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Trapping energy of a magnetic monopole in magnetic materials

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Abstract. Analytical methods to investigate the interaction of magnetic monopoles with known magnetic media have been developed. Trapping energies of monopoles inside ferro-magnetic or superconducting materials of size greater than about 10^{-6} cm are found to be of the order of several kiloelectron volts. These are two to three orders of magnitude higher than in paramagnetic materials. Thus if stable magnetic monopoles exist at all in the universe, they are perhaps trapped in these magnetic materials. The effect of the finite size of the magnetic bodies is taken into account explicitly in our calculations of the trapping energy.

Keywords. Magnetic monopoles; classical electrodynamics; trapping energy.

1. Introduction

Dirac (1931) suggested the existence of the magnetic charge (monopole). He showed that the existence of the monopole leads to the quantization of charge. The magnetic charge g of the monopole being related to the electronic charge e by $eg = n\hbar c/2$, where n is an integer (Dirac 1948). Schwinger (1966) on the other hand has argued that the product eg should be an integral multiple of $\hbar c$ only, with minimum $|g| = \hbar c/|e|$. Since the early suggestion of Dirac, many theoretical and experimental studies on magnetic monopoles have been carried out. But the results of experimental attempts to establish the existence of a monopole have been negative till now. Even the recent claim of Price *et al* (1975) about the observation of a monopole, with $|g| = \hbar c/|e| \simeq 137|e|$, in a baloon experiment seems to be not very convincing.

The present study is motivated by the fact that there is a strong possibility that stable monopoles, if they exist, would most likely be found trapped in magnetic materials in the universe (Gotto 1957). In order to understand this process, it would be necessary to investigate the nature of the interaction energy of the magnetic monopole with a magnetic material, and in particular calculate its trapping energy. Since, in general the problem is quite complicated analytically for an arbitrary geometrical shape of the magnetic material, we will only consider the case of magnetic spheres of any given radius R. We calculate trapping energies in different kinds of magnetic materials; ferromagnets, paramagnets and perfect diamagnets (super-conductors). As expected, the trapping energies in ferromagnets and superconductors are large, but even in paramagents it is of the order of several electron volts. It will take a magnetic field of several kilogauss to extract a monopole from ferromagnetic spheres of radius greater than 10^{-6} cm.

Before we consider the calculation of trapping energy in the next section of this article, in order to understand the important differences in the nature of fields produced by electric and magnetic charges, we briefly review here, the basic equations of electrodynamics in the presence of a moving magnetic monopole.

With the inclusion of external magnetic charges $\rho^{(m)}$ and magnetic currents $J^{(m)}$, macroscopic Maxwell's equations in an usual medium can be written as

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} - \frac{4\pi}{c}\boldsymbol{J}^{(m)}, \quad \boldsymbol{B} \equiv \boldsymbol{H} + 4\pi \boldsymbol{M}, \tag{1.1}$$

$$\vec{\nabla} \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\boldsymbol{J}^{(e)}, \quad \boldsymbol{D} \equiv \boldsymbol{E} + 4\pi \boldsymbol{P}, \tag{1.2}$$

$$\vec{\nabla} \cdot \boldsymbol{B} = 4\pi\rho^{(m)} \tag{1.3}$$

$$\vec{\nabla} \cdot \boldsymbol{D} = 4\pi\rho^{(e)}, \tag{1.4}$$

with the equations of continuity

$$\frac{\partial \rho^{(e)}}{\partial t} + \vec{\nabla} \cdot \boldsymbol{J}^{(e)} = 0, \quad \frac{\partial \rho^{(m)}}{\partial t} + \vec{\nabla} \cdot \boldsymbol{J}^{(m)} = 0. \tag{1.5}$$

Consistent with the relativistic invariance, the corresponding Lorentz force acting on a moving monopole having velocity $\boldsymbol{v}$ is

$$\boldsymbol{F}^{(m)} = g\left(\boldsymbol{B} - \frac{\boldsymbol{v}}{c} \times \boldsymbol{E}\right), \tag{1.6}$$

where g is the charge of the monopole. It is clear from eq. (1.6) that only the magnetic field does work on the monopole g, just as only the electric field does work on the electric charge e.

From eqs (1.1)–(1.4) one easily finds that for the free space ($\boldsymbol{P} = \boldsymbol{M} = 0$), the fields (denoted by the superscript m) produced by the magnetic charges and currents can be obtained from the corresponding fields of the electric charges by the transformations

$$\boldsymbol{H}^{(m)}(\boldsymbol{r}, t) = \boldsymbol{E}^{(e)}(\boldsymbol{r}, t), \quad \boldsymbol{E}^{(m)}(\boldsymbol{r}, t) = -\boldsymbol{H}^{(e)}(\boldsymbol{r}, t), \tag{1.7}$$

provided on the right hand side of eq. (1.7) we make the replacement $\rho^{(e)} \to \rho^{(m)}$, $J^{(e)} \to J^{(m)}$. Because of this similarity, the problem of a moving monopole does not present any additional complications in its solution. As the solution of the problem of electric charges is already available in most text books (*e.g.* Jackson 1962), note that in case of a uniformly moving monopole, the electric field $E^{(m)}$ is down by a factor $\beta = v/c$ compared to $H^{(m)}$, just like $H^{(e)}$ is down by a factor v/c compared to $E^{(e)}$ for the case of a uniformly moving electric charge.

