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# Nonlinear saturation of hot beam-plasma instability

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**Abstract.** The wave-particle interactions in a beam-plasma system in the presence of finite but small thermal motions of the particles are investigated in the linear as well as nonlinear regime. During the linear growth, the thermal motions are found to have a stabilizing effect. The nonlinear evolution is studied by using the Perturbed Orbit Formalism. Due to the thermal motions the nonlinear saturation of growth is found to take place at a level lower than that of the cold case. A detailed study of the energy balance shows that nonresonant particles pick up a bigger fraction of the energy lost by the streaming motion of the beam, thus leading to more efficient 'heating'.

**Keywords.** Perturbed orbits; nonlinear saturation; particle thermal velocities; energy balance.

## 1. Introduction

The evolution of beam-plasma systems, linear as well as nonlinear, has been recently studied extensively. In most of the theoretical studies the model chosen is that of a monoenergetic beam traversing through a cold plasma. In a real physical situation however neither the beam is monoenergetic nor the plasma is cold. The laboratory experiments correspond mostly to the case where the plasma and the beam particles have small but finite thermal velocities (Carr *et al* 1973 and Mizuno and Tanaka 1972). The influence of such thermal velocities on the linear growth of the instability and on the subsequent nonlinear processes, like the saturation and the energy transfer, is discussed in the present study.

In the linear regime the beam and the plasma temperature effects on beam-plasma instability for the high temperature case were discussed by Briggs (1971). He found that for high beam temperature ($v_b \gg un_{ob}/n_{op}$) the growth rate decreases as $1/v_b^2$ and for high plasma temperature ($v_p \gg u$) the instability can be excited only by the resistive medium effect rather than the reactive medium effect; $v_b$ and $v_p$ being the thermal speeds and $n_{ob}$ and $n_{op}$ the equilibrium densities of the beam and the plasma electrons respectively, and $u$ is the streaming velocity of the electrons.

The nonlinear stabilization of the beam-plasma instability due to particle trapping has been discussed, among others, by Onischenko *et al* (1970) and Drummond *et al* (1970). The effect of beam temperature was included in the work of Onischenko *et al* (1970). The saturation of the linear growth due to the diffusive interaction between the waves and the particles was discussed by Manheimer (1971) and

Gupta (1972). The experimental results of Apel (1969) support such a situation. The influence of the weak thermal motion of both beam and plasma in such a situation is studied here by using the Perturbed Orbit Formalism of Dupree (1966).

The complete dispersion relation, linear as well as nonlinear, is derived in section 2. The effects of thermal motions on the linear growth and on the oscillation frequency are obtained in section 3. In section 4 the diffusion coefficient governing the nonlinear interaction between the waves and the particles is derived. We get the saturation level by solving the complete dispersion relation in section 5. The redistribution of energy during the wave-particle interaction is discussed in section 6. Section 7 however deals with the discussion of the results and also of some associated problems.

## 2. Dispersion relation

Let us consider a collisionless beam-plasma system in which the plasma electrons are initially distributed as

$$f_{\text{op}}(v) = \frac{1}{(2\pi v_p)^{1/2}} e - \frac{v^2}{2v_p^2} \tag{2.1}$$

and the beam electrons, which stream through the stationary plasma with a velocity $u$, are governed by the distribution function

$$f_{\text{ob}}(v) = \frac{1}{(2\pi v_b)^{1/2}} e - \frac{(v-u)^2}{2v_b^2} \tag{2.2}$$

The system is taken to be isotropic, homogeneous, field-free and unbounded. Moreover the ions simply provide the neutralizing background. The electrostatic behaviour of such a system is described by the Vlasov equation,

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{e}{m}E\frac{\partial}{\partial v}\right)f_a(x, v, t) = 0. \tag{2.3}$$

In order to study the nonlinear behaviour of the system we will use the perturbed orbit formalism of Dupree (1966) and Weinstock (1969). On following Dupree's technique as used by Gupta (1972) for the beam-plasma system, we write

$$f_a(x, v, t) = \langle f_a(v, t)\rangle + \sum_k e^{ikx} f_{ak}(v, t) \tag{2.4}$$

and

$$E = \sum_k E_k(t)\, e^{ikx+i\beta_k} = \sum_k E_k \exp\{-i\Omega_k t + ikx + i\beta_k\}, \tag{2.5}$$

where $\beta_k$'s are the initial phases and the brackets $\langle \ldots\ldots\ldots \rangle$ indicate ensemble average over the initial phases. The turbulent wave-particle interactions are represented by a diffusion process in the velocity space and the ensemble average distribution function evolve according to the equation,

$$\frac{\partial}{\partial t}\langle f_a(v, t)\rangle = \frac{\partial}{\partial v} D(v) \frac{\partial}{\partial v}\langle f_a(v, t)\rangle, \tag{2.6}$$

where the diffusion coefficient is given by

$$D(v)=\frac{e^2}{m^2}\sum_k |E_k(t)|^2\int_0^\infty d\tau\,\exp\left\{i(\Omega_k-kv)\tau-\frac{1}{3}k^2D(v)\tau^3\right\} \tag{2.7}$$

On using eqs (2.4) to (2.6) in eq. (2.3), we obtain the dispersion relation

$$\epsilon(\Omega_k,k)\equiv 1+\sum_\alpha\frac{\omega_\alpha^2}{ik}\int dv\,\frac{\partial\langle f_\alpha\rangle}{\partial v}\int_0^\infty d\tau\exp\left\{i(\Omega_k-kv)\tau-\frac{1}{3}k^2D\tau^3\right\}$$
$$=0, \tag{2.8}$$

which describes the nonlinear behaviour of the system. In the limit $D=0$ in the exponent, it reduces to the linear dispersion relation. While deriving eq. (2.8) the diffusion coefficient $D$ has been assumed to be independent of $v$. Such a restriction, however, is not necessary in the formalism of Weinstock (1969).

