cdot \frac{\left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2} + \Gamma}{\Gamma - \left(1 + \Gamma^2 - \frac{2m}{r}\right)^{1/2}} - \frac{\Gamma\,(2\Gamma^2 + 3)}{2\,(1 + \Gamma^2)^{3/2}}$$

$$\times \ln \frac{(1+\Gamma^2)^{1/2} + \left(1+\Gamma^2 - \frac{2m}{r}\right)^{1/2}}{(1+\Gamma^2)^{1/2} - \left(1+\Gamma^2 - \frac{2m}{r}\right)^{1/2}}$$

$$- \frac{\Gamma}{1+\Gamma^2} \cdot \frac{r}{2m}\left(1+\Gamma^2 - \frac{2m}{r}\right)^{1/2}. \tag{17}$$

It is easy to check that $G(r_1, \Gamma) = 0$ and $G(r, \Gamma) \to \infty$ as $r \to 2m -$. The question is, how can we match (14) and (16) across the event horizon at $r = 2m$?

For this we put $r = 2m + \epsilon$ in (14) and $r = 2m - \epsilon$ in (16) and consider the asymptotic value of t as $\epsilon \to 0$ in both the cases. We find that for $r = 2m + \epsilon$

$$t \sim \{t_0 - 2mF(r_0, \Gamma)\} + 2m \ln\left(\frac{8m\Gamma^2}{\epsilon}\right)$$

$$- \frac{m\Gamma(2\Gamma^2+3)}{(1+\Gamma^2)^{3/2}} \ln \frac{(1+\Gamma^2)^{1/2} + \Gamma}{(1+\Gamma^2)^{1/2} - \Gamma} - \frac{2m\Gamma^2}{1+\Gamma^2}, \tag{18}$$

while for $r = 2m - \epsilon$

$$t \sim 2m \ln\left(\frac{8m\Gamma^2}{\epsilon}\right) - \frac{m\Gamma(2\Gamma^2+3)}{(1+\Gamma^2)^{3/2}} \ln \frac{(1+\Gamma^2)^{1/2} + \Gamma}{(1+\Gamma^2)^{1/2} - \Gamma} - \frac{2m\Gamma^2}{1+\Gamma^2}. \tag{19}$$

A comparison of the two expressions show that if we take principal parts in the integration of (11) across $r = 2m$ we will get

$$t_0 = 2mF(r_0, \Gamma). \tag{20}$$

The validity of the above limiting procedure may be justified by a transformation to the Kruskal-Szekeres co-ordinates. There the inward motion given by (11) may be described by the differential equation

$$\frac{dv}{du} = \frac{v\left(1+\Gamma^2 - \frac{2m}{r}\right)^{1/2} - \Gamma u}{u\left(1+\Gamma^2 - \frac{2m}{r}\right)^{1/2} - \Gamma v} \tag{21}$$

which holds throughout the motion for $r \gtrless 2m$. As $r \to 2m$ from either direction this tends to a finite limit. Initially u_0 and v_0 are known in terms of r_0 and t_0 and the integration in the (u, v) plane terminates when $u = 0$. The corresponding value of v is then identified with that given by (3) for $t = 0$, $r = r_1$. This identification gives a unique relation between t_0, r_0 and Γ. This procedure does not encounter the difficulty of the Schwarzschild barrier $r = 2m$ at any stage since (21) is well-defined throughout the integration range.

Consider now equation (19) and its form in the (u, v) co-ordinates. A simple calculation shows that as $\epsilon \to 0$ the trajectory of the tachyon passes through the point

$$u = v = \Gamma \exp\left\{-\frac{1}{4}\frac{\Gamma(2\Gamma^2+3)}{(1+\Gamma^2)^{3/2}} \ln \frac{(1+\Gamma^2)^{1/2} + \Gamma}{(1+\Gamma^2)^{1/2} - \Gamma} + \frac{1}{2(1+\Gamma^2)}\right\}. \tag{22}$$

The same limit is reached via (18) as $\epsilon \to 0$ provided (20) holds.

