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Flow of cholesteric liquid crystals—I: Flow along the helical axis

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Abstract. It is shown that the essential features of Helfrich's permeation model for flow along the helical axis of a cholesteric liquid crystal can be derived approximately on the basis of the Ericksen-Leslie theory.

Keywords. Cholesteric liquid crystal; Ericksen-Leslie theory.

1. Introduction

Cholesteric liquid crystals are known to exhibit a remarkable non-Newtonian behaviour, the apparent viscosity increasing by nearly a million times as the shear rate drops from a high to a very low value (Porter *et al* 1966, 1969). Helfrich (1969) has accounted for the very high apparent viscosity at low shear rates on the basis of a 'permeation model' which assumes that flow takes place along the helical axis without the helical structure itself moving because of anchoring effects at the walls, and that the velocity profile is flat rather than parabolic. He has shown that under these circumstances the apparent viscosity should be directly proportional to the square of the radius of the tube, inversely proportional to the square of the cholesteric pitch and independent of the shear rate. In the present paper we examine this problem from the point of view of the continuum theory and show that the main features of the permeation model do in fact follow as a natural consequence of the Ericksen-Leslie equations without having to make any special assumptions regarding the anchoring of the director at the boundaries.

2. Theory

Leslie (1968 *a*, 1969) has developed a continuum theory of the cholesteric state by extending the Ericksen-Leslie (Ericksen 1960, Leslie 1968 *b*) formulation for nematic liquid crystals. The equations of motion for an incompressible cholesteric liquid crystal, in the absence of temperature gradients, are

$$\rho \frac{dv_i}{dt} = \rho F_i + t_{ji,j} \tag{1}$$

$$\rho_1 \frac{d^2 n_i}{dt^2} = g_i + \pi_{ji,j} \tag{2}$$

Here v_i is the velocity at a point where the director orientation is n_i, F_i is the external body force per unit mass, g_i the director body force per unit volume, t_{ji} is the stress tensor per unit volume and π_{ji} the director stress tensor per unit volume. ρ is the density of the fluid and ρ_1 a material constant having the dimensions of moment of inertia. By using the entropy production inequality Leslie (1968 *a*) finds

$$t_{ji} = -p\delta_{ji} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + a e_{jkp}(n_p n_i)_{,k} + \hat{t}_{ji} \tag{3}$$

$$\pi_{ji} = \beta_j n_i + \frac{\partial W}{\partial n_{i,j}} + a e_{ijk} n_k \tag{4}$$

and

$$g_i = \gamma n_i - (\beta_j n_i)_{,j} - \frac{\partial W}{\partial n_i} - a e_{ijk} n_{k,j} + \hat{\ }_{i} \tag{5}$$

Here p and γ are arbitrary scalars and β_j an arbitrary vector arising out of the constraints of incompressibility and of constant director magnitude and a is a material constant. W is the elastic free energy per unit volume given by

$$\begin{aligned} 2W = 2K_2\,[\boldsymbol{n}\cdot\nabla\times\boldsymbol{n}] + K_{11}\,[\nabla\cdot\boldsymbol{n}]^2 \\ + K_{22}\,[\boldsymbol{n}\cdot(\nabla\times\boldsymbol{n})]^2 + K_{33}\,[(\boldsymbol{n}\cdot\nabla)\,\boldsymbol{n}]^2 \end{aligned} \tag{6}$$

Here K_2, K_{11}, K_{22} and K_{33} are the Frank elastic constants. The quantities $\hat{t}_{ji}$ and $\hat{g}_i$ are the hydrodynamic contributions to the stress tensor and the director body force respectively. They are given by

$$\hat{t}_{ji} = \mu_1 n_k n_p d_{kp} n_i n_j + \mu_2 n_j N_i + \mu_3 n_i N_j + \mu_4 d_{ji} + \mu_5 n_j n_k d_{ki} + \mu_6 n_i n_k d_{kj} \tag{7}$$

$$\hat{g}_i = \lambda_1 N_i + \lambda_2 n_j d_{ji} \tag{8}$$

where

$$\begin{aligned} d_{ij} &= d_{ji} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \\ w_{ij} &= -w_{ji} = \tfrac{1}{2}(v_{i,j} - v_{j,i}) \\ N_i &= \dot{n}_i - w_{ij} n_j \\ \lambda_1 &= \mu_2 - \mu_3 \\ \lambda_2 &= \mu_5 - \mu_6 \end{aligned} \tag{9}$$

and μ_1 to μ_6 are the viscosity coefficients. In nematics, $\lambda_1 < 0$; we shall assume this to be the case in the present discussion. We have now to solve eqs (1) and (2) to get the velocity and director profiles.