Equations (2.6) to (2.8) completely describe the evolution of the system within the limitations of the formalism used, *e.g.*, the mode coupling terms are neglected. In the initial stages the wave grows by extracting energy from the streaming motion of the beam. As the amplitude of the wave increases the particles start feeling its presence. This takes the system to the nonlinear regime and the orbits of the particles now get perturbed by the growing amplitude of the wave. Finally when the saturation takes place, the wave-particle interaction reaches an equilibrium state. This readustment of the waves and the particles against each other is the physical origin of the diffusion process envisaged in eqs (2.6) and (2.7).

The unstable waves start growing from the initial thermal noise level of the plasma and hence the diffusion process, which is dependent on the amplitude, may be neglected during the initial stage of the evolution. Thus putting $D=0$ and $\langle f_a\rangle_o = f_{oa}(v)$ into (2.8) we obtain the linear dispersion relation

$$\epsilon_L(\Omega_k,k)\equiv 1+\sum_{a=b,p}\frac{\omega_a^2}{ik}\int dv\frac{\partial}{\partial v}f_{oa}(v)\int_0^\infty d\tau\exp\{i(\Omega_k-kv)\tau\}$$
$$=1-\sum_{a=b,p}\omega_a^2\int dv\frac{f_{oa}(v)}{(\Omega_k-kv)^2}=0, \tag{2.9}$$

where $\omega_a^2=(4\pi n_{oa}e^2/m)$ is the plasma frequency.

## 3. Effects of thermal motions on the linear stability

Since we are dealing with low temperatures, we shall solve the dispersion relation (2.9) by iteration. Now for a cold beam plasma, we have

$$f_{op}(v)=\delta(v),\ f_{ob}(v)=\delta(v-u)$$

and the dispersion relation is

$$\epsilon_L^0 \equiv 1 - \frac{\omega_p^2}{\Omega_k^2} - \frac{\omega_b^2}{(\Omega_k - ku)^2} = 0. \tag{3.1}$$

This dispersion relation has been solved, to the lowest order, by Briggs (1971). On defining

$$\epsilon_p = 1 - \frac{\omega_p^2}{\Omega_k^2} \text{ and } \delta = \Omega_k - ku$$

and taking $\delta$ to be small compared to $\Omega_k$, from eq. (3.1) we get

$$\frac{\omega_b^2}{\delta^2} = (\epsilon_p)_{\Omega_k=ku} + \left(\frac{\partial \epsilon_p}{\partial \Omega_k}\right)_{\Omega_k=ku} \delta + \frac{1}{2}\left(\frac{\partial^2 \epsilon_p}{\partial \Omega_k^2}\right)_{\Omega_k=ku} \delta^2 + \ldots \tag{3.2}$$

The growth rate is maximum when $\Omega_k \simeq \omega_p \simeq ku$. On neglecting terms $0(\delta^2)$ eq. (3.2) immediately offers the solution

$$\Omega_0 = \omega_0 + i\gamma_0 \tag{3.3}$$

with

$$\omega_0 = \omega_p - \frac{1}{\sqrt{3}}\gamma_0, \quad \gamma_0 = \frac{\sqrt{3}}{2}\left(\frac{\eta}{2}\right)^{1/3} \omega_p;$$

and

$$\eta = \frac{\omega_b^2}{\omega_p^2} = \frac{\eta_{ob}}{\eta_{op}}.$$

In the present analysis $\eta^{1/3}$ will be taken as the parameters of smallness and all the terms up to second order in this parameter will be retained throughout. With this eq. (3.2) reduces to a quartic equation in $\delta$, which by iterative procedure can be further reduced to a cubic equation in $\delta$, namely

$$\delta_0^4 - \tfrac{2}{3}\,\omega_p \delta^3 + \tfrac{1}{3}\,\omega_b^2\,\omega_p^2 = 0, \tag{3.4}$$

where

$$\delta_0 = -\frac{1}{\sqrt{3}}\gamma_0 + i\gamma_0.$$

Equation (3.4) can now be solved to give

$$\Omega_k = \Omega_{LO} = \omega_{LO} + i\gamma_{LO}, \tag{3.5}$$

where

$$\omega_{LO} = \omega_p\left(1 - \frac{1}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{1}{3}\frac{\gamma_0^2}{\omega_p^2} - \frac{2}{3\sqrt{3}}\frac{\gamma_0^3}{\omega_p^3}\right), \tag{3.6}$$

and

$$\gamma_{LO} = \gamma_0\left(1 - \frac{1}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{5}{9\sqrt{3}}\frac{\gamma_0^3}{\omega_p^3}\right). \tag{3.7}$$