Since the Poynting vector still contains the product of $\boldsymbol{E}$ and $\boldsymbol{H}$, because of the symmetry described by eq. (1.7), we find that the total radiated power by an accelerated monopole is still given by Lienard's result for the electric charge case (except for the replacement $e \to g$):

$$P = \frac{2}{3}\frac{g^2}{c}\gamma^6 [\dot{\vec{\beta}}^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2], \quad \gamma^2 = (1 - \beta^2)^{-1}. \tag{1.8}$$

However, polarization characteristics are different. For example in the Thomson scattering of electromagnetic waves from a monopole, the scattering cross section will be

$$\frac{d\sigma^{(m)}}{d\Omega} = \left(\frac{g^2}{M_p c^2}\right)^2 \left| \frac{n \times (n \times (k \times (\hat{\epsilon}))^c}{\omega} \right|^2, \tag{1.9 a}$$

which should be compared with the corresponding result for the electric charge

$$\frac{d\sigma^{(e)}}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 | n \times (n \times \hat{\epsilon}) |^2, \tag{1.9 b}$$

where n is the unit vector in the scattering direction and $\hat{\epsilon}$ is the polarization vector of the incident electromagnetic field. A similar difference appears in the polarization characteristics of the Cherenkov radiation. Hagstron (1975) has suggested that this difference can be used in the detection of monopoles. Finally we would like to point out that the energy loss from monopoles in a medium is determined by the poles of the dielectric function $\epsilon(\omega)$ rather than zeros of $\epsilon(\omega)$. In fact one has for the energy loss per unit distance for a moving monopole in a medium, having collisions with impact parameters $b > a$

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{2g^2}{\pi c^2} \mathrm{Re} \int_0^\infty d\omega \, i\omega \, \lambda^* \, a \, K_0(\lambda a) \, K_1^*(\lambda a)$$

$$\times \left(\frac{c^2}{v^2} - \epsilon(\omega)\right), \quad \lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\,\epsilon, \quad \mathrm{Im}\,\lambda < 0. \tag{1.10}$$

Thus the energy loss expression contains implicity, the information about transverse excitations of the material medium. For frequencies ω and velocities v such that $\lambda a \ll 1$, the integral occurring in eq. (1.10) can be done explicitly (Tompkins 1965) by assuming the oscillator model for dielectric function $\epsilon(\omega)$. One indeed finds that the energy loss is almost independent of velocity for low velocities. To a very good approximation, one can say that in this region, the ratio of the energy loss by a monopole of charge g and the energy loss by an electric charge Ze is of the order of $(v/c)^2 (g/Ze)^2$. Details of the important examples of the interaction of monopoles and electromagnetic fields can be found in the papers of Cole (1951), Bauer (1951), Tompkins (1965), Dooher (1971).

2. Trapping energy in a magnetic medium

In this section we consider some static problems involving the interaction of a stationary magnetic monopole with a magnetic medium. Such questions are relevant in the detection of monopoles which may exist trapped in ferro-magnetic materials. The first estimates of trapping energy in an infinitely extended ferromagnet have been made by Gotto (1957) and Gotto *et al* (1963). The relevant Maxwell equations in this case are

$$\vec{\nabla} \cdot B = 4\pi\rho^{(m)} = 4\pi g \delta(r - r_0), \tag{2.1 a}$$

$$H = B - 4\pi M, \tag{2.1 b}$$

$$\vec{\nabla} \times H = 0. \tag{2.1 c}$$

We define the trapping energy as the difference of the electromagnetic energies in the presence and in the absence of the magnetic medium

$$W_{\text{trap}} = \frac{1}{4\pi} \int d^3 r \int \{\boldsymbol{H} \cdot \delta \boldsymbol{B} - \boldsymbol{H}^{(v)} \cdot \delta \boldsymbol{B}^{(v)}\}, \tag{2.2}$$

where the fields carrying the superscript v denote the vacuum fields in the absence of the magnetic medium. In general $\boldsymbol{H}(\boldsymbol{B})$ is not a linear function of $\boldsymbol{B}$ and hence for a non-linear medium the calculation of W_{trap} is more involved. However, we show below that by suitable transformations the trapping energy can be calculated from the magnetostatic potential by a simple integration over the strength of the monopole. The magnetostatic potential can be introduced because $\vec{\nabla} \times \boldsymbol{H} = 0$ *i.e.*, $\boldsymbol{H} = -\vec{\nabla} \Phi$. The expression for trapping energy now becomes

$$W_{\text{trap}} = \frac{1}{4\pi} \int d^3 r \int -\vec{\nabla} \Phi \cdot \delta \boldsymbol{B} - \frac{1}{8\pi} \int d^3 r\, B^{(v)^2}, \tag{2.3a}$$

which on integration by parts becomes (integration by parts does not lead to any surface terms because Φ and normal component of $\boldsymbol{B}$ are continuous across any surface).

