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Energy moments of scattering phase shift for partly non-local momentum dependent interaction

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Abstract. The phase shift sum rules recently derived by Puff are extended for a general partly non-local momentum dependent potential. In achieving this, one must use Fredholm determinant of the outgoing solution of the Schrodinger equation instead of the Jost function as was done by Puff. The constants appearing in the moment relations are explicitly defined in terms of the momentum representation of the interaction.

Keywords. Phase shift moments; partly non-local momentum; dependent interaction; Fredholm determinant; Phase equivalence; Jost function.

1. Introduction

In the non-relativistic scattering theory, the zeroth moment of the derivative of a phase shift is Levinson's theorem (Levinson 1949). The next moment relation was first derived by Newton (1956). The generalization of this phase shift sum rule to higher moments and partial waves, in the case of local potential have been derived by Buslaev and Faddeev (1960), Percival (1962), Percival and Roberts (1963) and by Roberts (1964). All these authors used different methods to derive the same sum rules. Recently, Puff (1975) rederived these moment relations, again, for local potentials, using the subtracted dispersion relations satisfied by the Jost function and the Born series expansion of the Jost solution. He also pointed out the sum rules satisfied by the real part of the Jost function. All the derivations mentioned above used only the local potentials. Due to this reason, the sum rules depend on the volume integrals and the derivatives at the origin of the powers of the potentials. The potentials frequently used in nuclear physics are often partly non-local and momentum dependent. Through this point of view, the generalization of the phase shift sum rules for such a general class of potentials will be quite useful. Some of the sum rules derived from local potential may not even be valid for the non-local potentials, especially those for the real part of the Jost function. In our earlier work (Warke and Bhaduri 1971; Warke 1972) on Jost function, it was found that the general Jost function expression, valid for partly non-local, momentum dependent potential, is the ratio of the Fredholm determinant of the outgoing solution of the Schrodinger equation to that of the regular solution. In the special case of a local potential, it reduces to the Fredholm determinant for the outgoing solution. These differences indicate that the sum rules will depend on the type of the potential, if the Jost solution is used to derive them. Our approach, besides being different from those used earlier, also gives the potential dependent constants appearing in the sum rules for a general potential. In the particular case of the local potential, these constants correspond to the volume integrals and the derivatives of the potential at the origin (Verde 1955; Puff 1975).

2. Derivation

For a partly non-local momentum dependent potentials to be phase equivalent to each other, it is sufficient that the Fredholm determinants of the outgoing solutions of these potentials be equal. Their Jost functions need not be the same. This implies that one should not use Jost function, as was done by Puff, in deriving the sum rules for non-local or momentum dependent potentials. Instead, the Fredholm determinant $D^+(k) = \exp\{Tr\log(1 - G^+ V)\}$ should be used in case of such a general potential. Here G^+ is the non-interacting Green function and the potential V is measured in units of $2\mu/\hbar^2$. The traces in the exponent of $D^+(k)$ are taken in the momentum space representation, in which the potential matrix is defined for a general interaction. We consider only the $l = 0$ channel scattering. The generalization to higher partial waves can be similarly carried out. We choose the orthonormal basis states

$$\langle r \mid p\rangle = \sqrt{\frac{2}{\pi}}\sin pr \qquad \text{so that} \qquad \langle p \mid p'\rangle = \delta(p - p'). \tag{1}$$

With this normalization, the summation over the momentum variable is equivalent to just the integration over that variable. In order to study the large k behaviour of $\log D^+(k)$ which will be required for the derivation of the sum rules, we expand the exponent.

$$Tr\log(1 - G^+ V) = -Tr\{G^+ V + \tfrac{1}{2}(G^+ V)^2 + \tfrac{1}{3}(G^+ V)^3 + \ldots\}. \tag{2}$$

This follows from the observation that $G^+ V$ introduces a factor $1/k$ every time it appears in the expansion. Thus the exponent correct up to fourth order in $1/k$ is

