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Fredholm approximants for potential scattering

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Abstract. An approximate method based on Fredholm approximants is used to solve Lippmann-Schwinger equation for s-and p-wave Yukawa and s-wave exponential potential scattering problems. The method is then applied to a problem of π–π scattering involving the use of a well known equivalent potential. The equation for partial wave amplitudes is solved to generate the first three π–π resonances, ρ, f_0, and g, in a self-consistent manner.

Keywords. Lippmann-Schwinger equation; phase-shifts; Yukawa potential; π–π scattering.

1. Introduction

In an earlier paper (Sharma and Bondyopadhyay 1974) we had introduced a method for solving Lippmann-Schwinger equation approximately by the use of Fredholm approximants. However, the unitary diagonal Fredholm approximants constructed there involved successive higher dimensional multiple integrals in its successive higher orders, so that only the first two diagonal approximants seem to be easily amenable to numerical computation. This paper is devoted to a study of the numerical evaluation of these approximants. We first test their applicability by calculating scattering phase shifts for Yukawa and exponential potential and compare with exact numerical calculations. Our final application is in a problem of π-π scattering involving the solution of Lippmann-Schwinger equation with a complicated energy dependent potential derived on the basis of analyticity and Regge behaviour of the π-π amplitude (Balàzs 1965).

Details of the Fredholm approximant method have been discussed earlier (Sharma and Bondyopadhyay 1974). The Lippmann-Schwinger Noyes (Noyes 1965) system of equations are

$$t_l(k) = \lambda\, V_l(\mathrm{k}) \left[1 + 4\pi\lambda \int_0^a \frac{p'^2\, dp'\, V_l(k, p') f(p')}{p'^2 - k^2} \right]^{-1} \tag{1}$$

and

$$f(p, k) = \frac{V_l(p, k)}{V_l(k)} - 4\pi\lambda \int_0^a \frac{p'^2\, dp'}{p'^2 - k^2} \times$$

$$\times \left[V_l(p, p') - \frac{V_l(p', k)\, V_l(k, p')}{V_l(k)} \right] f(p') \tag{2}$$

For convenience of numerical computation, we also make a transformation

$$p = \gamma x/1 - x$$

where γ is any suitable constant. Expressions for the first two unitary approximants constructed from eqs (1) and (2) and made use of in this paper are as follows:

$$t_l(k)_{[1,1]} \equiv \frac{\lambda V_l(k)}{1 - \lambda(\sigma_1 - 4\pi VGF)} \tag{3}$$

$$t_l(k)_{[2,2]} = \frac{\lambda V_l(k)[1 - \lambda\sigma_1]}{1 - \lambda[\sigma_1 - 4\pi VGF] + \lambda^2[\frac{1}{2}(\sigma_1^2 - \sigma_2) - 4\pi(VGF\sigma_1 - VGKGF)]}$$

where (4)

$$VGF \equiv \int dp'\, V(k,p')\, G(kp')\, F(p',k)$$

$$VGKGF \equiv \int dp' \int dp\, V(k,p')\cdot G(k,p') \times$$

$$\times K(k,p',p)\cdot G(k,p)\cdot F(p,k),$$

etc.

These are respectively the expressions (14) and (15) of SB and all the symbols appearing in these expressions have been fully explained there.

2. Yukawa potential

We first apply these equations to phase shift calculations involving Yukawa potential. Partial wave projection of Yukawa potential in momemtum space is given by the expression

$$V_l(p,p') = \frac{-g}{4\pi^2 pp'} \cdot Q_l\left(\frac{p^2 + p'^2 + \mu^2}{2pp'}\right) \tag{5}$$

where μ and g are respectively the range and coupling parameters. Equations (1) and (2) lead to the following phase shift representation for the on-shell amplitude.