$$W_{\text{trap}} = \frac{1}{4\pi} \int d^3 r \int \Phi \vec{\nabla} \cdot \delta \boldsymbol{B} - \frac{1}{8\pi} \int d^3 r\, \boldsymbol{B}^{(v)^2}. \tag{2.3 b}$$

In the calculation of trapping energies, one can imagine that the change in B is brought out by changing the strength of the monopole. Thus one can write

$$\vec{\nabla} \cdot \delta \boldsymbol{B} = \vec{\nabla} \cdot \boldsymbol{B} \frac{dg}{g}, \tag{2.3 c}$$

assuming that $\vec{\nabla} \cdot \boldsymbol{B}$ is a linear function of g (eq. (2·1 *a*)). Finally, on using (2·3 *c*), (2·1 *a*), eq. (2·3 *b*) reduces to

$$W_{\text{trap}} = \int_0^g dg\, [\Phi(\boldsymbol{r}_0) - \Phi^{(v)}(\boldsymbol{r}_0)], \tag{2.4}$$

where $\boldsymbol{r}_0$ is the position of the monopole. If the medium under consideration is linear *i.e.*, if

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{2.5}$$

then one finds that Φ is a linear functional of g and (2.4) leads to

$$W_{\text{trap}} = \tfrac{1}{2} g\, (\Phi(\vec{r}_0) - \Phi^{(v)}(\vec{r}_0)). \tag{2.6}$$

It would be noticed that for a linear medium Φ satisfies the equation

$$\nabla^2 \Phi = -\frac{4\pi\rho^{(m)}}{\mu} = -\frac{4\pi g}{\mu} \delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{2·7}$$

If we consider an infinitely extended magnetic medium characterized by a linear permeability μ, then from eqs (2 .6) and (2.7) it is clear that

$$W_{\text{trap}} = \frac{1}{2} g^2 \left(\frac{1}{\mu} - 1\right) \lim_{r \to 0} \left(\frac{1}{r}\right). \tag{2.8}$$

Thus the trapping energy diverges for a point monopole in an infinitely extended linear magnetic medium. Considerations about the finiteness of the medium (see appendix A) still lead to a divergence which could be removed either by treating the induced magnetisation of magnetic medium in a non-linear fashion or by giving a finite size to the monopole. We have to follow the first route since ferromagnetic or the paramagnetic medium does indeed show a non-linear behaviour.

2.1 *Trapping energy in ferromagnetic materials*

For ferromagnetic substances, the magnetization attains the saturation value M_s if the applied field is greater than the critical field H_s. For fields $H < H_s$, the magnetization is in general a complicated function of H. However it would not be a very bad approximation if in the range, $0 \leqslant H \leqslant H_s$, one takes the magnetization to depend linearly on H. Only for fields very close to H_s, the non-linear character is important. However for most ferromagnets one also has the relation $4\pi M_s \gg H_s$. In the ideal case $H_s \to 0$. Hence it seems that we can adopt the following constitutive relation as the model of the ferromagnet (figure 1):

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi M(H)\frac{\boldsymbol{H}}{H}, \tag{2.9}$$

where

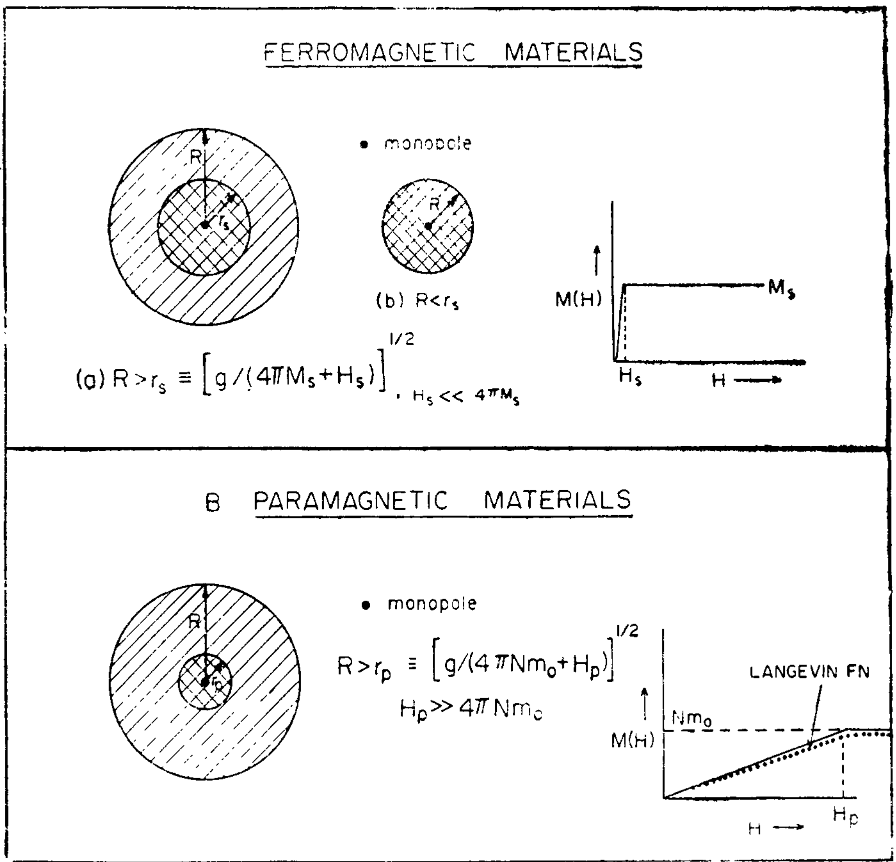

Figure 1. Trapping of a monopole in ferromagnetic or paramagnetic materials.