$$\begin{aligned}
&Tr\log(1 - G^+ V)\\
&= \frac{\pi i}{k}\langle k \mid V \mid k\rangle - P\int\frac{\langle p \mid V \mid p\rangle\, dp}{k^2 - p^2} + \frac{1}{2}\left(\frac{\pi}{2k}\right)^2 \langle k \mid V \mid k\rangle^2\\
&\quad + \frac{\pi i}{2k} P\int\frac{\langle k \mid V \mid p\rangle\langle p \mid V \mid k\rangle\, dp}{k^2 - p^2} - \frac{1}{2} P\int\frac{\langle p \mid V \mid p'\rangle^2\, dp dp'}{(k^2 - p^2)(k^2 - p'^2)}\\
&\quad - \frac{i}{3}\left(\frac{\pi}{2k}\right)^3 \langle k \mid V \mid k\rangle^3 + \left(\frac{\pi}{2k}\right)^2 \langle k \mid V \mid k\rangle\, P\int\frac{\langle k \mid V \mid p\rangle^2\, dp}{k^2 - p^2}\\
&\quad + \frac{\pi i}{2k} P\int\frac{\langle k \mid V \mid p\rangle\langle p \mid V \mid p'\rangle\langle p' \mid V \mid k\rangle\, dp dp'}{(k^2 - p^2)(k^2 - p'^2)}\\
&\quad - \frac{1}{3} P\int\frac{\langle p \mid V \mid p'\rangle\langle p' \mid V \mid p''\rangle\langle p'' \mid V \mid p\rangle\, dp dp' dp''}{(k^2 - p^2)(k^2 - p'^2)(k^2 - p''^2)}\\
&\quad - \frac{1}{4}\left(\frac{\pi}{2k}\right)^4 \langle k \mid V \mid k\rangle^4 + \left(\frac{\pi}{2k}\right)^2 \langle k \mid V \mid k\rangle
\end{aligned}$$

$$\times\ P\int \frac{\langle k\mid V\mid p\rangle\langle p\mid V\mid p'\rangle\langle p'\mid V\mid k\rangle\, dpdp'}{(k^2-p^2)(k^2-p'^2)}$$

$$-\frac{1}{4}P\int \frac{\langle p\mid V\mid p'\rangle\langle p'\mid V\mid p''\rangle\langle p''\mid V\mid p'''\rangle\langle p'''\mid V\mid p\rangle\, dpdp'dp''dp'''}{(k^2-p^2)(k^2-p'^2)(k^2-p''^2)(k^2-p'''^2)}$$

$$+\ \ldots\ . \tag{3}$$

In the above equation $P\int\ \ldots$ denotes the principle value multi-dimensional integrals over the variables shown by their differentials. From the knowledge of even and odd functions of k and the order of $G^{+}V$, the integrals in the above equation can be shown to have the following asymptotic expansions

$$\pi\langle k\mid V\mid k\rangle = C_1^{(1)} + C_3^{(1)}/(2k)^2 + 0\,(1/k^4)$$

$$P\int \frac{\langle p\mid V\mid p\rangle\, dp}{k^2-p^2} = C_2^{(1)}/(2k)^2 + C_4^{(1)}/(2k)^4 + 0\,(1/k^6)$$

$$\pi P\int \frac{\langle k\mid V\mid P\rangle\langle p\mid V\mid k\rangle\, dp}{k^2-p^2} = C_3^{(2)}/(2k)^2 + 0\,(1/k^4)$$

$$P\int \frac{\langle p\mid V\mid p'\rangle^2\, dpdp'}{(k^2-p^2)(k^2-p'^2)} = C_2^{(2)}/(2k)^2 + C_4^{(2)}/(2k)^4 + 0\,(1/k^6)$$

$$\pi P\int \frac{\langle k\mid V\mid p\rangle\langle p\mid V\mid p'\rangle\langle p'\mid V\mid k\rangle\, dpdp'}{(k^2-p^2)(k^2-p'^2)} = C_3^{(3)}/(2k)^2 + 0\,(1/k^4)$$

$$P\int \frac{\langle p\mid V\mid p'\rangle\langle p'\mid V\mid p''\rangle\langle p''\mid V\mid p\rangle\, dpdp'dp''}{(k^2-p^2)(k^2-p'^2)(k^2-p''^2)} = C_4^{(3)}/(2k)^4 + 0\,(1/k^6)$$

$$P\int \frac{\langle p\mid V\mid p'\rangle\langle p'\mid V\mid p''\rangle\langle p''\mid V\mid p'''\rangle\langle p'''\mid V\mid p\rangle\, dpdp'dp''dp'''}{(k^2-p^2)(k^2-p'^2)(k^2-p''^2)(k^2-p'''^2)}$$

$$= C_4^{(4)}/(2k)^4 + 0\,(1/k^6) \tag{4}$$

Substituting eq. (4) in eq. (3), and combining together terms of the same order in $1/k$, eq. (3) correct to order $1/k^4$ is

