

Ground state of liquid helium 3: Momentum space calculations of the two body energy

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Abstract. The reaction matrix elements needed to study the ground state properties of normal liquid Helium 3 are obtained by a matrix inversion method in momentum space within the Brueckner-Goldstone formalism. Fully self-consistent calculations of the hole-line potential and the two-body interaction energies are performed with self-consistent occupation probabilities for Frost-Musulin potential.

1. Introduction

Brueckner-Goldstone formalism has been used extensively to understand ground state properties of normal liquid Helium-3 (Brueckner and Gammel 1958, Burkhardt 1968, Ostgaard 1969, Ghassib *et al.* 1974, 1974 *a*). This theory was introduced originally to investigate properties of nuclear matter. Since then due to the work of Bethe, Brandow, Day, Rajaraman and others (Bethe 1971), Brueckner-Goldstone theory has reached a high level of sophistication and seems to be reasonably successful in explaining the properties of nuclear matter and of finite nuclei. The convergence properties of Brueckner Goldstone series are also well studied. In the course of the calculations for nuclear matter, a number of approximations and assumptions have emerged, which are proved to be satisfactory for that system.

These assumptions and approximations have been carried over to the study of liquid helium-3 with little critical examination. The results do not yet provide a satisfactory basis for understanding of the properties of liquid He³. Since liquid He³ is a relatively dense system, it is not *a priori* clear that similar approximations are valid in this case. Using co-ordinate space methods some of these points have been critically examined by Ghassib *et al.* (Ghassib *et al.* 1974).

Our aims in this work are the following: (1) to use momentum space methods developed by Haftel and Tabakin (Haftel and Tabakin 1970) to calculate the Brueckner reaction matrix as opposed to the co-ordinate space methods employed so far. The momentum space methods are ideally suited to study off-shell effects. The Brueckner reaction matrix elements then form the input for the two body calculations. (2) to estimate the importance of self-consistent occupation probabilities first introduced by Brandow (1967). The calculations using the self-consistent occupation probabilities form the main subject of the present paper. We use the Frost-Musulin potential (Frost-Musulin 1954). For the momentum

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space calculations it is very convenient to have a potential which has an analytic Fourier transform. In Section 2 we show how the Frost-Musulin potential can be well represented by a sum of three Yukawa potentials. In this paper we will be concerned only with the normal phase of liquid helium 3 at $T = 0^\circ \text{K}$. The observed density ρ of the system is 0.0166 particles per \AA^3 which corresponds to a Fermi momentum k_f of 0.79 \AA^{-1} . At this density the experimental binding energy per particle is -2.47°K (Strongin *et al.* 1962). The numerical calculations of the Brueckner reaction matrix and the two body energies with and without self-consistent occupation probabilities are outlined in Section 3. The results are discussed in Section 4.

2. Two body potential

The input for any many body calculations is the two body interaction assuming that non-additive many body interactions are unimportant. The He-He interatomic interaction is not very well known except for a few salient features. It is commonly believed that the interaction can be represented by a simple local static potential. Even with these restrictions the potential is not well determined. The only sound theoretical knowledge concerns the long range part of the potential, which is the usual Vanderwall's term proportional to $1/r^6$. There are very few *ab initio* calculations (Bertoncini and Wahl 1970, Schaefer *et al.* 1970) of the potential for small and medium interparticle distance. These seem to give a potential with a minimum around 2.8 to 3.0 \AA and a depth of 10 – 13°K . For short distances the potential is repulsive and is of the order of few electron Volts.

In the absence of a well established, reliable potential, a number of empirical potentials have been proposed (Bruch and McGee 1967). One of the potentials that have been widely used in recent work on liquid He^3 is the Frost-Musulin potential (Frost-Musulin 1954) which is given by

$$\begin{aligned} V_{FM}(r) &= -V_1 [1 + 8.01 (1 - 2.98/r)] \\ &\quad \times \exp [8.01 (1 - r/2.98)] \quad r \leq 3.5 \text{ \AA} \\ &= -V_2 \left[\frac{1.41}{r^6} + \frac{3.82}{r^8} \right] \quad r \geq 3.5 \text{ \AA} \\ V_1 &= 12.54^\circ \text{K}; \quad V_2 = 7250^\circ \text{K} \end{aligned} \quad (2.1)$$