$$t_e(k) = -\frac{1}{2\pi^2} \cdot \frac{e^{i\delta}\sin\delta}{q}$$

To take into account threshold effects we would prefer to work with the amplitude $t_l(k)/q^{2l}$. Figures (1) and (2), show the results of phase shift calculations using equations (1) and (2) and the potential (5). Figures 1 and 2 also show side by side the results of exact numerical solution of Lippmann-Schwinger equation. It is seen that both *s*- and *p*-wave phase shifts are satisfactory and the accuracy improves with the order of the approximant. The problem here refers to the scattering of two particles of mass $m = \frac{1}{2}$ each and μ has been chosen to be equal to $(\frac{1}{7})m$, *i.e.*, almost in the proportion of pion exchange in nucleon-nucleon scattering.

The phase-shift suddenly changes by 2π when a bound state of the potential goes in and this is seen to happen in the phase shift curves for both s- and *p*-waves. A resonance occurs when the phase shift rises through $\pi/2$ as is seen for *p*-wave

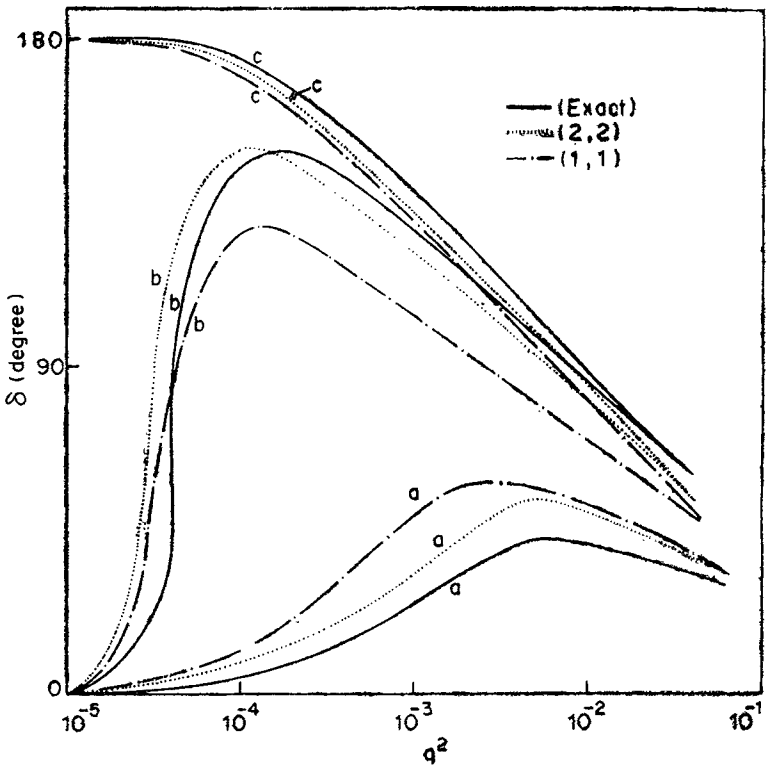

Figure 1. The p-wave phase shift *versus* the energy for three values of coupling constant (*a*) $g = 3{\cdot}5$, (*b*) $g = 7{\cdot}95$, (*c*) $g = 8{\cdot}15$ with $\mu = 1/14$.

in figure 1. Stern (1972) has described a method in which Lippmann-Schwinger equation is first solved for negative energies where there are no singularities and then the solution continued analytically by the Padé approximants to the positive energy region. Similar analytic continuation methods using continued fractions have also been used (Haymaker and Schlessinger 1970). There are two stages of numerical computation involved in these; first, in solving the integral equation in unphysical energy regions and then fitting these solutions by Padè approximants. The total amount of numerical work is consequently much greater. Our method which is simpler also yields an approximate analytic expression for the amplitude in the scattering region. A different way of computing Padè approximants for Lippmann-Schwinger equation has also been described by Caser *et al.* (1969), which also involves more detailed numerical computation. Their phase-shift analysis however is in good agreement with ours.

