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Cylindrically symmetric matter distribution with a magnetic field in Einstein-Cartan theory

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Abstract. In this paper we have extended our earlier studies of solutions of Einstein-Cartan equations to the case where a magnetic field co-exists with the matter distribution. We have obtained an exact solution of Einstein-Cartan-Maxwell equations representing a static cylinder of perfect fluid with an axial magnetic field H and a non-zero spin density K, satisfying the equation of state $\rho = \gamma\,(p_r + p_z - H^2/4\pi)$, γ being a constant. We notice that as a consequence of field equations there exists a direct relation between the pressure p, and the spin density K, indicating that an increase in pressure would enormously increase the spin density.

Introduction

Recently we have obtained some solutions (Prasanna 1975 *a* and *b*) for perfect fluid distributions in the framework of Einstein-Cartan theory, wherein the spin density of the fluid distribution gives rise to torsion of space-time. However, in obtaining these solutions we had not considered the contribution from the magnetic field towards the canonical energy-momentum tensor, treating the system as though the magnetic field was switched off after obtaining the necessary spin alignment. We now consider the more realistic picture of a static cylindrically symmetric perfect fluid distribution with an axial magnetic field which produces the spin polarisation of the particles composing the fluid. We follow in this the same notation and conventions as from our earlier papers (Prasanna 1975 *a* and *b*) and hence would refer the reader to them for details.

Since we are having a magnetic field we now have for field equations apart from the Einstein-Cartan equations,

$$\left.\begin{aligned} &R_i^{\,j} - \tfrac{1}{2} R\delta_i^{\,j} = -\,x t_i^{\,j}, \\ &Q^i_{jk} - \delta_j^{\,i}\, Q^l_{lk} - \delta_k^{\,i}\, Q^l_{jl} = -\,x s^i_{jk}, \end{aligned}\right\} \tag{1}$$

the generalized Maxwell equations (Prasanna 1975 *c*)

$$\left.\begin{aligned} &\nabla_k F_{jl} + \nabla_l F_{kj} + \nabla_j F_{lk} = F_{km} Q^m_{jl} + F_{lm} Q^m_{kj} + F_{jm} Q^m_{lk}, \\ &2\nabla_k F^{ik} + (Q^i_{km} - \delta_k^{\,i}\, Q^l_{lm} - \delta_m^{\,i}\, Q^l_{kl})\, F^{km} = J^i, \end{aligned}\right\} \tag{2}$$

with

$$\nabla_i J^i = -\,Q^k_{ik} J^i.$$

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where ∇_k denotes the usual covariant derivative, but taken with respect to the asymmetric connection Γ^i_{jk} (Prasanna 1975 *a*)

We represent the space-time under consideration by the usual cylindrically symmetric metric,

$$ds^2 = - e^{2\mu-2\nu}(dr^2 + dz^2) - r^2 e^{-2\nu} d\phi^2 + e^{2\nu} dt^2 \tag{3}$$

where $\mu = \mu(r)$ and $\nu = \nu(r)$. The symmetric energy-momentum tensor $\bar{T}_i^{\,j}$ is given by $\bar{T}_i^{\,j} = M_i^{\,j} + E_i^{\,j}$ where $M_i^{\,j}$ denotes the matter part and $E_i^{\,j}$ the electromagnetic part, defined as

$$E_i^{\,j} = -\frac{1}{4\pi}(F^{jk} F_{ik} - \tfrac{1}{4}\delta_i^{\,j} F^{kl} F_{kl}) \tag{4}$$

F_{ij} being the electromagnetic field tensor. We assume a perfect fluid distribution with anisotropic pressure so that,

$$M_{ij} = \mathrm{diag}(-p_r, -p_\phi, -p_z, \rho). \tag{5}$$

Since we have only an axial magnetic field the non-zero components of the field tensor F_{ij} are $F_{12} = -F_{21} = H$ (say), and hence $E_i^{\,j}$ has the components

$$E_1^{\,1} = E_2^{\,2} = -E_3^{\,3} = -E_4^{\,4} = -H^2/8\pi, \qquad E_i^{\,j} = 0, \qquad i \neq j. \tag{6}$$

As the spins align along the direction of the magnetic field, we have for a classical description of spin, (Trautman 1973) the components of the tensor of spin density s^i_{jk} to be $s^4_{12} = -s^4_{21} = K$ (say), rest being zero. With these we get for the canonical energy-momentum tensor $t_i^{\,j}$ given by (Prasanna 1975 *a*)

$$t_i^{\,j} = T_i^{\,j} + \tfrac{1}{2} g^{jm} \nabla_k (s^k_{im}), \tag{7}$$

the non-zero components:

$$t_1^{\,1} = -p_r - H^2/8\pi, \qquad t_2^{\,2} = -p_\phi - H^2/8\pi, \qquad t_3^{\,3} = -p_z + H^2/8\pi$$

$$t_4^{\,4} = \rho + H^2/8\pi, \qquad t_4^{\,2} = t_2^{\,4} = K e^{\nu-\mu} \nu'/2. \tag{8}$$