In order to include the thermal motion in the system, we use the distributions (2.1) and (2.2) into (2.9) to obtain the dispersion relation

$$\epsilon_L(\Omega_k, k) \equiv 1 - \sum_a \frac{\omega_a^2}{2k^2 v_a^2} Z'(\zeta_a) = 0, \tag{3.8}$$

where $Z'(\zeta_a)$ is the derivative of the dispersion function $Z(\zeta_a)$ (Fried *et al* 1961). The arguments of $Z$ are defined by

$$\zeta_a = \frac{\Omega_k - ku\delta_{ab}}{\sqrt{2}\, kv_a}, \quad a = p,\ b. \tag{3.9}$$

$\delta_{ab}$ in (3.9) is the kronecker delta. Since the thermal velocities involved are small, we can use the inequality

$$|\zeta_a|^2 \ll 1, \quad i.e., \quad \left(\frac{kv_p}{\omega_p}\right)^2 \ll 1 \quad \text{and} \quad \left(\frac{kv_b}{\gamma_0}\right)^2 \ll 1.$$

These inequalities hold good for the experiments of Mizuno and Tanaka (1972) and Carr *et al* (1973). We take both these quantities to be of order $\epsilon = \eta^{1/3}$; this implies that $(v_b/v_p) \sim \epsilon$. On using the asymtotic expansion for $Z(\zeta_a)$ and on retaining terms $0(\epsilon^2)$, eq. (3.8) reduces to

$$\epsilon_L(\Omega_k, k) \equiv \epsilon_p - \chi_b = 0, \tag{3.10}$$

where

$$\epsilon_p = 1 - \frac{\omega_p^2}{\Omega_k^2} - \frac{3k^2 v_p^2 \omega_p^2}{\Omega_k^4} \tag{3.11}$$

and

$$\chi_b = \frac{\omega_b^2}{(\Omega_k - ku)^2} + \frac{3k^2 v_b^2 \omega_b^2}{(\Omega_k - ku)^4}.$$

For low temperatures we can take $\Omega_k = \Omega_{LO} + \triangle$ ; thus eq. (3.10) becomes

$$\epsilon_p(\Omega_{LO} + \triangle, k) = \chi_b(\Omega_{LO} + \triangle, k). \tag{3.12}$$

Taylor expanding both sides of this equation around $\Omega_k = \Omega_{LO}$ and keeping term $0(\epsilon^2)$, we get

$$a\triangle^2 - b\triangle + c = 0, \tag{3.13}$$

with

$$a = \frac{3\omega_b^2}{(\Omega_{LO} - ku)^4}, \qquad b = \frac{2\omega_b^2}{(\Omega_{LO} - ku)^3} + \frac{2\omega_p^2}{\Omega_{LO}^3}$$

and

$$c = \frac{3k^2 v_b^2 \omega_b^2}{(\Omega_{LO} - ku)^4} + \frac{3k^2 v_p^2 \omega_p^2}{\Omega_{LO}^4}. \tag{3.14}$$

In deriving eq. (3.13), we have made use of eq. (3.1). On substituting eq. (3.5) into eq. (3.13), we obtain $\Omega_L = \omega_L + i\gamma_L$, where

$$\omega_L = \omega_p\left(1 - \frac{1}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{1}{3}\frac{\gamma_0^2}{\omega_p^2} - \frac{2}{3\sqrt{3}}\frac{\gamma_0^3}{\omega_p^3} + \frac{1}{\sqrt{3}}\frac{\gamma_0}{\omega_p}\frac{k^2 v_p^2}{\omega_p^2} + \frac{1}{2}\frac{k^2 v_p^2}{\omega_p^2} - \frac{\sqrt{3}}{16}\frac{k^4 v_p^4}{\gamma_0 \omega_p^3} - \frac{\sqrt{3}}{4}\frac{k^2 v_b^2}{\gamma_0 \omega_p}\right) \tag{3.15}$$

and

$$\gamma_L = \gamma_0\left(1 - \frac{1}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{k^2 v_p^2}{\omega_p^2} - \frac{3}{16}\frac{k^4 v_p^4}{\gamma_0^2 \omega_p^2} - \frac{3}{4}\frac{k^2 v_b^2}{\gamma_0^2}\right). \tag{3.16}$$

In the above expression for $\omega_r$ the last two terms are of order $\epsilon^3$ : in order to be consistent with our approximations, we must neglect these terms. From eq. (3.15), it is apparent that thermal motions increase the oscillation frequency. Because of the finite thermal velocity, the plasma electrons become more mobile and respond to the oscillations more easily thereby increasing its frequency. The effect on the growth of the amplitude of these oscillations is, however, just the opposite. The growth corresponds to increasing the space density of charge or leading to stronger charge bunching. When the electrons possess random thermal velocities they are less easily localized in space than when they are cold. Consequently the growth rate decreases with increasing thermal motion, as depicted by eq. (3.16).

## 4. Diffusion coefficients and nonlinear dispersion relation

As the beam-plasma system evolves, the nonlinear effects become more and more important. The feedback action of the growing wave on the particles may be considered, as indicated before, to be a diffusion process in the velocity space. This diffusion brings about the saturation of the growth.

The growth rate of the instability peaks around $\omega_p \simeq ku$. So from the initial thermal noise present in the plasma, a number of waves centred around the above start growing. However, this fastest growing wave begins to suppress the growth of the neighbouring waves and thus generate a narrow spectrum. The random thermal motion of the particles coupled with this narrow but finite spectrum of waves provide the stochasticity required for describing the system by a diffusive process which is fully described by eq. (2.6).