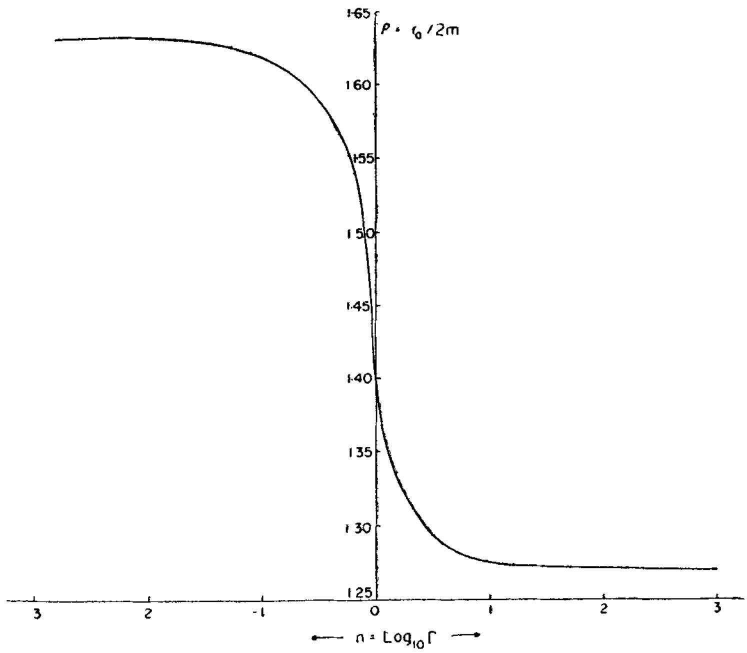

Figure 2. Curve showing the dependence of $r_0/2m$ on $\Gamma (=10^n)$. The curve, plotted on a logarithmic scale for Γ has asymptotes at $r_0 \simeq 3{\cdot}27m\,(\Gamma \to 0)$ and at $r_0 \simeq 2{\cdot}56\,m\,(\Gamma \to \infty)$.

We now return to the question raised earlier. A tachyon at (r_0, t_0) heading towards $r = 0$ will bounce and pass through the point $r = r_0$ again at $(r_0, -t_0)$. If $t_0 = 0$ the onward and returning tachyons will meet at $r = r_0$ at the same time. If $t_0 > 0$ the returning tachyon arrives at $r = r_0$ earlier than the onward moving tachyon, and vice-versa for $t_0 < 0$. So the space-like surface $t_0 = 0$ is of some significance.

In figure 2 we have plotted the dependence of r_0 on Γ for $t_0 = 0$. We see that as $\Gamma \to 0$ the value of r_0 reaches an asymptotic value of $\sim 3{\cdot}27\,m$ which is the solution of the equation $r_0 = 2m\rho$,

$$4\left(1 - \frac{1}{\rho}\right)^{-1/2} - 3 \ln \frac{\rho^{1/2} + (\rho - 1)^{1/2}}{\rho^{1/2} - (\rho - 1)^{1/2}} - \rho^{1/2}(\rho - 1)^{1/2} = 0. \tag{23}$$

Similarly the asymptotic value of $r_0 \sim 2{\cdot}56\,m$ is obtained in the limit of $\Gamma \to \infty$ as the solution of $r_0 = 2m\rho$,

$$\rho + \ln(\rho - 1) = 0. \tag{24}$$

For $r_0 > 3{\cdot}27\,m$ we can therefore say that a tachyon will arrive later than its time of departure, whatever be its energy. For $r_0 < 2{\cdot}56m$ the tachyon always arrives earlier than its time of departure. For $2{\cdot}56 < r_0/m < 3{\cdot}27$ the temporal order of arrival and departure depends on Γ.

3.2 *Non-radial trajectories*

We next consider the tachyon heading towards the condensed object, with a finite angular momentum. We then have the following well-known integrals of (8):

$$\left(1-\frac{2m}{r}\right)\frac{dt}{d\sigma}=\Gamma,\quad \theta=\frac{\pi}{2},\quad r^2\frac{d\phi}{d\sigma}=h, \tag{25}$$

where h is the angular momentum per unit mass. The first integral (9) then gives

$$\left(\frac{dr}{d\sigma}\right)^2=-V^2(r)+\Gamma^2 \tag{26}$$

where

$$V^2(r)=\left(1-\frac{2m}{r}\right)\left(\frac{h^2}{r^2}-1\right). \tag{27}$$

For motion to be possible we require $V^2(r)\leqslant\Gamma^2$ throughout the trajectory. In the low energy-Newtonian approximation we get $r\gg 2m$, $h^2\ll r^2$, $\Gamma^2\ll 1$, so that (26) becomes

$$\left(\frac{dr}{dt}\right)^2=1-\frac{2m}{r}-\frac{h^2}{r^2}. \tag{28}$$

In this case motion is possible only if

$$r\geqslant r_N=m+(m^2+h^2)^{1/2}. \tag{29}$$

The tachyon is therefore repelled by the object and bounces back at $r=r_N$.