$$M(H) = \frac{M_s}{H_s} H \quad \text{for } H \leqslant H_s, \tag{2.10a}$$

$$= M_s \qquad \text{for } H \geq H_s. \tag{2.10b}$$

Due to the assumed non-linear nature of the function $M(H)$; a general expression for the trapping energy in an arbitrary geometry is rather difficult to get (the problem of a linear medium which can be treated exactly is discussed in appendix A). In what follows we assume that the monopole is located at the *centre* of a *sphere* of the ferromagnetic material of radius R. Because of the spherical symmetry we can look for the spherically symmetric solutions, *i.e.*, the magnetic field has radial component and depends on r only. On combining (2.1) and (2.9) we obtain

$$\operatorname{div}\left\{\operatorname{grad}\Phi\left(1 + \frac{4\pi M(|\nabla\Phi|)}{|\nabla\Phi|}\right)\right\} = -4\pi g\delta(\mathbf{r}),$$

$$H = -\operatorname{grad}\Phi, \qquad H_r = -\frac{\partial\Phi}{\partial r}, \tag{2.11}$$

where Φ depends only on r. The divergence equation in (2.11) can be written as

$$\frac{\partial}{\partial r}\left\{r^2\left[H_r + \frac{4\pi M(|H_r|)}{|H_r|} H_r\right]\right\} = g\,\delta(r), \tag{2.12}$$

which on integration gives

$$H_r + \frac{4\pi M(|H_r|)}{|H_r|} H_r = (g + A)/r^2 \tag{2.13}$$

where A is a constant of integration. In the region of the origin $A = 0$. Let r_s be the radius of the sphere inside which the fields are such that $H > H_s$. Now we consider two different situations for the solution depends on the relation of the radius of the sphere to r_s.

2.1.1 $r_s < R$

In this case we have to consider three different regions (i) $r \leqslant r_s$ in which (2.10 *a*) applies (ii) $r_s \leqslant r \leqslant R$ in which the behaviour of the magnetic medium is linear (iii) $r \geqslant R$, the vacuum region. It is clear that in the region I the constant of integration A must be zero, otherwise the strength of the monopole at $r = 0$ will be $(g + A)$. Hence for region I

$$H_r^{(1)}\left(1 + \frac{4\pi M_s}{|H_r^{(1)}|}\right) = g/r^2, \tag{2.14}$$

which leads to

$$H_r^{(1)} = g/r^2 - 4\pi M_s. \tag{2.15}$$

From (2.14) it is clear that $H_r^{(1)}$ is positive, however (2.15) does not necessarily imply so. The field decreases as one goes away from the centre. But for $r = r_s$ the field by definition is H_s and hence

$$r_s^2 = g/(H_s + 4\pi M_s). \tag{2.16}$$

The potential in the region (i) will be

$$\Phi^{(1)} = \frac{g}{r} + 4\pi M_s r + \alpha . \tag{2.17}$$

In the region ii, (2.13) gives [the term g/r^2 is absent as it arose from the integration of $\delta(r)$]

$$H_r^{(2)} = \frac{A}{r^2}\left(1 + \frac{4\pi M_s}{H_s}\right)^{-1} \tag{2.18}$$

For $r = r_s$ it should reduce to H_s and hence $A = g$. The potential in the region II will be

$$\Phi^{(2)} = \frac{g}{r}\left(1 + \frac{4\pi M_s}{H_s}\right)^{-1} + \beta. \tag{2.19}$$

Finally for the region outside the sphere R, one has

$$H_r^{(3)} = \frac{A}{r^2}, \qquad \Phi^{(3)} = A/r. \tag{2.20}$$

The constant A will again be g as at $r = R$, the normal component of $\boldsymbol{B}$ is to be continuous. The potential $\Phi^{(3)}$ has no constant term as it should vanish at $r = \infty$. The constants α and β are obtained from the continuity of potential at $r = r_s$, $r = R$. Whence one finds that

$$\beta = \frac{g}{R}\left(\frac{4\pi M_s}{H_s + 4\pi M_s}\right),$$

$$\alpha = -8\pi M_s r_s + \frac{g}{R}\left(\frac{4\pi M_s}{H_s + 4\pi M_s}\right). \tag{2.21}$$