For Brueckner theory calculations it is very convenient to have a potential that has a Fourier transform. For Lennard-Jones and Yntema-Schneider potentials Fourier-transforms do not even exist. Even for the others Fourier transforms do not have simple analytic properties. To obtain a simple form in momentum space we adopted a procedure of simulating a given potential by a sum of three Yukawa potentials:

$$V_R(r) = \sum_{i=1}^3 \tilde{V}_i \{ \exp [-a_i r] \} / r \quad (2.2)$$

The six parameters $\{ \tilde{V}_i, a_i; i = 1, 3 \}$ in (2.2) are determined by reproducing (1)

σ , the distance at which the two-body potential changes from repulsion to attraction (2) $V'(\sigma)$ the slope of the potential at $r = \sigma$ (3) r_m the position of the minimum in $V(r)$ (4) $V(r_m)$ the value of the potential at the minimum r_m ; (5) the value of the potential at some typical distance $r_i \sim 3.5 A$; (6) the sixth parameter is determined by minimizing the Born correction

$$\int_{\sigma}^{\infty} [V(r) - V_R(r)] r^2 dr$$

Using this procedure a fit of (2.2) to the Frost-Musulin potential of (2.1) gives the following values for the six parameters:

$$\begin{aligned} \tilde{V}_1 &= 2.4206 \times 10^8 \text{ }^\circ\text{K} & a_1 &= 2.945 A^{-1} \\ \tilde{V}_2 &= -1.200 \times 10^8 \text{ }^\circ\text{K} & a_2 &= 2.683 A^{-1} \\ \tilde{V}_3 &= -35.4709 \text{ }^\circ\text{K} & a_3 &= 0.544 A^{-1} \end{aligned} \quad (2.3)$$

This potential reproduces the Frost-Musulin potential of (2.1) very well in the range $\sigma < r < 12 A$. For smaller distances $r < \sigma$ it underestimates $V_{FM}(r)$ but this should not affect the calculations of the ground state energy and other thermodynamic properties of the system as we shall see in Section 3. In the very long range limit $r \gg 15 A$ it underestimates the attraction. We shall see in Section 3 that any differences in the long range limit can be treated by perturbation theory in the calculation of the reaction matrix elements for the Frost-Musulin potential.

For the L th partial wave the momentum representation of our two-body potential $V_R(r)$ of (2.2) is given by

$$\begin{aligned} V_L(k | k') &= \int_0^{\infty} dr r^2 j_L(kr) j_L(k'r) V_R(r) \\ &= \left(\frac{1}{2kk'} \right) \sum_{i=1}^3 \tilde{V}_i Q_L(Z_i) \end{aligned} \quad (2.4)$$

where $\{Q_L(Z_i)\}$ are the Legendre function of the second kind with

$$Z_i = (k^2 + k'^2 + a_i^2) / (2kk'); \quad i = 1, 2, 3 \quad (2.5)$$

and j_L are the spherical Bessel functions.

Finally we note that we have similarly been able to represent the Lennard-Jones potential by a sum of three Yukawa potentials. For the calculations to be described in this paper unless explicitly mentioned otherwise lengths are given in units of Angstroms, momenta in A^{-1} and energies in units of A^{-2} , the conversion factor being $\hbar^2/m = 16.36^\circ K A^2$.

3. Two body energy calculations for liquid helium 3

In this section we describe the two body energy calculations for liquid Helium 3 with self-consistent occupation probabilities using the Brueckner Goldstone formulation. We shall not give the details of this formulation as it has been given at many places in the literature (*see* for example Day 1967).