The orbit integral of eq. (2.8), namely,

$$I = \int_0^\infty d\tau \exp\{i(\omega_k - kv + i\gamma_k)\tau - \tfrac{1}{3}k^2 D\tau^3\},$$

can be rewritten as

$$I(y) = \frac{1}{(k^2 D)^{1/3}} \int_0^\infty d\tau \exp\{-y\tau - \tfrac{1}{3}\tau^3\}, \tag{4.1}$$

with

$$y = [\gamma_k + i(k\boldsymbol{v} - \omega_k)](k^2 D)^{-1/3}.$$

If we consider the number of trapped particles to be small, then the orbit integral of (4.1) can be expanded (Gupta 1972) as

$$(k^2 D)^{1/3} I(y) = \frac{1}{y}\left[1 - \frac{8}{9\xi^2} + \cdots\right], \quad \text{for} \quad |\xi| = |\tfrac{2}{3}y^{3/2}| \gg 1.$$

Consequently, we obtain

$$I(v) \simeq \frac{1}{\gamma_k + i(kv - \omega_k)} - \frac{2k^2 D}{\{\gamma_k + i(kv - \omega_k)\}^4}. \tag{4.2}$$

Putting this into eq. (2.7), we get the diffusion coefficient for the untrapped particles

$$D(v) = \frac{e^2}{m^2} \sum_k |E_k(t)|^2 \frac{\gamma_k}{\gamma_k^2 + (kv - \omega_k)^2}. \tag{4.3}$$

In the limit of small $\gamma_k$ this expression reduces to that of quasilinear diffusion coefficient, thus indicating that the resonance broadening in (4.3) is due to finite $\gamma_k$ which corresponds to strong turbulence.

On using eq. (4.2) for the orbit integral and writing

$$\langle f_a(v, t)\rangle \simeq f_{0a}(v) + \langle f_a(v, t)\rangle_1,$$

the nonlinear dispersion relation of (2.8) simplifies to

$$\epsilon_{NL}(\Omega_k, k) \equiv \epsilon_L(\Omega_k, k) + \sum_a \chi_a(\omega_k, \gamma_k) = 0, \tag{4.4}$$

where $\epsilon_L(\Omega_k, k)$ is given by eq. (3.8) and

$$\chi_a(\omega_k, \gamma_k) = \frac{\omega_a^2}{ik}\int dv \left\{\frac{\frac{\partial}{\partial v}\langle f_a\rangle_1}{\gamma_k + i(kv-\omega_k)} - \frac{2k^2 D\frac{\partial}{\partial v}f_{0a}(v)}{\{\gamma_k + i(kv-\omega_k)\}^4}\right\}, \tag{4.5}$$

with

$$\langle f_a\rangle_1 = \frac{1}{2\gamma_k}\frac{\partial}{\partial v}D(v)\frac{\partial}{\partial v}f_{0a}(v). \tag{4.6}$$

Equation (4.5), with the help of eq. (4.3), can be written in terms of the dispersion function $Z(\zeta_a)$. Once again for $|\zeta_a| \gg 1$, we can use the asymtotic expansion for $Z(\zeta_a)$. Then after some straightforward but cumbersome algebra, we get

$$\begin{aligned}\chi_a(\omega_k, \gamma_k) &= \frac{\omega_a^2\,\omega_B^4}{2^4\gamma_k^4 k^2 v_a^2}\left\{\frac{1}{\zeta_a^2} + 2\sqrt{2}\,i\beta\frac{1}{\zeta_a^3} + \left(-6\beta^2 + \frac{3}{2}\right)\frac{1}{\zeta_a^4}\right.\\ &\quad + (-8\sqrt{2}\,i\beta^3 + 6\sqrt{2}\,i\beta)\frac{1}{\zeta_a^5} + \left(10\beta^4 - 30\beta^2 + \frac{15}{4}\right)\frac{1}{\zeta_a^6}\\ &\quad \left. - \frac{1}{\zeta_a^{*2}} - \frac{3}{2\zeta_a^{*4}} - \frac{15}{4\zeta_a^{*6}} + 4i\sqrt{\pi}\,\zeta_a^* e^{-\zeta_a^{*2}}\right\},\end{aligned} \tag{4.7}$$

where

$$\omega_B^4 = \left|\frac{ekE_k(t)}{m}\right|^2, \quad \beta = \frac{\gamma_k}{kv_a}$$

and $\zeta_a^*$ is the complex conjugate of $\zeta_a$. $\omega_B$ defines the bounce frequency of a particle of mass $m$ and charge $e$ bouncing in the potential of the electric field $E_k(t)$. Now for $kv_p \ll \omega_p$ and $kv_b \ll \gamma_0$,

$$\zeta_a^* e^{-\zeta_*^2} \to 0$$

and hence eq. (4.7) reduces to

$$\begin{aligned}\chi_a(\omega_k, \gamma_k) &= \frac{\omega_a^2\,\omega_B^4}{8\gamma_k^4}\left\{\frac{1}{(\Omega_k - ku\delta_{ab})^2} + \frac{4i\gamma_k}{(\Omega_k - ku\delta_{ab})^3}\right.\\ &\quad + (-12\gamma_k^2 + 3k^2 v_a^2)\frac{1}{(\Omega_k - ku\delta_{ab})^4} + \frac{-32i\gamma_k^3 + 24i\gamma_k k^2 v_a^2}{(\Omega_k - ku\delta_{ab})^5} +\end{aligned}$$

$$+\frac{40\gamma_k^4-120\gamma_k^2 k^2 v_a^2+15k^4 v_a^4}{(\Omega_k-ku\delta_{ab})^6}-\frac{1}{(\Omega_k^*-ku\delta_{ab})^2}$$

$$-\frac{3k^2 v_a^2}{(\Omega_k^*-ku\delta_{ab})^4}-\frac{15k^4 v_a^4}{(\Omega_k^*-ku\delta_{ab})^6}\Big\}. \tag{4.8}$$