Using the expressions for $R_i^{\,j}$ and R from reference (2), and $t_i^{\,j}$ as given above we get the field equations to be

$$-e^{(2\nu-2\mu)}\left(\frac{\mu'}{r} - \nu'^2\right) - \frac{x^2 K^2}{4} = -x(-p_r - H^2/8\pi) \tag{9}$$

$$-e^{(2\nu-2\mu)}(\mu'' + \nu'^2) - \frac{x^2 K^2}{4} = -x(-p_\phi - H^2/8\pi) \tag{10}$$

$$-e^{(2\nu-2\mu)}\left(\nu'^2 - \frac{\mu'}{r}\right) - \frac{x^2 K^2}{4} = -x(-p_z + H^2/8\pi) \tag{11}$$

$$e^{(2\nu-2\mu)}\left(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2\right) + \frac{x^2 K^2}{4} = -x(\rho + H^2/8\pi) \tag{12}$$

$$e^{(\nu-\mu)}(K' + K\mu' - K\nu') = -K e^{\nu-\mu} \nu', \tag{13}$$

$$e^{(\nu-\mu)}(K' + K\mu' + K\nu') = K e^{\nu-\mu} \nu', \tag{14}$$

$$J^2 = -2e^{-\mu}(He^{\mu})_{,1}, \tag{15}$$

$$\nabla_k F_{ij} + \nabla_j F_{ki} + \nabla_i F_{jk} = 0. \tag{16}$$

(13) and (14) imply the relation $K' + K\mu' = 0$, solving which we get $K = Ae^{-\mu}$,

A being a constant of integration. It is apparent from (9) and (11) that p_r and p_z both cannot be simultaneously zero when K is non-zero. In fact we have

$$p_r + p_z = \left(\frac{4\pi G}{c^2}\right) K^2, \tag{17}$$

a direct relation between the pressure and the spin density. p_z being the stress in the direction along which the spins are aligned and p_r the stress in the direction perpendicular to it, it is indeed interesting to note that if the spins are aligned then one of these stresses should be non-zero. Rewriting the equation other way we get,

$$K = (c/\sqrt{4\pi G})\,(p_r + p_z)^{\frac{1}{2}} \tag{18}$$

showing that any increase in pressure would increase the spin density enormously. Since this relation is a direct consequence of the field equations, it may be considered as a prediction of the theory. However, one should remember that the model under consideration is that of an infinite cylinder and thus should consider only the qualitative nature of this result. (It may be noticed that the above arguments are equally valid even for the case when the magnetic field is assumed to be zero, *i.e.*, $H = 0$).

Looking at the system (9)–(12) we find that there are seven quantities to be determined but only four equations. Hence we need three further relations. Supposing we assume that one of the components of pressure to be zero, from the continuity conditions at the boundary, namely, (i) metric potentials are C^1, (ii) H is continuous, (iii) K is discontinuous, we find that only p_ϕ could be zero. Using this, and adopting natural units, the equations reduce to,

$$\left.\begin{aligned}
H^2 &= e^{(2\nu-2\mu)}\,(\mu'' + \nu'^2) + 16\,\pi^2 A^2 e^{-2\mu},\\
8\pi p_r &= e^{(2\nu-2\mu)}\left(\frac{\mu'}{r} - \mu'' - 2\nu'^2\right),\\
8\pi p_z &= e^{(2\nu-2\mu)}\left(\frac{-\mu'}{r} + \mu'' + 2\nu'^2\right) + 32\,\pi^2 A^2 e^{-2\mu},\\
8\pi \rho &= 2e^{(2\nu-2\mu)}\left(-\mu'' + \nu'' + \frac{\nu'}{r} - \nu'^2\right).
\end{aligned}\right\} \tag{19}$$