We first outline the method of computation of the Brueckner G-matrix for the potential of Eq. (2.2). Since the potential matrix elements (2.4) are all finite, the most convenient method to obtain the reaction matrix is to directly solve the Brueckner-Goldstone equation in momentum space by matrix inversion following Haftel and Tabakin (1970) (hereafter referred to as HT). We will only write down the main equations and outline the numerical procedure. The Brueckner-Goldstone equation for the L th partial wave is given by [for central potentials of the type (2.2)]

$$\langle k | G_L(P) | k_0 \rangle = V_L(k | k_0) - \frac{2}{\pi} \times \int_0^\infty dk' \frac{V_L(k | k') \bar{Q}(k' P) (k')^2 \langle k' | G_L(P) | k_0 \rangle}{[E_>(k' P) - E_<(k_0 P)]} \quad (3.1)$$

where $V_L(k | k_0)$ is given by (2.3); k , k_0 and k' are the relative momenta and P is the centre of mass momentum;

$$k = (k_1 - k_2)/2; \quad P = (k_1 + k_2)/2 \quad (3.2)$$

and \bar{Q} is the angle averaged Pauli operator as defined in HT, *i.e.*,

$$\bar{Q}(k' P) = \begin{cases} 0 & k' < (k_f^2 - P^2)^{\frac{1}{2}} \\ (k'^2 + k_f^2 - P^2)/(2k_f P) & (k_f^2 - P^2)^{\frac{1}{2}} < k' < k_f + P \\ 1 & k' > k_f \end{cases} \quad (3.3)$$

The energies $E_>$ and $E_<$ correspond to the energies of two He^3 atoms above and below the Fermi sea. In the effective mass approximation these may be written as

$$\begin{aligned} E_>(k' P) &= (k')^2 + P^2 \\ E_<(k_0 P) &= (k_0^2 + P^2)/m^* + 2U_0 \end{aligned} \quad (3.4)$$

$$U_0 = -\Delta k_f^2$$

where m^* is the effective mass and U_0 is the depth of the one-body potential for the hole states which are to be determined self-consistently (*see* later in this section). It is convenient to define a quantity called the starting energy by

$$\Gamma = P^2 \left(1 - \frac{1}{m^*}\right) - \frac{k_0^2}{m^*} + 2\Delta k_f^2 \quad (3.5)$$

We now indicate the procedure to obtain the reaction matrix elements

$\langle k | G_L(P) | k_0 \rangle$ for a given starting energy

$$\langle k | G_L(P) | k_0 \rangle = V_L(k | k_0) - \frac{2}{\pi} \times \int_0^\infty dk' \frac{V_L(k | k') \bar{Q}(k' P) (k')^2 \langle k' | G_L(P) | k_0 \rangle}{(k')^2 + \Gamma} \quad (3.6)$$

In order to solve (3.6) directly the integral equation is reduced to a matrix equation by mapping the infinite interval of the intermediate k' values onto a finite range by the transformation

$$k' = (k_f^2 - P^2)^{\frac{1}{2}} \frac{(1-x)}{2} + (k_f + P)(1+x)/2$$

$$(k_f^2 - P^2)^{\frac{1}{2}} \leq k \leq k_f + P; -1 \leq x \leq 1 \quad (3.7)$$

$$k' = k_f + P + R(1+x)/(1-x)$$

$$k_f + P < k' < \infty; -1 \leq x \leq 1 \quad (3.8)$$

where R is a suitable scale parameter. The number of Gaussian points for integration in the ranges $(k_f^2 - P^2)^{\frac{1}{2}} < k' < k_f + P$ and $k_f + P < k' < \infty$ and R are to be chosen such that the resulting G -matrix elements are accurate to say within 1%. We find that it is sufficient to choose 5-10 grid points in the range $(k_f^2 - P^2)^{\frac{1}{2}} < k' < k_f + P$ and 40 points in the range $(k_f + P) < k' < \infty$ with $R = 2.0 A^{-1}$. The quantity $\langle k | G_L(P) | k_0 \rangle$ can then be obtained easily by matrix inversion.