Separating the real and the imaginary parts of eq. (4.8), we obtain

$$Re\,\chi_a=\frac{\omega_a^2\,\omega_B^4}{\{(\omega_k-ku\delta_{ab})^2+\gamma_k^2\}^6}\{-5\,(\omega\;-ku\delta_{ab})^6$$

$$-37\,(\omega_k-ku\delta_{ab})^4\gamma_k^2+117\,(\omega_k-ku\delta_{ab})^2\gamma_k^4-11\gamma_k^6$$

$$+2k^2 v_a^2\,[105\,(\omega_k-ku\delta_{ab})^4-126\,(\omega_k-ku\delta_{ab})^2\gamma_k^2+9\gamma_k^4]\} \tag{4.9}$$

and

$$\mathrm{Im}\chi_a=\frac{2\omega_a^2\,\omega_B^4\,(\omega_k-ku\delta_{ab})\,\gamma_k}{\{(\omega_k-ku\delta_{ab})^2+\gamma_k^2\}^6}\Big\{3\,(\omega_k-ku\delta_{ab})^4+54\,(\omega_k-ku\delta_{ab})^2\gamma_k^2$$

$$-29\gamma_k^4+\frac{6k^2 v_a^2}{\gamma_k^2}\Big[5\,(\omega_k-ku\delta_{ab})^4-26\,(\omega_k-ku\delta_{ab})^2\gamma_k^2$$

$$+9\gamma_k^4+\frac{5}{8}\frac{k^2 v_a^2}{\gamma_k^2}\big(-3\,(\omega_k-ku\delta_{ab})^4+10\,(\omega_k-ku\delta_{ab})^2\gamma_k^2$$

$$-3\gamma_k^4\big)\Big]\Big\}. \tag{4.10}$$

In the limit of vanishing thermal velocity of the particles, expressions (4.9) and (4.10) reduce to those of Gupta (1972) for the cold case.

## 5. Nonlinear saturation

The dispersion relation (4.4) may be solved to obtain the nonlinear effects on the growth and the oscillation characteristics of the system under consideration. Let us write

$$\Omega_k=\omega_k+i\gamma_k=\Omega_L+\delta\Omega_k, \tag{5.1}$$

where $\Omega_L$ is defined in eq. (3.15) and $\delta\Omega_k=\delta\omega_k+i\delta\gamma_k$ with $|\delta\Omega_k|\ll|\Omega_L|$.

Now eq. (4.4) may be Taylor expanded to give

$$-\delta\Omega_k\left(\frac{\partial\epsilon_L}{\partial\Omega}\right)_{\Omega=\Omega_L}\simeq\chi_p\,(\omega_L,\gamma_L)+\chi_b\,(\omega_L,\gamma_L). \tag{5.2}$$

On separating the real and the imaginary parts of eq. (5.2) we get two simultaneous equations whose solutions are:

$$\delta\omega_k=-\frac{b_1}{a_1}\left(1+\frac{a_2}{a_1}\frac{c_1}{b_1}-\frac{a_2^2}{a_1^2}\right) \tag{5.3}$$

and

$$\delta\gamma_k=-\frac{c_1}{a_1}\left(1-\frac{a_2}{a_1}\frac{b_1}{c_1}-\frac{a_2^2}{a_1^2}\right) \tag{5.4}$$

where

$$a_1 = -\operatorname{Re}\left(\frac{\partial \epsilon_L}{\partial \Omega}\right)_{\Omega=\Omega_L}$$

$$= -\frac{6}{\omega_p}\left(1 + \frac{2}{\sqrt{3}}\frac{\gamma_0}{\omega_p} - 2\frac{\gamma_0^2}{\omega_p^2} + \frac{\sqrt{3}}{4}\frac{k^2 v_p^2}{\gamma_0\omega_p} + \frac{33}{2}\frac{k^2 v_p^2}{\omega_p^2} - \frac{3}{16}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} + \frac{3}{4}\frac{k^2 v_b^2}{\gamma_0^2}\right), \tag{5.5}$$

$$a_2 = -\operatorname{Im}\left(\frac{\partial \epsilon_L}{\partial \Omega}\right)_{\Omega=\Omega_L}$$

$$= \frac{12}{\omega_p}\left(\frac{\gamma_0}{\omega_p} + \sqrt{3}\frac{\gamma_0^2}{\omega_p^2} - \frac{3}{8}\frac{k^2 v_p^2}{\gamma_0\omega_p} + \frac{3\sqrt{3}}{16}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} - \frac{3\sqrt{3}}{8}\frac{k^2 v_b^2}{\gamma_0^2} - \frac{3\sqrt{3}}{8}\frac{k^2 v_p^2}{\omega_p^2}\right), \tag{5.6}$$