3. Exponential potential

We have considered only s-wave scattering by the exponential potential. The s-wave projection of the Fourier transform for the potential

$$V(r) = -\lambda\, e^{-\mu r}$$

is

$$V_0(p, p') = \frac{-\lambda\mu}{\pi^2} \cdot \frac{1}{[(g+g')^2+\mu^2]\,[(g-g')^2+\mu^2]} \tag{6}$$

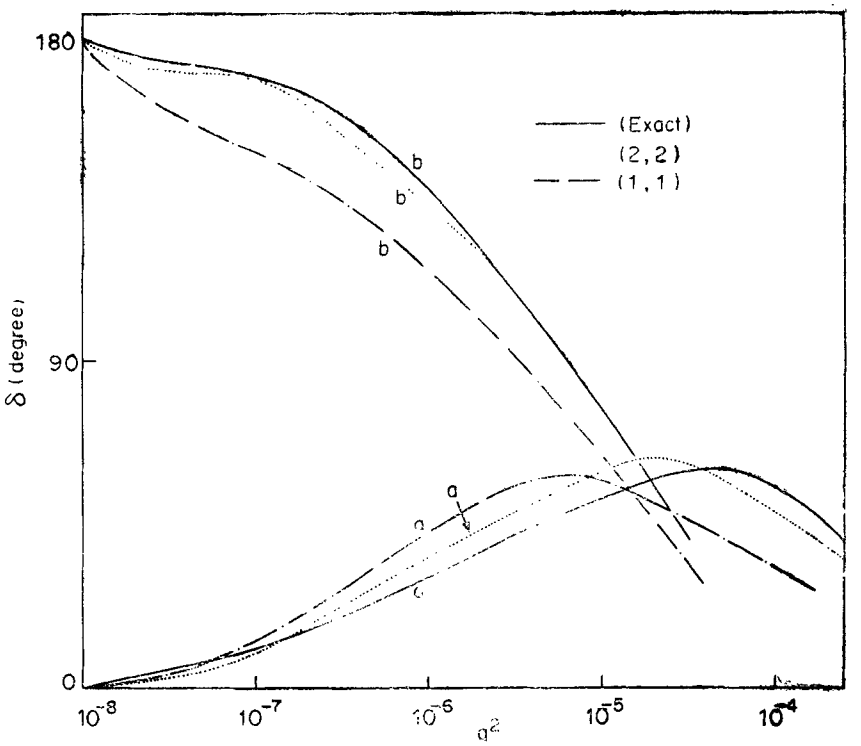

Figure 2. The *s*-wave phase shift *versus* the energy-energy for two values of coupling constant (*a*) $g = 1{\cdot}54$, (*b*) $g = 2{\cdot}5$ with $\mu = 1/14$.

The parameters λ and μ have been chosen to be identical with a set of parameters used by Moiseiwitsch (1970) which refer to a phase-shift analysis of neutron-proton-scattering (Swan 1960) in singlet (table 1) and triplet (table 2) states respectively with the nuclear interaction represented by an exponential potential. A comparison of the phase shifts are shown for a few values of the parameter k in the tables. A fairly good agreement can be seen to prevail in both the orders.

4. Application to π-π scattering

Finally we apply this method to a system with a truly complicated potential. The method of construction of t and u channels ρ-exchange equivalent potential for the π-π scattering problems has been discussed by Balàzs (1965). Later f_0 meson exchange effects were also incorporated. Details of the construction of the potential have been reported elsewhere (Sharma and Bondyopadhyay 1972, Balàzs and Patil 1968). The important advantage here over the relativistic N/D technique is that a self-consistency can indeed be achieved for ρ meson both as regards its mass as well as width. The problem of width is indeed notorious in Chew-Mandelstam (1960) N/D formalism in that there is usually a difference of a factor of 4 for the input and output widths and this persists even in extensive multi-channel calculations. The advantage of the equivalent potential over the N/D method has sharply focussed attention on it. Yet another advantage lies in the fact that while constructing equivalent potential from the crossed channels exchanges certain terms can be replaced by delta functions and this can curb the

Table 1. The s-wave singlet phase shifts for exponential potential as a function of the wave number k, for $\lambda = -2{\cdot}64$ and $\mu = 1{\cdot}408$.