Using the above procedure we have evaluated the reaction matrix elements for the potential $V_R(r)$ of Eq. (2.3). The reaction matrix elements for the Frost-Musulin potential $V_{FM}(r)$ can then be obtained from the reaction matrix elements obtained from $V_R(r)$ by using the relation

$$G^{FM} = G + (V_{FM} - V_R) - G \frac{Q}{e} (V_{FM} - V_R) - (V_{FM} - V_R) \frac{Q}{e} G^{FM}$$

$$+ G \frac{Q}{e} (V_{FM} - V_R) \frac{Q}{e} G^{FM} \quad (3.9)$$

$$\simeq \tilde{G}$$

where

$$\tilde{G} = G + (V_{FM} - V_R) \quad (3.10)$$

In table 1 we have given typical reaction matrix elements $\langle k | G_L(P) | k \rangle$ for a typical centre of mass momentum, and various starting energies. In table 2 we compare the reaction matrix elements obtained from (3.10) with the reaction matrix elements for the Frost-Musulin $V_{FM}(r)$ obtained directly for S-waves using co-ordinate space method (Ghassib *et al.*). (We have also used the momentum space method outlined in this section to obtain S-wave reaction matrix elements directly for the Frost-Musulin potential for which Fourier transform exists. These results agree with the co-ordinate space calculations). As evident from table 2 the agreement is satisfactory. We have found similar agreement in the case of off-shell t -matrix also*. Since the potential $V_R(r)$ of (2.2) has an analytic Fourier transform it makes the numerical calculations considerably simpler. This simplification is especially useful for performing higher order calculations as well as for calculations using phase-equivalent potentials.

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Table 1. Partial wave reaction matrix elements $\langle k_0 | G_L(P) | k_0 \rangle$ for *s*- and *p*-waves in Angstroms Fermi momentum $k_f = 0.76 \text{ \AA}^{-1}$ and $P/k_f = 0.5$.

k/k_f	$\Gamma \rightarrow$	0.125		0.375		0.625	
	$L =$	0	1	0	1	0	1
0.1		3.16	-11.94	26.59	-11.67	48.77	-11.41
0.5		54.69	-94.24	74.91	-88.15	94.05	-82.31
0.9		70.77	+1.44	84.52	+17.10	98.65	+31.02

Table 2. Diagonal partial wave reaction matrix elements (in Angstroms) G^{FM} of (3.9) and G of (3.10) for $k_f = 0.75$ and centre of mass momentum $P = 0$ with the starting energies being given by (3.5) with $m^* = 2.5$ and $\Delta = 0.4$.

L	$k_0/k_f = 0.5$		$k_0/k_f = 1.0$	
	G^{FM}	G	G^{FM}	G
0	47.10	47.72	55.0	55.12
1	-88.80	-88.45	+40.60	41.14
2	-11.5	-11.54	-28.7	-28.62
3	-7.1	-8.03	-52.8	-52.55

Once the reaction matrix element have been determined the two body binding energy per particle $(BE)_2$ can be calculated from:

$$(BE)_2 = 0.3k_f^2 + W_2 \tag{3.11}$$

$$W_2 = \frac{3}{k_f^3} \int_0^{k_f} k^2 U(k) \left[1 - \frac{1}{2} P_k \right] dk \tag{3.12}$$

$$U(k) = \frac{8}{\pi k} \int_0^{(k_f - k)/2} k_0 dk_0 \int_{|k_0 - k|}^{(k_0 + k)} PdP \langle k_0 | G_\Gamma(P) | k_0 \rangle Pk_2$$

$$+ \frac{4}{\pi k} \int_{|k_f - k|/2}^{(k_f + k)/2} k_0 dk_0 \int_{|k_0 - k|}^{[\frac{1}{2}(k_f^2 + k^2) - k_0^2]^{1/2}} PdP \langle k_0 | G_\Gamma(P) | k_2 \rangle Pk_2$$

$$k < k_f$$

$$U(k) = 0 \quad \text{for } k \geq h_f \tag{3.13}$$

$$Pk_1 = \left\{ 1 - \sum_{k_2} \nu_{k_2} Pk_2 \left[\frac{\partial}{\partial \Gamma} \langle k_0 | G_\Gamma(P) | k_0 \rangle \right]_{\Gamma = \epsilon_{h_1} + \epsilon_{h_2}} \right\}^{-1} \tag{3.14}$$

$$k_0 = (k_1 - k_2)/2 \tag{3.15}$$

$$\epsilon_{\mathbf{k}} = \frac{k^2}{2} + U(\mathbf{k}) \tag{3.16}$$

$$\langle k_0 | G_{\Gamma}(P) | k_0 \rangle = \sum_L (2L + 1) [2 - (-1)^L] \langle k_0 | G_L(P) | k_0 \rangle. \tag{3.17}$$