$$\log D^{+}(k) = Tr\log(1-G^{+}V) = i\delta(k) + \log|D^{+}(k)|$$

$$= \frac{i}{2k}C_1^{(1)} + \frac{i}{(2k)^3}[C_3^{(1)} + C_3^{(2)} + C_3^{(3)} - \tfrac{1}{3}C_1^{(1)3}]$$

$$+\frac{1}{(2k)^2}[-C_2^{(1)} + \tfrac{1}{2}C_1^{(1)2} - \tfrac{1}{2}C_2^{(2)}]$$

$$+\frac{1}{(2k)^4}[-C_4^{(1)} - \tfrac{1}{4}C_4^{(4)} - \tfrac{1}{3}C_4^{(3)} - \tfrac{1}{2}C_4^{(2)} - \tfrac{1}{4}C_1^{(1)4}$$

$$+ C_1^{(1)}C_3^{(1)} + C_1^{(1)}C_3^{(2)} + C_1^{(1)}C_3^{(3)}] + 0\,(1/k^5). \tag{5}$$

In the particular case of a local potential, one finds after a tedious algebra,

$$C_1^{(1)} = \int_0^\infty V(r)\,dr = U_0, \qquad C_3^{(1)} = V'(0), \qquad C_2^{(1)} = -V(0),$$

$$C_4^{(1)} = V''(0), \qquad C_3^{(2)} = \int_0^\infty V^2(r)\,dr = W_0, \; C_2^{(2)} = U_0^2,$$

$$C_4^{(2)} = 2V'(0)\,U_0 - 4V^2(0), \; C_3^{(3)} = \tfrac{1}{3}\,U_0^3, \qquad C_4^{(3)} = 3U_0W_0,$$

$$C_4^{(4)} = \tfrac{1}{3}\,U_0^4. \tag{6}$$

Using these values of the coefficients $C_i^{(n)}$ in eq. (5), we obtain the asymptotic expansion of $\log D^+(k)$ for a local potential correct to order $1/k^4$.

$$\log D^+(k) = \frac{i}{2k}\,U_0 + \frac{i}{(2k)^3}\,(V'(0) + W_0) + \frac{V(0)}{(2k)^2} + \frac{\big((2V^2(0) - V''(0)\big)}{(2k)^4} + \ldots \tag{7}$$

This expression agrees with that derived by Verde (1955) and by Puff (1975). In the case of a partly non-local momentum dependent potential, the coefficients $C_i^{(n)}$ are to be obtained from the defining eq. (4) using the known momentum representation of the potential. It is noticed that in general ten $C_i^{(n)}$ are required for the specification of the expansion of $\log D^+(k)$ up to order $1/k^4$. The evaluation of the constants depends on the singularity structure of the interaction matrix elements. In the case of one term separable non-local potentials $C_i^{(n)}$ can be expressed only in terms of

$$\pi\,\langle k \mid V \mid k\rangle = C_1^{(1)} + C_3^{(1)}/(2k)^2 + 0\,(1/k^4)$$

$$P\int \frac{\langle p \mid V \mid p\rangle\, dp}{k^2 - p^2} = C_2^{(1)}/(2k)^2 + C_4^{(1)}/(2k)^4 + 0\,(1/k^6)$$

$$C_2^{(2)} = 0, \; C_3^{(2)} = C_2^{(1)}\,C_1^{(1)}, \quad C_4^{(2)} = C_2^{(1)2}, \; C_3^{(3)} = C_4^{(3)} = C_4^{(4)} = 0.$$

The expression for the $\log D^+(k)$ for this simple potential becomes,

$$\begin{aligned}\log D^+(k) = {} & \frac{i}{2k}\,C_1^{(1)} + \frac{i}{(2k)^3}\,[C_3^{(1)} + C_2^{(1)}\,C_1^{(1)} - \tfrac{1}{3}\,C_1^{(1)3}] \\ & + \frac{1}{(2k)^2}\,[-C_2^{(1)} + \tfrac{1}{2}\,C_1^{(1)2}] \\ & + \frac{1}{(2k)^4}\,[-C_4^{(1)} - \tfrac{1}{2}\,C_2^{(1)\,2} - \tfrac{1}{4}\,C_1^{(1)4} + C_1^{(1)}\,C_3^{(1)} + C_1^{(1)2}\,C_2^{(1)}] \\ & + \ldots\end{aligned} \tag{8}$$