The trapping energy is now obtained by substituting eq. (2.17) in eq. (2.4). It is interesting to note that Φ has terms which depend on $g^{1/2}$ as well as on g. A simple integration leads to

$$W_{\text{trap}} = -\frac{4}{3} g^{3/2} \frac{(4\pi M_s)}{(H_s + 4\pi M_s)^{1/2}} + \frac{1}{2}\frac{g^2}{R}\frac{4\pi M_s}{(H_s + 4\pi M_s)} \tag{2.22}$$

For $R \to \infty$, this gives

$$W_{\text{trap}}(\infty) = -\frac{4}{3} g^{3/2} \frac{4\pi M_s}{(H_s + 4\pi M_s)^{1/2}}, \tag{2.23}$$

whereas for $H_s \ll 4\pi M_s$, one finds

$$W_{\text{trap}} \to -\frac{4}{3}\frac{g^2}{r_s}\left(1 - \frac{3}{8}\frac{r_s}{R}\right). \tag{2.24}$$

It should be noted that the expression (2.24) in the limit $R \to \infty$ goes over to $-4g^2/3r_s$, which is in agreement with Gotto (1957). The surface dependent contribution in eq. (2.24) is identical to the R dependent contribution in (A 11) or in (A 4). The surface dependent contribution arises only from the linear behaviour of the ferromagnet. The limit $\mu \to \infty$ applies because of $4\pi M_s \gg H_s$. For $g = 137\,e$, and the typical value of $M_s \sim 20$ kilogauss, the trapping energy (2.24), for $R > r_s$, is of the order of several kilo-electron volts. The numerical value of $r_s = (g/4\pi M_s)^{\frac{1}{2}}$ in a ferromagnetic material is of the order of 10^{-6} cm.

We now consider the other case, *i.e.*, when the radius of the sphere is less than the radius r_s:

2.1.2 $r_s > R$

In this case the magnetic field inside and outside the sphere is given by

$$H_r^{(1)} = g/r^2 - 4\pi M_s, \quad H_r^{(2)} = g/r^2. \tag{2.25}$$

These solutions clearly satisfy the boundary condition that normal component of $\boldsymbol{B}$ is continuous. The potential distribution is

$$\Phi^{(1)} = \frac{g}{r} + 4\pi M_s (r - R), \qquad \Phi^{(2)} = g/r. \tag{2.26}$$

The trapping energy is now to be obtained by using eqs (2.26), (2.17) in eq. (2.4). The integration in the present case is to be done as follows

$$W_{\text{trap}} = \int_0^{g_0} dg'\, \Phi\,(r = 0, g') + \int_{g_0}^{g} dg'\, \Phi\,(r = 0, g') - \frac{g}{2}\,\Phi^{(v)}\,(r = 0), \tag{2.27 a}$$

where g_0 is defined by $g_0 = R^2 (H_s + 4\pi M_s)$. For the region $0 < g' < g_0$, the solution (2.17) applies whereas for $g_0 < g' < g$, the solution (2.26) applies. The final result for the trapping energy is

$$W_{\text{trap}} = \frac{1}{6} R^3 (4\pi M_s)(H_s + 4\pi M_s) - 4\pi M_s\, Rg\,, \tag{2.27 b}$$

which in the limit $H_s \ll 4\pi M_s$ goes to

$$\begin{aligned} W_{\text{trap}} &\to \frac{1}{6} R^3\, (4\pi M_s\,)^2 - 4\pi M_s\, Rg \\ &= (-g^2/r_s)\left(R/r_s - \frac{R^3}{6r_s^3}\right). \end{aligned} \tag{2.27 c}$$

The trapping energy of course goes to zero as $R \to 0$. However, eqs (2.24) and (2.27 c) show that for $R \gtrsim r_s \sim 10^{-6}$ cm, the trapping energy in a ferromagnetic material is of the order of kilo electron volts.

2.2 *Trapping energy in a paramagnetic material*

For a paramagnetic substance it is well known that magnetization as a function of temperature is given by [Kittel (1974, p. 503)]

$$M = Nm_0\, L\left(\frac{m_0\,|\,H\,|}{k_B\, T}\right), \tag{2.28}$$

where k_B is the Boltzmann constant, T the temperature and $L(x)$ is the Langevin function defined by

$$L(x) = \coth x - \frac{1}{x} \to \frac{x}{3} \text{ for } x \ll 1; \to 1 \quad \text{for } x \to \infty. \tag{2.29}$$

In view of (2.29) one can introduce a linear susceptibility defined by

$$\chi_p = \left(\frac{Nm_0^2}{3k_BT}\right). \tag{2.30}$$

We now approximate (figure 1) the Langevin function by

$$\begin{aligned} M &= \frac{Nm_0^2}{3k_BT} H \qquad &\text{for } H \leqslant H_p \\ &= Nm_0 \qquad &\text{for } H \geqslant H_p, \end{aligned} \tag{2.31}$$

where H_p is defined by

$$\frac{Nm_0^2}{3k_BT} H_p = Nm_0, \quad \text{i.e.,} \quad H_p = \frac{3k_BT}{m_0}. \tag{2.32}$$