$$b_1 = -\operatorname{Re}\chi_b(\omega_L, \gamma_L)$$

$$= -\frac{5\times 3^3}{2^5}\frac{\omega_b^2\,\omega_B^4}{\gamma_0^6}\left(1 + \frac{17\sqrt{3}}{10}\frac{\gamma_0}{\omega_p} + \frac{9\sqrt{3}}{40}\frac{k^2 v_p^2}{\gamma_0\omega_p}\right), \tag{5.7}$$

$$b_2 = -\operatorname{Re}\chi_p(\omega_L, \gamma_L)$$

$$= -5\frac{\omega_B^4}{\omega_p^4}, \tag{5.8}$$

$$c_1 = -\operatorname{Im}\chi_b(\omega_L, \gamma_L)$$

$$= -\frac{3^4\sqrt{3}}{2^6}\frac{\omega_b^2\,\omega_B^4}{\gamma_0^6}\left(1 - \frac{5}{\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{11\sqrt{3}}{4}\frac{k^2 v_p^2}{\gamma_0\omega_p} - \frac{29}{2}\frac{\gamma_0^2}{\omega_p^2} + \frac{27}{2}\frac{k^2 v_p^2}{\omega_p^2} + \frac{111}{32}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} - \frac{17}{4}\frac{k^2 v_b^2}{\gamma_0^2}\right) \tag{5.9}$$

and

$$c_2 = -\operatorname{Im}\chi_p(\omega_L, \gamma_L)$$

$$= -\frac{6\gamma_0\omega_B^4}{\omega_p^5}\left[1 + 10\frac{k^2 v_p^2}{\gamma_0^2}\left(1 - \frac{5}{8}\frac{k^2 v_p^2}{\gamma_0^2}\right)\right]. \tag{5.10}$$

On substituting eqs (5.5) — (5.10) and retaining $\epsilon^2$ terms, we immediately get the nonlinear contribution to the oscillation frequency and growth rates which are given by

$$\delta\omega_k = -\frac{5\sqrt{3}}{4}\frac{\omega_B^4}{\gamma_0^3}\left(1 + \frac{13\sqrt{3}}{30}\frac{\gamma_0}{\omega_p} + \frac{\sqrt{3}}{5}\frac{k^2 v_p^2}{\gamma_0\omega_p}\right) \tag{5.11}$$

and

$$\delta\gamma_k = -\frac{9}{8}\frac{\omega_B^4}{\gamma_0^3}\left(1 - \frac{1}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{5}{\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0\omega_p} - \frac{49}{18}\frac{\gamma_0^2}{\omega_p^2} - \frac{37}{3}\frac{k^2 v_p^2}{\omega_p^2} + \frac{101}{32}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} - \frac{15}{2}\frac{k^2 v_b^2}{\gamma_0^2}\right). \tag{5.12}$$

The nonlinear growth rate is thus given by

$$\gamma_k = \gamma_L + \delta\gamma_k = \gamma_L\left(1 - \frac{9}{8}\frac{\omega_B^4}{\gamma_0^4}\mu\right), \tag{5.13}$$

where

$$\mu = 1 + \frac{2}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{5}{\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0\omega_p} - \frac{5}{2}\frac{\gamma_0^2}{\omega_p^2} - \frac{29}{3}\frac{k^2 v_p^2}{\omega_p^2} + \frac{107}{32}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} - \frac{27}{4}\frac{k^2 v_b^2}{\gamma_0^2}. \tag{5.14}$$

The growth of the instability saturates when $\gamma_k = 0$ which happens only if

$$1 = \frac{9}{8}\frac{\omega_B^4}{\gamma_0^4}\mu. \tag{5.15}$$

In the limit of vanishing thermal velocities of the particles and considering only the zero order term, eq. (5.15) reduces to the condition obtained by Gupta (1972) for the cold beam-plasma case.

The saturation level of the electric field fluctuations can now be obtained from eq. (5.15) and is given by

$$|E_k(t)|^2 = \frac{8}{9}\gamma_0^4\left|\frac{m}{ek}\right|^2\frac{1}{\mu} = |E_k(t)|_0^2\,\mu^{-1},$$

where $|E_k^{(t)}|^2$ is the saturation level for the cold case. The above expression gives a saturation spectrum $|E_k(t)|^2 \sim k^{-2}$. This agrees with the Langmuir turbulence spectrum obtained by Kingsep *et al* (1973). The effect of the thermal motions on the saturation level is seen from the above relation which may be re-written as

$$|E_k(t)|^2 = |E_k(t)|_0^2\left(1 - \frac{2}{3\sqrt{3}}\frac{\gamma_0}{\omega_p} - \frac{143}{54}\frac{\gamma_0^2}{\omega_p^2} - \frac{5}{\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0\omega_p} + \frac{107}{9}\frac{k^2 v_p^2}{\omega_p^2} + \frac{479}{96}\frac{k^4 v_p^4}{\gamma_0^2\omega_p^2} + \frac{27}{4}\frac{k^2 v_b^2}{\gamma_0^2}\right). \tag{5.16}$$

The dominant term arising from the thermal motion is the term $(-5k^2 v_p^2/\sqrt{3}\gamma_0\omega_p)$ and consequently the saturation level is lowered with respect to the level for the zero thermal velocity case. The finite thermal velocity of the particles enhances the diffusive interaction between the waves and the particles.