The full relativistic case has been discussed before by Honig *et al* (1974) and hence we only briefly summarize the results. The potential function V has a maximum which is non-negative. A tachyon approaching the black hole from afar ($\dot{r}<0$) will fall into the blackhole if $\Gamma>V_{max}$. For $\Gamma<V_{max}$ the potential barrier is high enough to bounce back the incoming tachyon; for $h<2m$ the bounce occurring *inside* the Schwarzschild barrier. For $\Gamma=V_{max}$ there are unstable circular orbits located at $r=r_{max}$ where $V_{max}=V(r_{max})$. It is easily seen that $r_{max}\leqslant 3\,m$. The interesting thing to note is that a tachyon of sufficiently low angular momentum will bounce from the black hole—unlike the case for light or particles of positive rest mass.

Finally, consider the angular motion of the tachyon which bounces at the potential barrier. What is the angular deflection suffered by its trajectory as a result of encountering the black hole? From (25) and (26) we get

$$\left(\frac{d\phi}{dr}\right)^2=\frac{h^2}{r^4(\Gamma^2-V^2)}. \tag{30}$$

Using (30) and writing $\zeta=m/r$ we get with $h=ml$,

$$\left(\frac{d\phi}{d\zeta}\right)^2=\frac{1}{2}\left\{\zeta^3-\frac{1}{2}\zeta^2-\frac{1}{l^2}\zeta+\frac{1+\Gamma^2}{2l^2}\right\}^{-1}=\frac{1}{2f(\zeta)}\,(\text{say}). \tag{31}$$

A tachyon coming from infinity and bouncing back to infinity therefore suffers a deflection of the amount

$$\triangle\phi=\sqrt{2}\int_0^{a}\frac{d\zeta}{\{f(\zeta)\}^{1/2}}, \tag{32}$$

where α is the smaller of the two positive roots of $f(\zeta) = 0$. In the case of a bounce $\Gamma < V_{max}$ and there are two positive roots and one negative root. The angle $\triangle\phi$ becomes very large when the two positive roots of $f(\zeta) = 0$ come close together. Writing the three roots as

$$\alpha = p - \epsilon, \quad \beta = p + \epsilon, \quad \gamma = \tfrac{1}{2} - 2p, \tag{33}$$

where, in the limit $\epsilon \to 0$, $p \to m/r_{max}$; we see that in this limit $\triangle\phi$ diverges as

$$\triangle\phi \sim \frac{2}{\sqrt{6p-1}} \ln \left\{ \frac{6p-1}{p} \cdot \frac{5p-1-(6p-1)^{1/2}(4p-1)^{1/2}}{\epsilon} \right\}. \tag{34}$$

This result is derived in the Appendix. When $p \to \frac{1}{3}$ we have $\Gamma \to \infty$, $l \to \infty$ and we arrive at the case of the photon with the limiting circular orbit at $r_{max} = 3m$. In that case (34) becomes

$$\triangle\phi \sim 2 \ln \left(\frac{8 - 4\sqrt{3}}{\epsilon} \right). \tag{35}$$

This is analogous to the formula derived by Chitre *et al* (1975) for photons.

Thus the highly energetic tachyon with large angular momentum satisfying $\Gamma < V_{max}$ can bounce even from inside the Schwarzschild radius after making several revolutions about $r = 0$.

4. Astrophysical applications

Highly collapsed objects or black holes have been discussed extensively by astrophysicists. These may be considered as good approximations to the gravitating singularity discussed here. So far as a tachyon is concerned it is bounced at a finite radial co-ordinate, r_1 in many of the cases of interest. Hence we may regard the black hole as having a 'radius' $r_b < r_1$ and apply the above results.

In the case of photons or particles of real mass, the general result is that once their trajectory crosses into the Schwarzschild radius, it cannot come out. Such orbits eventually wind up into the singularity (or into the black hole). For tachyons, however, the phenomenon of bounce presents an interesting situation. By identifying the two points (u, v) and $(-u, -v)$ in the Kruskal-Szekares diagram we are able to recover tachyons of sufficiently low angular momentum even if they have penetrated the Schwarzschild barrier. In terms of S. I. units with c and G given their respective values, the limit on the angular momentum per unit mass parameter may be expressed as follows:

$$h \leqslant 10^{16} \left(\frac{m}{M_\odot} \right) \text{cm}^2\ \text{sec}^{-1}, \tag{36}$$

Where M. = mass of the sun. since a tachyon has velocity $v > c$ its impact parameter h/v must be very small to satisfy the above inequality. Indeed, it must be so fired that it is directed within the Schwarzschild sphere of the black hole:

$$\frac{h}{v} < \sim 3 \times 10^5 \left(\frac{m}{M_\odot} \right) \text{cm}. \tag{37}$$

Such a condition is not hard to satisfy if the source of tachyons is located in the vicinity of the black hole. Such tachyons can then act as probes to the interior of the black hole.