2.1. *Flow between parallel plates*

We shall consider the flow of the liquid crystal between two parallel plates, caused by a pressure gradient. We choose a right-handed cartesian system such that the plates occupy the planes $x = \pm h/2$. We seek solutions of the form

$$\begin{aligned} n_x &= \cos(qz + \phi)\cos\theta & v_x &= 0 \\ n_y &= \sin(qz + \phi)\cos\theta & v_y &= 0 \\ n_z &= \sin\theta & v_z &= v \end{aligned} \tag{10}$$

with $\theta = \theta(x, z)$, $\phi = \phi(x, z)$, $v = v(x)$.

This gives a cholesteric of pitch $P = 2\pi/q$ with the helical axis along z for $\theta = \phi = v = 0$.

We consider very low pressure gradients and retain only first powers in v, θ and ϕ. Then

$$n_x = C - \phi S$$
$$n_y = S + \phi C$$
$$n_z = \theta$$

where $C = \cos qz$, $S = \sin qz$. Equation (2) reduces to (neglecting director inertia and products like $v\theta$; $v_{,x}\, \theta_{,1}$; $\theta_{,xz}\, \theta$; $v_{,x}\, \theta_{,z}$, etc.)

$$\theta_{,xz}(K_{11} - K_{22}S^2) - \phi_{,xx}(K_{11}S + K_{33}SC^2) - \phi_{,zz}(K_{22}S) - \theta_{,z}\,[(K_{33} + 3K_{22})\,qSC] - \phi_{,z}(2K_{22}qC) - \lambda_1 vqS + \gamma\,(C - S\phi) = 0 \quad (11)$$

$$\theta_{,zz}(K_{22}SC) + \phi_{,xx}(K_{33}C^3) + \phi_{,zz}(K_{22}C) + \theta_{,x}(K_{33}qC^2 + K_{22}\,(C^2 - 2S^2)\,q] - \phi_{,z}\,(2K_{22}qS) + \lambda_1 vqC + \gamma\,(S + C\phi) = 0 \quad (12)$$

$$\theta_{,xx}(K_{22}S^2 + K_{33}C^2) + \phi_{,xz}\,[(K_{22} - K_{11})\,S] + \theta_{,zz}(K_{11}) - \phi_{,x}\,[(K_{11} + K_{33})\,qC] - \theta\,(K_{33}q^2) + \frac{\lambda_2 - \lambda_1}{2}\, v_{,x}\, C + \gamma\theta = 0 \quad (13)$$

In the above equations

$$\theta_{,x} = \partial\theta/\partial x,$$

$$\theta_{,xz} = \partial^2\theta/\partial x\partial z, \quad \text{etc.}$$

Similarly under the same approximation eq. (1) reduces to

$$p_{,x} = -\left[\left(\frac{\mu_6 + \mu_3}{2}\right) + \mu_2\right] qSCv_{,x}$$

$$p_{,y} = \left[\mu_2 C^2 - \frac{\mu_3}{2} + \frac{\mu_6}{2}(C^2 - S^2)\right] qv_{,x} \quad (14)$$

$$p_{,z} = \theta_{,xz}\,[(K_{11} - K_{22})\,qS] - \phi_{,xx}\,[(K_{11}S^2 + K_{33}C^2)\,q] - \phi_{,zz}(K_{22}q) - \theta_{,x}\,[(K_{22} + K_{33})\,q^2C] + v_{,xx}\left[\frac{\mu_4 + (\mu_5 - \mu_2)\,C^2}{2}\right] \quad (15)$$

From eqs (11) and (12) we get

$$\theta_{,xz}\,[(K_{11} - K_{22})\,S] - \phi_{,xx}(K_{11}S^2 + K_{33}C^2) - \phi_{zz}(K_{22}) - \theta_{,x}\,[(K_{22} + K_{33})\,qC] - \lambda_1 vq = 0 \quad (16)$$

From eqs (15) and (16) we find

$$v_{,xx}\left[\frac{\mu_4 + (\mu_5 - \mu_2)\,C^2}{2}\right] + v\,(\lambda_1 q^2) - p_{,z} = 0 \quad (17)$$