As expected, the expansion coefficients of $\log D^+(k)$ depend only on those of the first Born term. Similar expression for $\log D^+(k)$ can easily be derived for the rank two separable potentials. For a general potential one can find out the expansion coefficients of $\log D^+(k)$ from the momentum space representation of a given potential from eq. (4). Let this expansion be

$$\begin{aligned}\log D^+(k) &= \frac{i}{2k}\,A + \frac{1}{(2k)^2}\,B + \frac{i}{(2k)^3}\,C + \frac{1}{(2k)^4}\,D + \ldots \\ &= F_4(k) + \ldots\end{aligned} \tag{9}$$

$$\begin{aligned}F_4(k) &= F_3(k) + \frac{1}{(2k)^4}\,D = F_2(k) + \frac{iC}{(2k)^3} + \frac{D}{(2k)^4} \\ &= F_1(k) + \frac{B}{(2k)^2} + \frac{iC}{(2k)^3} + \frac{D}{(2k)^4}.\end{aligned}$$

From the definition of $F_n(k)$, it is clear that

$$\log D^+(k) - F_n(k) = 0\,(1/k^{n+1}). \tag{10}$$

Let the contour $\mathcal{E}$ be a large semi-circle, centre $k = 0$ in the upper half k plane, indented at $k = 0$ and the contour integral I be defined as

$$I = \frac{1}{2\pi i} \oint_{\mathbf{E}} k^m \frac{d}{dk} [\log D^+(k) - F_n(k)]\, dk. \tag{11}$$

It is well known that $D^+(k)$ has zeros in the upper half k-plane only at the bound states corresponding to $\mathcal{E}_l = -k_l^2$. The subtracted term does not have any singularities except at $k = 0$. We choose m so that the integrant approaches $1/|k|^2$ on the semi-infinite circle and $1/|k|$ as $|k| \to 0$. From these arguments and the construction of $F_n(k)$, it is clear that $m = n$. With this choice, the integrant in I is an analytic function of k in the upper half k-plane, except for simple poles at ik_l, for a class of potentials for which $D^+(k)$ is analytic. The contour integral I can easily be evaluated and one obtains

$$I = (i)^n \sum_{l=1}^{N} k_l^n = \frac{-nf_n}{2} + \frac{1}{2\pi i} \int_{-\infty+i\mathbf{E}}^{\infty+i\mathbf{E}} dk\, k^n \frac{d}{dk} [\log D^+(k) - F_n(k)]. \tag{12}$$

The constant f_n is the coefficient of $1/k^n$ in $F_n(k)$. For a particular case of $n = 0$, the above equation reduces to Levinson's theorem. Integrating by parts the right hand side of eq. (12)

$$(i)^n \sum_l k_l^n = -\frac{nf_n}{2} - \frac{n}{2\pi i} \int_{-\infty}^{\infty} k^{n-1} [\log D^+(k) - F_n(k)]\, dk. \tag{13}$$

The constant terms arising from the integration by parts vanish because of the relation in eq. (10). Using further the properties,

$$D^+(-k) = D^+(k)^* \qquad \text{and} \qquad F_n(-k) = F_n^*(k), \tag{14}$$

the following sum rules can easily be derived:

$$\frac{1}{\pi} \int_0^{\infty} k^{n-1} [\delta(k) + \operatorname{Im} F_n(k)]\, dk = \frac{(-1)^{n/2}}{n} \sum_l k_l^n + \frac{Re_{J_n}}{2}, \tag{15}$$

for n even $\geqslant 2$.

$$\frac{1}{\pi} \int_0^{\infty} k^{n-1} [\log |D^+(k)| - Re\, F_n(k)]\, dk = \frac{(-1)^{(n-1/2)}}{n} \sum_l k_l^n + \frac{I_m J_n}{2}, \tag{16}$$

for n odd $\geqslant 1$.

With the substitution of $F_n(k)$ and J_n from eq. (9) into eqs (15) and (16), we obtain the required sum rules for $\delta(k)$ and $|D^+(k)|$

3. Conclusion

The phase shift sum rules first derived by Newton and Percival for local potential are extended for the general partly non-local momentum dependent potential. In extending the sum rules to such a general potential the Fredholm determinant of the outgoing solution must be used instead of the Jost function used by Puff in the case of a local potential. The constants appearing in the sum rule expressions are explicitly defined in terms of the momentum representation of the potential. In the particular case of a local potential, these constants are the same as those derived by Puff. For a special case of one term separable non-local interaction, all these constants can be expressed in terms of the expansion coefficients of the first Born term.

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