We can also visualize the possibility of a black hole producing and ejecting tachyons. Little is known at present about the interactions which produce tachyons from ordinary matter. It is conceivable that the peculiar space-time effects inside the Schwarzschild radius may be conducive to the generation of tachyons. As the above calculations show, tachyons so produced will be able to come out of the Schwarzschild barrier. We therefore conjecture that if ordinary matter is dropped into a black hole and converted to tachyons inside the Schwarzschild barrier, some of the resulting tachyons may re-emerge. Thus, the detection attempts for tachyons should be directed towards black holes, or where there is indirect evidence for the location of black holes, *e.g.*, Cygnus X–1.

Finally we note that in the Newtonian approximation as well as in the relativistic radial free-fall, a tachyon appears to be repelled by the central object. Although Raychaudhuri (1974) has shown that between tachyons themselves there is a force of attraction, it is still a matter of conjecture as to whether the presence of tachyons can be conducive to halting the gravitational collapse of a massive object. The quantum theory of tachyons associates negative energies with tachyon fields (Sudarshan 1972). Since this violates the energy conditions of the established singularity theorems (Hawking and Ellis 1973) it seems likely that a composite object made of matter and tachyons may bounce. Future investigations will be directed towards the resolution of this problem.

Appendix

We evaluate the integral (34) when $f(\zeta) = 0$ has roots α, β, γ given by (33), with $\epsilon \ll p$. Write

$$\zeta = (p - \epsilon) \sin^2 \theta, \tag{A 1}$$

to get

$$\Delta\phi \sim 2\sqrt{2} \int_0^{\pi/2} \frac{\sin\theta \, d\theta}{(2\epsilon + p\cos^2\theta)^{1/2} (3p - \frac{1}{2} - p\cos^2\theta)}. \tag{A 2}$$

The symbol $\sim$ represents approximation involved in ignoring ϵ in comparison with p. Using the transformation

$$\mu = \left(\frac{2p}{6p-1}\right)^{1/2} \cos\theta, \qquad \mu_0 = \left(\frac{2p}{6p-1}\right)^{1/2}, \tag{A 2}$$

we rewrite $\Delta\phi$ in the form

$$\Delta\phi \sim 2\sqrt{2} \int_0^{\mu_0} \frac{d\mu}{(1-\mu^2)^{1/2} \{2\epsilon + (3p - \frac{1}{2})\mu^2\}^{1/2}}. \tag{A 3}$$

Finally write

$$\mu = \cos\psi, \qquad \mu_0 = \cos\psi_0$$

to express (A 3) in the form of an elliptic integral

$$\Delta\phi = \frac{4}{(6p-1)^{1/2}} \int_{\psi_0}^{\pi/2} \frac{d\psi}{(1 - k^2\sin^2\psi)^{1/2}}, \quad k^2 \simeq 1 - \frac{4\epsilon}{6p-1} \tag{A 4}$$

The result (36) then follows by using the asymptotic value of the complete elliptic integral when k^2 is close to unity.

References

Barashenkov V S 1974 *Usp. Fiz. Nauk.* **114** 133
Chitre S M, Narlikar J V and Kapoor R C 1975 *G.R.G. Journal* **6** 477
Davies P C W 1975 *Nuovo Cim.* **25B** 571
Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Spacetime* (Cambridge)
Honig E, Lake K and Roeder R C 1974 *Phys. Rev.* **D10** 3155
Israel W 1966 *Nature* **211** 466
Kruskal M D 1960 *Phys. Rev.* **119** 1743
Narlikar J V and Sudarshan E C G 1976 *Mon. Not. Roy. Astr. Soc.* **175** 105
Raychaudhuri A K 1974 *J. Math. Phys.* **15** 856
Sudarshan E C G 1972 *Fields and Quanta* **2** 175
Szekeres G 1960 *Publ. Mat. Debrecen,* **7** 285.