Here $\epsilon_{\mathbf{k}}$ is the hole line energy and $U(\mathbf{k})$ is the hole-line potential. The quantities $\{P_h\}$ are called the self-consistent occupation probabilities and were introduced by Brandow [Brandow 1967] in the Brueckner-Goldstone formulation. These $\{P_h\}$ have to be determined from Eq. (3.14). The effective mass approximation for the hole-energies implies that $\epsilon_{\mathbf{k}}$ and $U(\mathbf{k})$ have the form

$$U(\mathbf{k}) = \frac{k^2}{2} \left(1 - \frac{1}{m^*}\right) + U_0 \tag{3.18}$$

$$U_0 = -\Delta k_f^2$$

$$\epsilon_{\mathbf{k}} = \frac{k^2}{2m^*} + U_0 \tag{3.19}$$

Because of the complicated angular dependence of $p_{\mathbf{k}}$ on \mathbf{k} it is very difficult to do in general the integrations in (3.13) and (3.14). To proceed further we make the simplifying assumption:

$$P_{\mathbf{k}} = \exp[-a - bk^2] \tag{3.20}$$

where a and b are two parameters to be determined by solving the integral equation of (3.14). Using (3.20) we can rewrite the integral equations of (3.13) and (3.14) as

$$\begin{aligned} U(\mathbf{k}) = & \frac{8}{\pi k} \exp[a + bk^2] \int_0^{(k_f - k)/a} k_0 dk_0 \int_{|k_0 - k|}^{(k_0 + h)} PdP \langle k_0 | G_{\Gamma}(P) | k_0 \rangle \\ & \times \exp[-2a - 2b(P^2 + k_0^2)] \\ & + \frac{4}{\pi k} \exp[a + bk^2] \int_{|k_f - k|/2}^{(k_f + h)/2} k_0 dk_0 \int_{|k_0 - k|}^{[\frac{1}{2}(k_f^2 + k^2) - k_0^2]^{1/2}} PdP \\ & \cdot \langle k_0 | G_{\Gamma}(P) | k_0 \rangle \exp[-2a - 2b(P^2 + k_0^2)] \end{aligned} \tag{3.21}$$

$$\begin{aligned} & \left\{ \exp(-a - bk^2) - \frac{8}{\pi k} \int_0^{(k_f - k)/2} k_0 dk_0 \int_{|k_0 - k|}^{(k_0 + k)} PdP \right. \\ & \exp[-2a - 2b(P^2 + k_0^2)] \cdot \frac{\partial}{\partial \Gamma} \langle k_0 | G_{\Gamma}(P) | k_0 \rangle \\ & \left. + \frac{4}{\pi k} \int_{|k_f - k|/2}^{(k_f + k)/a} k_0 dk_0 \int_{|k_0 - k|}^{[\frac{1}{2}(k_f^2 + k^2) - k_0^2]^{1/2}} PdP \right. \\ & \left. \exp[-2a - 2b(P^2 + k_0^2)] \cdot \frac{\partial}{\partial \Gamma} \langle k_0 | G_{\Gamma}(P) | k_0 \rangle \right\} = 1 \tag{3.22} \end{aligned}$$

It is clear from Eqs (3.11)–(3.22) that the two-body binding energy calculations involve the self-consistent evaluation of the hole-line potential and the occupation probabilities. We now discuss briefly the self-consistency procedure. Starting with some initial values of m^* and Δ , and some input values of a and b we solve Eq. (3.22) for output values of a and b . The procedure is repeated until self-consistency of a and b is reached for a given (m^*, Δ) . Using the self-consistent values of a and b , the output hole-line potential $U(k)$ is obtained from (3.21). The output self-energy $U(k)$ is then used to obtain new values of (m^*, Δ) . The double self-consistency procedure is then repeated until full self-consistency is achieved for (m^*, Δ) and (a, b) . The reaction-matrix elements $\langle k_0 | G_{\Gamma}(P) | k_0 \rangle$ and their derivatives required in (3.21) and (3.22) have been obtained from interpolation from the reaction-matrix elements $\langle k_0 | G_{\Gamma}(P) | k_0 \rangle$ over a suitably chosen range of Γ obtained by using the procedure discussed at the beginning of this section. Once P_k and $U(k)$ have been obtained self-consistently the binding energy is obtained from (3.11) and (3.12).