Wave number k	[1, 1]	[2, 2]	Exact
0·25	1·234	1·210	1·103
0·50	0·9600	0·942	0·934
0·75	0·773	0·777	0·780
1·00	0·654	0·658	0·664
1·25	0·564	0·583	..
1·50	0·497	0·502	0·507
1·75	0·449	0·464	..
2·00	0·397	0·402	0·408

Table 2. The s-wave triplet phase shifts for exponential potential as a function of the wave number k, for $\lambda = -5{\cdot}2$ and $\mu = 1{\cdot}587$.

Wave number k	[1, 1]	[2, 2]	Exact
0·25	2·380	2·321	2·111
0·50	1·660	1·632	1·585
0·75	1·293	1·300	1·285
1·00	1·083	1·090	1·087
1·25	0·908	0·910	..
1·50	0·833	0·840	0·836
1·75	0·748	0·754	..
2·00	0·679	0·687	0·680

s-dependence of the potential, leading to convergent integrals for the solution of the integral equation. The problem of simultaneous self-consistent generations of ρ and f_0 has been investigated by Balàzs and Vaidya (1965) and Everett (1968). There exists another π-π resonance of mass 1680 MeV, $I^G(J^P) = 1^+(3^-)$ and width 180 MeV with the so called g resonance also contributing to the exchange potential for π-π scattering. However, recent data indicate that its π-π width is rather small, of the order of 26% of the total. In spite of the highly inelastic nature of this resonance, its inclusion may still contribute significantly to the potential because the total width is rather large. We therefore attempt to solve the problem of simultaneous self-consistent generation of ρ and f_0 with the inclusion of g-meson exchange force. Masses and partial widths of these resonances appear in the potential the expression for which is as follows:

$$V_l(p, p') = G\,[3C_1 q_{t1}^2\, p_1(\cos\theta_{t1})\, Q_l(a_\rho) + + 5C_2 G_1\, p_{t2}^4\, p_2(\cos\theta_{t2})\, Q_l(a_{f_0}) + + 7\,C_3 G_2 q_{t3}^6\, p_3(\cos\theta_{t3})\, Q_l(a_g)]\, \frac{1}{2\pi^2 (pp')^{l+1}\sqrt{s}} \tag{7}$$

where C_1, C_2 and C_3 are the isospin recoupling coefficients obtained from the following crossing matrix

$$\begin{vmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & -\frac{5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{vmatrix}$$

An extra factor of 2 has been incorporated in these elements to signify identical contribution from t and u channels. In (7)

$$a_i = \frac{p^2 + p'^2 + m_i^2}{2pp'}, \qquad \cos\theta_{ti} = 1 + \frac{2s}{m_i^2 - 4}$$

$$S = 4(q^2 + 1) \quad , \quad q^2{}_{ti} = \frac{m_i^2}{4} - 1$$

where m_i, $i = 1, 2, 3$ stands for the masses of ρ, f_0 and g respectively. The constant G is related to the decay width of ρ by the formula

$$\Gamma = \frac{q_R^{2l+1} G}{E_R^2}$$

R denotes the resonance position and $G_1 = \frac{G_{f_0}}{G}$ and $G_2 = \frac{G_G}{G}$.