For most of the paramagnetic materials $\chi_p = Nm_0/m_p \ll 1$. The model (2.31) is identical to (2.10) with $M_s \to Nm_0$, $H_s \to H_p$, $r_s \to r_p$. However, now the limit $H_p \gg 4\pi Nm_0$ applies. Hence the trapping energy for paramagnetic materials will be given by eqs (2.23) and (2.27) respectively. In the limit $\chi_p \ll 1$, for $R > r_p$, eq. (2·23) goes over to

$$W_{\text{trap}} = -\frac{4}{3} g^{3/2} (4\pi Nm_0)^{1/2} (4\pi\chi_p)^{1/2} + \frac{1}{2}\frac{g^2}{R}\frac{4\pi\chi_p}{(1+4\pi\chi_p)} \tag{2.33}$$

Since for most paramagnetic materials, $4\pi Nm_0 \approx 10$ kilo gauss, $\chi_p \sim 10^{-6}$, for $R \gtrsim r_p$ the trapping energy is of the order of a few electron volts. Instead of using the approximate model (2·31) for the paramagnetic material, we can also use the full Langevin function, provided we assume $\chi_p \ll 1$ (Gotto 1957). In this case we may approximate $M(H)$ by

$$M = N m_0 L\left(\frac{3B}{H_p}\right) \tag{2.34}$$

It is then easily shown that the trapping energy is

$$\begin{aligned} W_{\text{trap}} &= -\int_0^R dr \int_0^g (4\pi Nm_0)\, L\left(\frac{3g'}{r^2H_p}\right) dg' \\ &= -\int_0^\infty dr \int_0^g 4\pi Nm_0 L\left(\frac{3g'}{r^2 H_p}\right) dg' + W^{(s)}, \end{aligned} \tag{2.35}$$

where $W^{(s)}$ is the R-dependent contribution given by

$$W^{(s)} = \int_0^g 4\pi N m_0 \, dg' \sqrt{\frac{3g'}{H_s}} \int_{R\sqrt{\frac{H_p}{3g'}}}^{\infty} L\left(\frac{1}{x^2}\right) dx. \tag{2.36}$$

The first term in eq. (2.36) is almost identical to the first term in eq. (2.35). The leading term in eq (2.36) is obtained by letting

$$L\left(\frac{1}{x^2}\right) \sim \frac{1}{3x^2} :$$

$$W^{(s)} \approx \frac{g^2}{2R}(4\pi\chi_p), \tag{2.37}$$

which is in agreement with the surface dependent contribution in eq. (2.33) Finally note that in case $R < r_p$, the trapping energy is given by

$$W_{\text{trap}}(R < r_p) = -(4\pi N m_0)^{1/2} (4\pi\chi_p)^{1/2} g^{3/2}\left(\frac{R}{r_p} - \frac{R^3}{6r_p^3}\right),$$

$$r_p \simeq (4\pi\chi_p g/4\pi N m_0)^{1/2}. \tag{2.38}$$

Since for most paramagnetic materials, r_p turns out to be less than 10^{-8} cm this limit is not of any practical interest, except that it avoids the mathematical singularity in the expression (2.33), if r_p is finite.

2.3. *Trapping energy in a perfect diamagnet (superconducting material)*

Finally we consider the trapping energy of a monopole inside a superconducting material. For the purpose of this paper, we will characterise the superconductivity by saying that inside a superconductor, magnetic induction B is zero. We will again assume that the monopole is situated at the centre of the superconducting sphere of radius R. The fields near the monopole are very strong, which wil destroy the superconductivity, and hence in the region $0 \leqslant r \leqslant r_c$ it will behave like an ordinary metal (figure 2). Here r_c is defined by

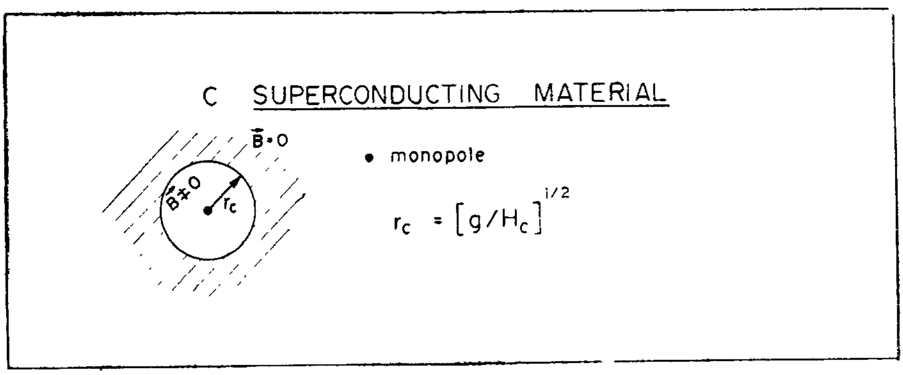

Figure 2. Trapping of a monopole inside a superconductor (perfect diamagnet).