We now assume the equation of state,

$$\rho = \gamma\,(p_r + p_z - H^2/4\pi), \tag{20}$$

and the ansatz,

$$\nu = \ln\,(A_1 + A_2\, r^2/a^2), \tag{21}$$

where r, A_1, A_2, a, are constants, a being the radius of the cylinder. Using (19) and (21) in (20) we get a simple differential equation for μ,

$$(1-\gamma)\,\mu'' = \frac{4A_2 a^2}{(A + A_2\, r^2/a^2)} + (\gamma - 2)\,\frac{4A_2{}^2\, r^2/a^4}{(A_1 + A_2\, r^2/a^2)^2}, \tag{22}$$

solving which we get,

$$\mu = 2\nu - \left(\frac{2\gamma}{\gamma-1}\right)\left\{\left(\sqrt{\frac{A_2}{A_1}}\,\frac{r}{a}\right)\tan^{-1}\left(\sqrt{\frac{A_2}{A_1}}\,\frac{r}{a}\right)\right\} - \frac{C_1\, r}{(\gamma - 1)} - \frac{(\gamma \ln A_1 + C_2)}{(\gamma - 1)} \tag{23}$$

Hence we have for the physical quantities,

$$
\left.\begin{aligned}
8\pi p_r &= e^{(2\nu-2\mu)}\left\{\left(\frac{-2\gamma}{\gamma-1}\right)\left[\sqrt{\frac{A_2}{A_1}}\frac{1}{ar}\tan^{-1}\left(\sqrt{\frac{A_2}{A_1}}\frac{r}{a}\right)\right.\right.\\
&\quad\left.\left. -\frac{A_2}{a^2}\frac{(A_1-A_2r^2/a^2)}{(A+A_2r^2/a^2)}\right]-\frac{C_1}{(\gamma-1)r}\right\},\\
8\pi p_\theta &= -8\pi p_r + 32\pi^2A^2e^{-2\mu},\\
8\pi\rho &= (8rA_1A_2e^{-2\mu})/a^2(\gamma-1),
\end{aligned}\right\} \tag{24}
$$

$$
H^2 = e^{-2\mu}\left\{16\pi^2A^2-\frac{4A_1A_2}{a^2(\gamma-1)}\right\}; \; J^2=0, \textit{ i.e.,} \quad J^i=0. \tag{25}
$$

We have now five constants A_1, A_2, C_1, C_2, and A to be determined and we have exactly five continuity conditions, *viz.*, μ, ν, μ', ν' and H are continuous at the boundary $r=a$.

Since the region $r>a$ *i.e.*, outside the cylinder is empty (devoid of matter) we have here only a pure magnetic field coupled to the gravitational field. Thus the field equations are given by,

$$
e^{(2\nu-2\mu)}\left(\frac{\mu'}{r}-\nu'^2\right)=H^2; \quad e^{2\nu-2\mu}(\mu''+\nu'^2)=H^2;
$$

$$
e^{(2\nu-2\mu)}\left(2\nu''-\mu''+\frac{2\nu'}{r}-\nu'^2\right)=H^2. \tag{26}
$$

There are some known solutions of these equations and we use the one given by Ghosh and Sengupta (1965) as follows:

$$
\mu = 2\nu+\lambda(\lambda-2)\ln r+\ln B; \quad \nu=\ln\left\{r^\lambda+\frac{c^2\,r^{2-\lambda}}{4(1-\lambda)^2}\right\}
$$

$$
H^2 = C^2/B^2\,r^{2\lambda(\lambda-2)}\left\{r^\lambda+\frac{C^2\,r^{2-\lambda}}{4(1-\lambda)^2}\right\}^4 = C^2e^{-2\mu}. \tag{27}
$$

Now matching our solution as given by (21) (23) and (25) with (27) at the boundary $r=a$, we get the various constants as,

$$
\left.\begin{aligned}
A_1 &= \frac{1}{2}\left\{(2-\lambda)\,a^\lambda+\frac{\lambda C^2\,a^{2-\lambda}}{4(1-\lambda)^2}\right\},\\
A_2 &= \frac{1}{2}\left\{\lambda a^\lambda+\frac{(2-\lambda)\,C^2\,a^{2-\lambda}}{4(1-\lambda)^2}\right\},\\
C_1 &= \frac{\lambda(2-\lambda)(\gamma-1)}{a}-\frac{2\gamma}{a}\left\{\frac{A_2}{(A_1+A_2)}+\sqrt{\frac{A_2}{A_1}}\tan^{-1}\sqrt{\frac{A_2}{A_1}}\right\},\\
C_2 &= (\gamma-1)\left[\lambda(2-\lambda)(\ln a-1)-\ln B\right]+\frac{2\gamma A_2}{(A_1+A_2)}-\gamma\ln A_1,\\
A &= \frac{1}{4\pi}\left\{\left(\frac{1}{\gamma-1}\right)\left[\gamma C^2+\frac{\lambda(2-\lambda)}{a^2}\left(a^\lambda+\frac{C^2\,a^{2-\lambda}}{4(1-\lambda)^2}\right)^2\right]\right\}^{1/2}.
\end{aligned}\right\} \tag{28}
$$

It is obvious that in order to have the density positive and the magnetic field real we should have $\gamma > 1$.

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