We make a 'coarse-grained' approximation and replace $\frac{1}{2}[(\mu_4 + (\mu_5 - \mu_2)\,C^2]$ by an average value $\bar{\eta}$, and rewrite (17) as

$$\bar{\eta} v_{,xx} + v\lambda_1 q^2 - p_{,z} = 0 \tag{18}$$

A solution of eq. (18) with the boundary conditions $v(\pm h/2) = 0$ is

$$v(x) = \frac{p_{,z}}{\lambda_1 q^2}\left[1 - \frac{\cosh kx}{\cosh kh/2}\right] \tag{19}$$

where

$$k = \left(\frac{-\lambda_1 q^2}{\bar{\eta}}\right)^{\frac{1}{2}}.$$

The velocity is symmetric about $x = 0$. The amount of liquid flowing per second in the z direction is given by

$$Q = \int_{-h/2}^{h/2} v(x)\,dx = 2\int_{0}^{h/2} v(x)\,dx$$

$$= \frac{p_{,z} h}{\lambda_1 q^2}\left[1 - \frac{\tanh kh/2}{kh/2}\right]$$

Hence the apparent viscosity for this geometry is given by

$$\eta_{\text{app}} = \frac{-p_{,z} h^3}{12Q} = \frac{-\lambda_1 q^2 h^2}{12\left[1 - \frac{\tanh kh/2}{kh/2}\right]} \tag{20}$$

when $h = 100\,\mu$ and $P = 1\,\mu$, the velocity attains 0·99 of the maximum value within a thickness of about 0·5 μ of the boundary. Thus in all practical situations, the velocity profile is flat over most of the region between the plates* and

$$\eta_{\text{app}} \approx \frac{-\lambda_1 q^2 h^2}{12} \tag{21}$$

which is the analogue of Helfrich's equation. The apparent viscosity is extremely large.

2.2. *Poiseuille flow*

In cylindrical polar coordinates we seek solutions of the form

$$n_r = \cos(qz - \psi + \phi)\cos\theta$$

$$n_\psi = \sin(qz - \psi + \phi)\cos\theta$$

$$n_z = \sin\theta$$

where θ and ϕ are function of r, ψ and z. Considering very small pressure gradients we obtain to a first order in θ and ϕ

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* We observe from eq. (14) that $p_{,y} = p_{,z} = 0$ when $v_{,x} = 0$. This implies that secondary flow is absent over most of the region between the plates. Of course, the choice of the velocity field (10) and (23) is a physical assumption that is not strictly consistent with the basic equations but justified *a posteriori* by the smallness of the error terms. Evidently the analysis fails very close to the boundaries but as in Helfrich's model we neglect this boundary layer. We are indebted to a referee for emphasizing this point.

$$n_r = C - \phi S$$

$$n_\psi = S + \phi C$$

$$n_z = \theta \tag{22}$$

where

$$C = \cos(qz - \psi) \text{ and } S = \sin(qz - \psi).$$

For the velocity field we assume

$$v_r = 0,$$

$$v_\psi = 0$$

$$v_z = v(r) \tag{23}$$

From eqs' (1) and (2) we obtain for the velocity

$$p_{,z} = \frac{v_{,rr}}{2}[\mu_4 + (\mu_5 - \mu_2)C^2] + \frac{v_{,r}}{2r}[\mu_4 + (\mu_5 - \mu_2)S^2] + \lambda_1 v q^2$$

As before, replacing the coefficients of $v_{,rr}$ and $v_{,r}$ by the average value $\bar{\eta}$ we obtain

$$v_{,rr} + \frac{1}{r} v_{,r} + \frac{\lambda_1 q^2 v}{\bar{\eta}} - \frac{p_{,z}}{\bar{\eta}} = 0 \tag{24}$$

A well-behaved solution of eq. (24) is

$$\frac{\lambda_1 q^2 v}{\bar{\eta}} = \frac{p_{,z}}{\bar{\eta}} + A I_0(kr)$$

where A is a constant, I_0 is the modified Bessel function of the first kind and zero order and

$$k^2 = \frac{-\lambda_1}{\bar{\eta}} q^2.$$

Using the boundary condition $v(R) = 0$, we find

$$A = -\frac{p_{,z}}{\bar{\eta} I_0(kR)}$$

and

$$v(r) = \frac{p_{,z}}{\lambda_1 q^2}\left[1 - \frac{I_0(kr)}{I_0(kR)}\right]$$

where R is the radius of the tube.

The amount of liquid crystal flowing out per second

$$Q = \int_0^R 2\pi r v(r)\, dr$$

$$= \frac{2\pi p_{,z}}{\lambda_1 q^2}\left[\frac{R^2}{2} - \frac{R\, I_1(kR)}{k\, I_0(kR)}\right]$$

where I_1 is the modified first order Bessel function of the first kind.

The apparent viscosity

$$\eta_{app} = \frac{-\pi p_{,z} R^4}{8Q}$$

$$= \frac{-\lambda_1 q^2 R^2}{8\left[1 - \frac{2I_1(kR)}{(kR)\, I_0(kR)}\right]}$$

Again in practical situations, the velocity profile is almost flat except very near the boundaries and

$$\eta_{app} \approx \frac{-\lambda_1 q^2 R^2}{8}$$

which is Helfrich's equation.

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