$$g/r_c^2 = H_c \tag{2.39}$$

where H_c is the critical field. Since the monopole is static, in fact the region $0 \leqslant r \leqslant r_c$ behaves like vacuum. The solution of the boundary value problem in the present case is (Note that London's equations imply that only tangential $\boldsymbol{B}$ has to be continuous)

$$\begin{aligned} B_r &= g/r^2 \quad \text{for} \quad 0 \leqslant r \leqslant r_c, \quad R \leqslant r \leqslant \infty, \\ &= 0 \quad \text{for} \quad r_c \leqslant r \leqslant R. \end{aligned} \tag{2.40}$$

In view of the solution (2·40), for $R > r_c$, the trapping energy is

$$W_{\text{trap}} = -\frac{1}{2}\frac{g^2}{r_c}\left(1 - \frac{r_c}{R}\right). \tag{2.41}$$

Since the critical field H_c is of the same order as $4\pi M_s$ in ferromagnetic materials, the trapping energy in a superconducting material is of the same order of magnitude as in a ferromagnet, *i.e.*, several kiloelectron volts. Note that this trapping energy is identical to the case in which the superconductor were treated as a perfect magnetic conductor (eq. (A 11)). This equality of course holds only if the monopole is at the centre of the superconducting sphere.

So far we have assumed that the monopole is located at the centre of the magnetic sphere. We conclude this section by commenting on how the variational method can be used to treat the general non-linear problem in which the monopole is located at an arbitrary point. Let us define a functional $\tilde{F}(\Phi)$ by

$$\begin{aligned} \tilde{F}(\Phi) &= -\frac{1}{4\pi}\int d^3r \int \boldsymbol{B} \cdot d\boldsymbol{H} + \frac{1}{4\pi}\int 4\pi g^{(m)}(\boldsymbol{r})\, \Phi(\boldsymbol{r})\, d^3r \\ &= -\frac{1}{4\pi}\int d^3r \int^{H} 4\pi M\, dH - \frac{1}{4\pi}\int d^3r\, \frac{H^2}{2} \\ &\quad + \frac{1}{4\pi}\int 4\pi\, \rho^{(m)}(\boldsymbol{r})\, \Phi(\boldsymbol{r})\, d^3 r, \quad \boldsymbol{H} = -\vec{\nabla}\Phi, \end{aligned} \tag{2.42}$$

where the integration over $\boldsymbol{r}$ is over all the space. It is verified that a variation of the functional $\tilde{F}(\Phi)$ with respect to Φ yields

$$\nabla^2 \Phi + \vec{\nabla} \cdot \left(\frac{4\pi M(|\vec{\nabla}\Phi|)}{|\vec{\nabla}\Phi|} \vec{\nabla}\Phi \right) = -4\pi\rho^{(m)}(\boldsymbol{r}), \tag{2·43}$$

which is just the eqaution obtained from eq. (2.1) for the model medium (2.10). The variation also yields the boundary condition that $(\vec{\nabla}\Phi \cdot \boldsymbol{n})\,(1 + 4\pi M(|\vec{\nabla}\Phi|)/|\vec{\nabla}\Phi|)$ should be continuous across the boundary. If one wants to apply the variational principle over a domain V occupied by the medium only, the boundary condition has to be displayed explicitly in the function itself. Consider a simplified situation in which outside fields are such that $\boldsymbol{n} \cdot \vec{\nabla}\Phi_0/\Phi_0$ is a constant. In such a case the following variational function can be used

$$\tilde{\tilde{F}}(\Phi) = -\frac{1}{4\pi}\int d^3r \int^{H} 4\pi\, M\, dH - \frac{1}{4\pi}\int d^3r\, \frac{H^2}{2}$$

$$+ \frac{1}{4\pi}\int 4\pi\rho^{(m)}(\boldsymbol{r})\,\Phi(\boldsymbol{r})\, d^3 r + \frac{1}{4\pi}\int ds\, \frac{\boldsymbol{n}\cdot\vec{\nabla}\Phi_0}{\Phi_0}\,\frac{\Phi^2(\boldsymbol{r})}{2}, \qquad (2.44)$$

over the domain occupied by the medium. This can then be minimised explicitly by using suitable trial functions ϕ. This will immediately lead to an approximate determination of the trapping energy in a magnetic medium with monopole source at arbitrary positions.

3. Conclusions

In this paper we have shown quantitatively that a spherical ferromagnetic or superconducting grain of radius $R \approx 10^{-6}$ cm can effectively trap a magnetic monopole with trapping energies of several kiloelectron volts. It would take several kilogauss of external magnetic field to extract such monopoles from these grains. Thus all stable magnetic charges, if they exist at all, are perhaps trapped in ferromagnetic grains or bulk ferromagnetic materials in the universe. The effect of the finite size of the magnetic body had not been considered explicitly in previous calculations, which is by the very nature of the problem, the most essential point in understanding the trapping of monopoles.

As indicated in the introduction the radiation characteristics produced by moving monopoles differ from those of electrons in the polarization properties. We have also indicated that the energy loss is determined from the transverse excitations of the material medium; this may be important in the inelastic scattering of low energy monopoles.