On using eq. (5.15), eq. (5.11) becomes

$$\delta\omega_k = -\frac{10}{3\sqrt{3}}\gamma_0\left(1+\frac{19}{30\sqrt{3}}\frac{\gamma_0}{\omega_p}-\frac{22}{5\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0\omega_p}\right).$$

and thus the oscillation frequency at saturation is given by

$$\omega_{ks} = \omega_p\left(1-\frac{13}{3\sqrt{3}}\frac{\gamma_0}{\omega_p}-\frac{46}{27}\frac{\gamma_0^2}{\omega_p^2}+\frac{97}{18}\frac{k^2 v_p^2}{\omega_p^2}\right) \tag{5.17}$$

## 6. Energy balance

In the present problem of beam-plasma interaction the initial energy comprises of the streaming energy of the beam electrons ($T_b^{st} = \frac{1}{2} n_{0b} mu^2$), the thermal energy of the plasma electrons ($T_p^{th} = \frac{1}{2} n_{0p} mv_p^2$), the thermal energy of the beam electrons ($T_b^{th} = \frac{1}{2} n_{0b} mv_b^2$) and the initial fluctuation energy of the waves

$$\left(\mathscr{E}_i = \sum_k \frac{|E_k(t)|^2}{8\pi}\right).$$

Initially the system is taken to be in thermal equilibrium so that $\mathscr{E}_i$ is the thermal noise. Usually this energy is very small compared to the mean particle kinetic energy, *i.e.*

$$\mathscr{E}_i \ll T^{th} = T_p^{th} + T_b^{th}.$$

The wave grows by extracting energy from the streaming motion of the beam and as the amplitude becomes appreciable, it influences the orbits of the particles—thus changing their distribution and energy content. These physical processes are represented by the diffusion equation (2.6) with the diffusion coefficient given by eq. (4.3).

The streaming velocity of the beam after the nonlinear interaction is

$$u_s = \int dv \langle f_a(v, t)\rangle v$$

$$\simeq u + \frac{1}{2\gamma_k}\int dv\, f_{bs}(v)\, \frac{\partial}{\partial v} D(v).$$

After substituting for $D(v)$ from eq. (4.3), the integral may be expressed in terms of the dispersion function $Z(\zeta)$ and its derivatives. Once again if we use the asymtotic expansion for $Z(\zeta)$, we obtain

$$u_s = u + \frac{\omega_B^4}{k}\frac{(\omega_k - ku)}{\{(\omega_k - ku^2) + \gamma_k^2\}^2}\left\{1 + 6k^2 v_b^2 \frac{(\omega_k - ku)^2 - \gamma_k^2}{\{(\omega_k - ku)^2 + \gamma_k^2\}^2}\right\} \tag{6.1}$$

On using eqs (3.15) and (5.15) this gives

$$u_s = u - \frac{3\sqrt{3}}{16}\frac{\omega_B^4}{k\gamma_0^3}\left(1+\sqrt{3}\frac{\gamma_0}{\omega_p}+\frac{\gamma_0^2}{\omega_p^2}+\frac{9}{2}\frac{k^2 v_p^2}{\omega_p^2}\right); \tag{6.2}$$

the energy being

$$\tilde{T}_b^{st} = T_b^{st} - 2\mathscr{E}\left(1+\frac{3\sqrt{3}}{4}\frac{\gamma_0}{\omega_p}-\frac{1}{3}\frac{\gamma_0^2}{\omega_p^2}+\frac{23}{4}\frac{k^2 v_p^2}{\omega_p^2}\right), \tag{6.3}$$

where

$$\mathcal{E} = \sum_k \frac{|E_k(t)|^2}{8\pi}$$

is the fluctuation energy after the nonlinear process. The changes in the energy of the different components shall be expressed in terms of $\mathcal{E}$.

The thermal energies of the plasma and the beam particles may be obtained in the same way. They are given by

$$\tilde{T}_p^{\,\mathrm{th}} = \langle \tfrac{1}{2} n_{0p} m v^2 \rangle$$
$$= T_p^{\,\mathrm{th}} + \mathcal{E}\left(1 + \frac{2}{\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{2}{3}\frac{\gamma_0^2}{\omega_p^2} + 8\,\frac{k^2 v_p^2}{\omega_p^2}\right), \tag{6.4}$$

and

$$\tilde{T}_b^{\,\mathrm{th}} = \langle \tfrac{1}{2} n_{0b} m v^2 \rangle$$
$$= T_b^{\,\mathrm{th}} + \frac{7}{2\sqrt{3}}\frac{\gamma_0}{\omega_p}\,\mathcal{E}\left(1 + \frac{8}{21\sqrt{3}}\frac{\gamma_0}{\omega_p} + \frac{11}{7\sqrt{3}}\frac{k^2 v_p^2}{\gamma_0\omega_p}\right). \tag{6.5}$$

From eq. (6.3) it is obvious that the energy lost by the streaming motion of the beam increases because of the thermal motion. Also there is an increase in the thermal energies of the plasma and the beam, as is evident from eqs (6.4) and (6.5). On the other hand eq. (5.16) shows a decrease in fluctuation energy with thermal motion. Thus the finite temperature of the particles result in an enhanced transfer of energy from the streaming motion into the thermal motion. The quasi-linear theory predicts an equal sharing of the energy lost by the resonant particles among the nonresonant particles and the fluctuations (Davidson 1972). Equations (5.16), (6.3), (6.4) and (6.5) suggest a modification of this result: the particles pick up more energy than the waves during the nonlinear interaction.