In table 3 we summarize the results of the procedure discussed in the preceding paragraph, for the special case in which the occupation probabilities $\{P_k\}$ are all set to unity *i.e.* $a = b = 0$. This special case has been investigated earlier in the literature and we shall return to a comparison of our results with earlier work in the next section. We note from table 3 that the binding energy considered as a function of density exhibits a minimum of -0.1°K at a Fermi momentum of $k_f \sim 0.52 \text{ \AA}^{-1}$. These values are to be compared with the experimental values of -2.5°K at a Fermi momentum of 0.79 \AA^{-1} . The effective mass m^* is found to be ~ 1.3 whereas the experimental value is ~ 3.0 . We have repeated our self-consistent calculations including the self-consistent occupation probabilities and the results are presented in table 4. The minimum in the binding energy versus density does shift to higher densities occurring at $k_f \sim 0.60 \text{ \AA}^{-1}$ with a binding energy of -0.15°K . The effective mass also increases slightly. But the results are clearly still far from the corresponding experimental values.

To facilitate comparison with earlier work in literature we consider the special case in which the occupation probabilities have been set to unity and a fixed average centre of mass value is used in place of the centre of mass integration in (3.11) to (3.22). For this special case the hole line potential takes the form

$$\begin{aligned}
 U(k) = & \frac{16}{\pi^2} \int_0^{(k_f - k)/2} dk_0 k_0^2 \langle k_0 | G_{\Gamma}(\bar{P}) | k_0 \rangle \\
 & + \frac{8}{\pi^2} \int_{(k_f - k)/2}^{(k_f + k)/2} dk_0 \cdot k_0^2 \left[1 + \frac{k_f^2 - k^2 - 4k_0^2}{4kk_0} \right] \\
 & \cdot \langle k_0 | G_{\Gamma}(\bar{P}) | k_0 \rangle
 \end{aligned} \tag{3.23}$$

The self-consistency procedure is then repeated for a given \bar{P} . The results for the special case of $\bar{P}/k_f = 0.5$ are listed in table 5. Comparison of tables 3 with 5

Table 3. Self-consistent values of the effective mass, gap and binding energy per particle of Eq. (3.11) for the case when the occupation probabilities have been set to unity.

Fermi momentum k_f (\AA^{-1})	Kinetic energy $^{\circ}\text{K}$	m^*	Δ	Binding energy $^{\circ}\text{K}$
0.48	1.13	1.58	0.53	-0.09
0.52	1.33	1.48	0.53	-0.10
0.56	1.54	1.39	0.54	-0.10
0.60	1.77	1.32	0.53	-0.07
0.64	2.01	1.25	0.51	+0.02
0.68	2.27	1.19	0.48	+0.20
0.72	2.54	1.13	0.44	+0.46

Table 4. Self-consistent values of the effective mass m^* , gap Δ and binding energy per particle of Eq. (3.11) with occupation probabilities. The self-consistent values of the parameters a and b introduced in Eq. (3.20) are also given.

Fermi momentum k_f (\AA^{-1})	Kinetic energy K°	m^*	Δ	a	b	Binding energy $^{\circ}\text{K}$
0.48	1.13	1.50	0.48	0.11	0.07	-0.11
0.52	1.33	1.47	0.48	0.14	0.07	-0.13
0.56	1.54	1.33	0.47	0.17	0.06	-0.14
0.60	1.77	1.27	0.46	0.21	0.06	-0.14
0.64	2.01	1.21	0.44	0.25	0.06	-0.08
0.68	2.27	1.19	0.40	0.30	0.06	+0.11
0.72	2.54	1.12	0.37	0.34	0.07	+0.29

Table 5. Self-consistent values of gap Δ and binding energy per particle obtained from Eqs (3.11) and (3.23) for a fixed value of centre of mass momentum $P/k_f = 0.5$. For comparison we give the corresponding values obtained by Ghassib *et al.* (1974) in the last column.