Appendix A

Interaction energy between a monopole and a linear magnetic medium

We consider the form of the interaction energy when a monopole is placed (A) outside a magnetic sphere of permeability μ and radius R (B) inside the permeable sphere. In both the cases we assume that the medium outside the sphere is occupied by a medium of permeability μ_0. These boundary value problems are easily solved using the general solution of Laplace equation in spherical polar coordinates and the boundary conditions (i) Φ is continuous (ii) $-\mu\partial\Phi/\partial r$ is continuous. For the boundary value problem (A) one finds that the potential outside the sphere can be written as

$$\Phi(\boldsymbol{r}) = \frac{g}{\mu_0}\,|\boldsymbol{r} - \boldsymbol{r}_0|^{-1} + g\sum Y_{lm}(\theta,\phi)\, Y^{*}_{lm}(\theta_0,\phi_0)$$

$$\frac{4\pi r^{-(l+1)}}{(2l+1)}\,\frac{R^{2l+1}}{r_0^{l+1}}\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right)\left\{1 + \frac{\mu_0}{\mu}\,\frac{(l+1)}{l}\right\}^{-1}, \qquad (\text{A } 1)$$

where $\boldsymbol{r}_0$ denotes the position of the monopole. For the boundary value problem (B) the potential inside the sphere is found to be

$$\Phi(r) = \frac{g}{\mu} \mid r - r_0 \mid^{-1} + g \sum Y_{lm}(\theta, \phi) \, Y^*_{lm}(\theta_0, \phi_0)$$

$$\times \frac{4\pi}{(2l+1)} \frac{r^l r_0^l}{R^{2l+1}} \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \left\{ 1 + \frac{\mu l}{\mu_0 (l+1)} \right\}^{-1} \qquad \text{(A 2)}$$

The surface (R dependent) contributions to the interaction energies are

$$W_A^{(s)} = \frac{1}{2} g^2 \left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) \sum_{l=0}^{\infty} \frac{R^{2l+1}}{r_0^{2l+2}} \left(1 + \frac{\mu_0}{\mu} \frac{l+1}{l}\right)^{-1}, \qquad \text{(A 3)}$$

$$W_B^{(s)} = \frac{1}{2} g^2 \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right) \sum_{0}^{\infty} \frac{r_0^{2l}}{R^{2l+1}} \left(1 + \frac{\mu}{\mu_0} \frac{l}{(l+1)}\right)^{-1}, \qquad \text{(A 4)}$$

The R-independent contribution to the energy diverges as

$$\frac{1}{2} g^2 \left(\frac{1}{\mu_0} - 1\right) \frac{1}{r}\bigg|_{r\to 0}, \quad \frac{1}{2} g^2 \left(\frac{1}{\mu} - 1\right) \frac{1}{r}\bigg|_{r\to 0}. \qquad \text{(A 5)}$$

In case (A) with $\mu_0 = 1$, there is no divergent contribution and the interaction energy is

$$W_A = \frac{1}{2} g^2 \left(\frac{1}{\mu} - 1\right) \sum_{0}^{\infty} \frac{R^{2l+1}}{r_0^{2l+2}} \left(1 + \frac{l+1}{l\mu}\right)^{-1}. \qquad \text{(A 6)}$$

Similarly in case (B) with $\mu = 1$, the interaction energy is

$$W_B = \frac{1}{2} g^2 \left(\frac{1}{\mu_0} - 1\right) \sum_{0}^{\infty} \frac{r_0^{2l}}{R^{2l+1}} \left(1 + \frac{l}{\mu_0 (l+1)}\right)^{-1}. \qquad \text{(A 7)}$$

The series in (A 6) and (A 7) can be summed up in the following cases

$$W_A \xrightarrow[\mu \to \infty]{} -\frac{1}{2} g^2 \frac{R}{r_0^2} \left(1 - \frac{R^2}{r_0}\right)^{-1}, \qquad \text{(A 8)}$$

$$W_A \xrightarrow[\mu \to 0]{} \frac{1}{2} g^2 \frac{R}{r_0^2} \left(1 - \frac{R^2}{r_0^2}\right)^{-1} \left\{1 + \left(\frac{r_0^2}{R^2} - 1\right) \ln\left(1 - \frac{R^2}{r_0^2}\right)\right\}, \qquad \text{(A 9)}$$

$$W_B \xrightarrow[\mu_0 \to \infty]{} -\frac{1}{2} \frac{g^2}{R} \left(1 - \frac{r_0^2}{R^2}\right)^{-1}. \qquad \text{(A 10)}$$

The results (A 8), (A 10) are appropriate to the case of magnetic conductors.

Finally we also quote the result for the case of a monopole placed at the centre of a hollow cavity of radius r_0 inside a spherical magnetic conductor of radius R. The region outside the conductor is assumed to be vacuum. For this problem one finds for the interaction energy

$$W = -\frac{1}{2} \frac{g^2}{r_0} \left(1 - \frac{r_0}{R}\right). \qquad \text{(A 11)}$$

It is also of interest to note that this interaction energy is independent of the location of the cavity in the conductor (Batygin and Toptygin (1964), p. 217).

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