If the real part of the denominators of the approximants are denoted by $D(S)$ then the output width at the resonance position is given by

$$\Gamma = \frac{-2\pi^2 q_R^{2l+1} V(q_R, q_R)}{E_R \left.\frac{dD(S)}{dS}\right|_{S=S_R}} \tag{8}$$

The experimental (review of particle properties, Particle data Group 1974) masses and widths of the resonances (in units of $m_\pi = 1$) are as follows:

$m_\rho^2 = 30{\cdot}25$	$m_{f_0}^2 = 82{\cdot}3$	$m_g^2 = 144$
$\Gamma_\rho = 1{\cdot}07$	$\Gamma_{f_0} = 1{\cdot}23$	$\Gamma_{g \to \pi\pi} = 0{\cdot}34$

All the parameters are subject to variation to achieve input-output self-consistency. However, we have tried to keep them as close as possible to the experimental values while using them as input parameters. An exact solution of the Lippman-Schwinger equation with this potential is not quite feasible because of the existence of large number of parameters that need to be continuously varied. Considering the successful application of the method of Fredholm approximants for Yukawa and exponential potentials, we feel that this approximation method would give sufficiently reliable results for the π-π problem as well.

Actual experimental values of G, G_1 and G_2 are $1{\cdot}928$, $0{\cdot}03091$ and $0{\cdot}9865 \times 10^{-4}$ respectively. We first studied the self-consistency situation by varying the ρ width alone while all other parameters like masses of the resonances, G_1 and G_2 were assigned their experimental values. With all experimental inputs, the output widths of both ρ and f_0 are found to be nearly 20% larger than input values. Their output masses are slightly less; for ρ the output mass is less by 9%, for f_0 it

is less by 20%. It is found that raising the input partial widths of 9 to about 90 MeV will restore the output widths of ρ and f_0 to their experimental values, while their masses are only slightly affected. By increasing the width and decreasing the mass of ρ by about 10% and keeping all other parameters within their experimental limits, it is possible to generate ρ mass and width self-consistently but then the f_0 mass turns out to be 30% less. It is found that the inclusion of f_0 and g exchange forces are essential for the resonances to be reproduced more or less conforming to their experimental positions. ρ exchange alone is totally incapable of doing this. With ρ and f_0 parameters having their experimental values, we found that a spin 3 resonance can be self-consistently generated at mass 1635 MeV and width 120 MeV, *i.e.*, at these values both input and output quantities match. However, because of the presence of high degree of inelasticity near g-meson resonance it is clear that it would require a different approach than the simple one-channel model considered by us. There still exists some controversy regarding this resonance. Some have suggested (Barnham 1970, Holmes 1972) that there may exist several close lying resonances near g-meson mass. Self-consistent generation of a resonance in this region in terms of ρ and f_0 parameters seem to suggest that there may indeed be a resonance with a predominantly high value for the π-π width.

5. Conclusion

We have discussed the application of Fredholm approximant method to Lippmann-Schwinger-Noyes set of equations for several potential scattering problems. Accuracy obtained by Yukawa and exponential potentials is comparable to that obtained by Moiseiwitsch (1970) in his Fredholm and Padé approximants solution of the Schrödinger equations. Also since the lowest lying singularities of Bethe-Salpeter equation can be removed by Noyes method, it is clear that this method may be easily extended to the solution of the equation for low energy scattering problems. An extension to N/D equations is also visualized because of the very similar form of simultaneous equation sets encountered.

Attention to the possibility of a simultaneous bootstrap of ρ and f_0 was first drawn by Balàz's and Vaidya (1965). We have found that further inclusion of g-meson exchange force greatly facilitates this. The fact that a near self-consistent generation of the two lowest lying π-π resonances can be achieved is indeed a great success to the method of equivalent potential. It is also remarkable that a single channel calculation suffices, but this may not be unnatural considering that the inelasticities in π-π channels are quite small even up to f_0 meson mass and this has been implied in the earlier calculation of Sharma and Bondyopadhyay (1972). It is possible, however, that a multichannel calculation or a calculation based on inelastic unitary equations would improve upon the small discrepancies found to occur in the present approach, especially in the case of highly inelastic g-resonance.

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