## 7. Discussion and conclusions

The nonlinear interaction of the growing waves with the nontrapped particles is the cause of the nonlinear processes studied here. The saturation of the beam-plasma instability due to trapping as discussed by Drummond *et al* (1970) requires a single wave with large enough amplitude to trap the beam particles. Here we have discussed the diffusive interaction between the waves and particles. This could be considered as an alternative way of saturating the beem-plasma instability.

The basis for using the diffusion type interaction rather than the trapping process are :

(i) The waves discussed here are the ones that grow from the thermal noise and thus have a very small amplitude to begin with. In the experiments where the saturation due to particle trapping are observed [Mizuno *et al* (1972) and Carr *et al* (1973)], the wave is a large amplitude launched single wave. Also in these experiments the particles are found to be detrapped when other waves appear in the system. According to theoretical model of O'Neil *et al* (1972), similar detrapping of particles would take place due to nonlocal interactions.

(ii) When the waves grow up from the thermal noise, a number of waves around the, $\omega_s \simeq ku$, mode grow and thus generate a spectrum of waves. The presence of a spectrum of waves, though narrow, is a favourable condition for the diffusion process to occur. Moreover diffusion can take place at not-so-large amplitude of the wave because unlike the trapping process it does not need a large critical amplitude.

(iii) The random thermal motion of the particles also contribute to the stochasticity, which is the essential requirement for diffusion. The phenomenon in this sense is analogous to the linear Landau damping, which is absent for particle distributions represented by delta function.

Biskamp and Welter (1972) had shown that the diffusion model would be valid if $k \triangle (\omega/k)_s > \omega_B$, $\triangle (\omega/k)_s$ being the spread in the phase velocity at saturation. In our case, from eq. (5.17), we find that

$$k \triangle \left(\frac{\omega}{k}\right)_s \approx \frac{\triangle k}{k} \omega_p \approx \frac{\omega_B}{\epsilon} \frac{\triangle k}{k} .$$

In beam-plasma case, it was pointed out by Drummond *et al* (1970) that $(\triangle k/k) \sim \epsilon$ and hence it is a reasonable model to be used.

We have considered the beam particles to be nontrapped. As discussed by Gupta (1972), the required condition for non-trapping is

$$(kv - \omega_k)^2 > (k^2 D)^{2/3} - \gamma_k^2,$$

which can be rewritten as $2\gamma_0^2 > (k^2 D)^{2/3}$. And for nonresonant particles eq. (4.3), gives

$$(k^2 D)^{2/3} \simeq \left(\frac{e^2 k^2 \mid E_k(t) \mid^2}{m^2} \frac{\gamma_0}{\omega_p}\right)^{2/3} = \omega_B^2 \left(\frac{\omega_B \gamma_0}{\omega_p^2}\right)^{2/3},$$

Since at saturation $\omega_B \sim \gamma_0$, the condition for nontrapping is easily satisfied.

The beam-plasma instability discussed here results in the turbulence of the electron plasma or Langmuir waves. The electric potential $\phi$ of these turbulent waves are governed by the nonlinear Schröedinger equation (Asano and Taniuti 1969)

$$i \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \frac{\partial^2 \phi}{\partial \xi^2} - \frac{\partial \omega}{\partial \mid \phi \mid^2} \mid \phi \mid^2 \phi = 0.$$

When $(\partial^2 \omega / \partial k^2) (\partial \omega / \partial \mid \phi \mid^2) < 0$, the system is modulationally unstable. From eqs (3.15) and (5.11) it is seen that the Langmuir turbulence is modulationally stable; this is in agreement with the results of Asano and Tanuti (1969).

The thermal motions of the particles, which are inevitably present in a plasma have an appreciable effect on the evolution of the system. The presence of the thermal motion results in the reduction of the saturation level of the electric field. This is because of the fact that the diffusion in velocity space, which brings about the saturation, is strengthened by the randomness in the particle motion. The streaming motion of the beam loses more energy in the presence of thermal motion. This energy is shared by the fluctuations and the particles (thermal motion), the latter taking the bigger fraction due to the finite temperature. It is known that the nonlinear saturation of the beam-plasma instability results in a heating of the plasma electrons. The particle thermal motions produce a more efficient heating by slowing down the beam further and by bringing the level of saturation down. As is seen in the various results, the thermal velocity of the plasma particles have a more prominent effect than that of the beam particles.

Initially the system under consideration consists of a plasma, a beam traversing through it, and background fluctuations. The beam drives the resonant mode unstable and this instability is stabilized by the feedback effect of the growing waves on the particles. The end product of the analysis is a plasma with a larger spread in the thermal velocities, a beam with a reduced streaming velocity but larger thermal velocity traversing through the system, and a large-amplitude-stable wave (the saturated wave). This final configuration is stable as far as the saturated wave is concerned but not necessarily against other modes. In fact the resonant mode ($\omega_p \simeq ku_s$) may now be expected to grow. The nonlinear behaviour of this later time instability may be studied approximately by carrying out the complete analysis all over again. However, there are other factors like the presence of a large amplitude wave, broader spectrum of waves, which should also be taken into consideration.

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