Fermi momentum k_f (\AA^{-1})	Kinetic energy $^{\circ}\text{K}$	Δ	Binding energy $^{\circ}\text{K}$	Binding energy Ghassib <i>et al.</i> (1974)
0.68	2.27	0.35	0.22	-0.45
0.70	2.40	0.33	0.34	-0.38
0.72	2.54	0.30	0.50	-0.28
0.74	2.69	0.27	0.70	-0.13
0.76	2.83	0.23	0.86	+0.04

and 7 indicate the following: (1) The choice of $\bar{P}/h_f = 0.5$ gives roughly the same results as those obtained by complete centre of mass integration (2) The results seem to be insensitive to the choice of m^* provided the corresponding self-consistent value of Δ is used. These conclusions are in agreement with those of Ghassib *et al.* (Ghassib *et al.* 1974).

4. Discussion

In this paper we have used the matrix inversion method in momentum space to solve the Bethe-Goldstone equation. We have found it convenient to express a given two-body He-He interaction potential for *e.g.*, Frost-Musulin potential in terms of a sum of three Yukawa potentials. The reaction matrix elements for the Frost-Musulin potential have been obtained from those of the three Yukawa potential by a simple reference spectrum method [see Sec. 2 and Sec. 3, equation (3.10)]. The reaction matrix elements obtained by such a procedure do agree with those obtained directly using co-ordinate space methods. (see Table 2). These reaction matrix elements have then been used to perform calculations of the hole-line potential and binding energy with self-consistent occupation probabilities. The self-consistency procedure and the results have been presented in detail in Sec. 3. We now compare our results, for the case when the self-consistent occupation probabilities are set to unity, with the earlier work of Ghassib *et al.* (1974) using co-ordinate space methods. For the case when the effective mass m^* is fixed at 2.5 and the centre of mass momentum \bar{P} is fixed at $\bar{P}/k_f = 0.5$ our results disagree with the corresponding results of Ghassib *et al.* (see Table 5). We now discuss the reasons for this discrepancy and indicate possible errors in the work of Ghassib *et al.* We have found that our reaction matrix elements for $k_f = 0.75 \text{ \AA}^{-1}$ agree with those given by Ghassib *et al.* in their paper. Moreover we have been able to reproduce all the two-body interaction energies given by Ghassib *et al.* for fixed effective mass m^* and gap Δ . The discrepancy is arising only in the case of self-consistent calculations. Our self-consistency procedure is different from theirs. The self-consistency procedure adopted by Ghassib *et al.* consists in selecting the gap parameter Δ such that the expression for the binding energy obtained from Eqs (3.11)–(3.13) agrees with the one obtained by using the effective mass approximation:

$$(BE)_a = \frac{1}{2} \left[\left(1 + \frac{1}{m^*} \right) 0.3 - \Delta \right] k_f^2 \quad (4.1)$$

For the case when $m^* = 2.5$ binding energy cannot be attractive unless $\Delta > 0.42$. However the self-consistent values quoted by Ghassib *et al.* for the case of $k_f = 0.70 \text{ \AA}^{-1}$ do not satisfy this criterion.† The discrepancy seems to us to be due to using different units in solving graphically for self-consistent value of Δ using Equations (3.11)–(3.13) and (4.1)‡. (If these errors are corrected then their results can be shown to reduce to ours). Because these incorrect values of Δ have been used in the three body calculations, the three body contributions to the binding energy obtained by them are also in error.

† See table 11 of Ghassib *et al.* 1974

‡ See figure 6 of Ghassib *et al.* 1974

Finally we note that the calculations with self-consistent occupation probabilities have not been reported so far in the literature. Our results (see Sec. 3 and table 4) indicate that the minimum in the binding energy *versus* density curve shifts to higher densities. The results are still far off from experimental values. We believe that the use of phase-equivalent potentials (Haftel and Tabakin 1970) will considerably improve the situation. These calculations are in progress and preliminary results are quite encouraging.

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