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On energy-momentum tensors as sources of spin-2 fields

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Abstract. The improvement terms in the generalised energy-momentum tensor of Callan, Coleman and Jackiw can be derived from a variational principle if the Lagrangian is generalised to describe coupling between 'matter' fields and a spin-2 boson field. The required Lorentz-invariant theory is a linearised version of Kibble-Sciama theory with an additional (generally-covariant) coupling term in the Lagrangian. The improved energy-momentum tensor appears as the source of the spin-2 field, if terms of second order in the coupling constant are neglected.

Keywords. Energy-momentum tensor; Kibble-Sciama theory; matter fields; spin-2 fields.

The purpose of this note is to show that the 'improved energy-momentum tensor' of Callan, Coleman and Jackiw (1970; see also de Alfaro *et al* 1973) can be obtained from a Lagrangian approach as the *source* of a spin-2 boson field. The appropriate spin-2 theory that has this property is the linearised version of Kibble-Sciama theory, with an *additional term* corresponding to a coupling between the spin-2 boson field and other fields. The improved energy-momentum tensor $\theta^{\mu\nu}$ occurs in the canonical forms

$$D^{\mu} = \theta^{\mu\nu} x_{\nu}, \qquad C^{\mu\nu} = (2x^{\mu} x_{\rho} - \delta^{\mu}{}_{\rho} x^2)\, \theta^{\rho\nu} \tag{1}$$

of the dilatation and conformal currents. The currents associated with co-ordinate transformations $x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}$ are

$$j^{\mu} = -\mathcal{L}\xi^{\mu} + i\pi^{\mu}[G, \psi], \qquad \pi^{u} = \partial\mathcal{L}/\partial\partial_{\mu}\psi \tag{2}$$

where $i\,[G, \psi]$ is the change in the field ψ brought about by the transformation and a summation over all the fields occurring in $\mathcal{L}$ is implied in the second term in j^{μ}. In particular, the currents associated with dilatations, Lorentz rotations, and translations are given respectively by

$$\left.\begin{aligned}
&\xi^{\mu} = \epsilon x^{\mu}, \quad i\,(G, \psi) = \epsilon\,(l + x\partial)\,\psi, \quad j^{\mu} = \epsilon D^{\mu}, \quad D^{\mu} = \pi^{\mu}\,(l + x\partial)\,\psi, \\
&\xi^{\mu} = \epsilon^{\mu\nu} x_{\nu}, \quad i\,(G, \psi) = \epsilon_{\mu\nu}\,(x^{\mu}\partial^{\nu} + \tfrac{1}{2}\Sigma^{\mu\nu})\,\psi, \quad j^{\mu} = \epsilon^{\rho\sigma}\,(\pi^{\mu} x_{\rho}\partial_{\sigma}\psi \\
&\qquad + \tfrac{1}{2} S^{\mu}_{\rho\sigma}), \quad S^{\mu}_{\rho\sigma} = \pi^{\mu}\,\Sigma_{\rho\sigma}\psi, \\
&\xi^{\mu} = \epsilon^{\mu}, \quad i\,[G, \psi] = \epsilon^{\mu}\,\partial_{\mu}\psi, \quad j^{\mu} = \epsilon^{\nu}\, T^{\mu}_{\nu}, \quad T^{\mu}_{\nu} = -\mathcal{L}\delta^{\mu}_{\nu} + \pi^{\mu}\,\partial_{\nu}\psi.
\end{aligned}\right\} \tag{3}$$

Here, l is the scale dimension of the field ψ, $S^{\mu}_{\rho\sigma}$ is the density of intrinsic angular momentum (spin), and T^{μ}_{ν} is the *canonical* energy-momentum tensor (Noether's tensor).

If the Lagrangian is expressed in a curvilinear co-ordinate system or more generally in a curved space time, then a *tetrad* has to be introduced to deal with spinor fields. In this case, the Lagrangian can be written in terms of fields which transform as representations of the Lorentz rotations of the tetrad and are *invariants* for co-ordinate transformations, and $S^{\mu}_{\rho\sigma}$ is then the *total* current associated with the rotations of the tetrad rather than the *part* of the current associated with Lorentz rotations of the co-ordinate system. Moreover, with certain natural assumptions about the form of the Lagrangian, it turns out that $l = s + 1$ where s is the spin value of the field (see Bayen 1970 and Mack and Todorov 1968) and the current

$$l^{\mu} = l\pi^{\mu}\,\psi \tag{4}$$

is the current associated with conformal mappings, *i.e.* Weyl's gauge group (Weyl 1919; Lord 1972; Dirac 1973).

As is well known, the theory of a massless spin-2 boson in Minkowski space-time is algebraically equivalent to the linearised version of Einstein's theory of gravitation (Lord 1971). A natural generalisation of the theory of a linear spin-2 field in the presence of other fields is the corresponding linearisation of Kibble-Sciama theory (Kibble 1960; Sciama 1963). This theory in its usual form is a curved space-time theory with an asymmetric connection. The torsion is determined by the spin-density and the Einstein tensor constructed from the asymmetric affine connection turns out to be proportional to the *canonical* energy momentum tensor (Noether's tensor). By replacing the torsion on these generalised Einstein equations by spin-density terms, the theory can be readily shown to be equivalent, *to first order in the coupling constant,* to a theory obtained from Einstein's gravitational theory by replacing the energy-momentum tensor in Einstein's field equations by Belinfante's symmetrised tensor (Belinfante 1940). For integral-spin fields, and for Lagrangians containing only *first* derivatives of the fields, it can be shown that Belinfante's tensor is the same tensor as the one that appears on the right hand side of the field equations in Einstein's theory (Goedecke 1974; Lord 1975). Belinfante's tensor is

$$B_{\mu\nu} = T_{\mu\nu} + \tfrac{1}{2}\,\partial^{\lambda}\,(S_{\lambda\mu\nu} + S_{\mu\nu\lambda} + S_{\nu\lambda\mu})$$

(where $T_{\mu\nu}$ is the canonical tensor). (5)

The linearised version of the Lagrangian density of Kibble-Sciama theory is

$$(1/\kappa)\,[\xi^{\mu}_{\nu}\,(\partial_{\mu}\lambda^{\nu} + \partial_{\rho}\lambda^{\nu\rho}_{\mu}) + \tfrac{1}{2}\,(\lambda^{\mu\nu\rho}\,\lambda_{\rho\mu\nu} + \lambda_{\rho}\lambda^{\rho})] + \mathcal{L}. \tag{6}$$

In this expression, ξ^{μ}_{ν} and $\lambda^{\mu\nu}_{\rho}\ (= -\lambda^{\nu\mu}_{\rho})$ are treated as independent dynamical fields in the variation (we have used the abbreviation $\lambda^{\rho} = \lambda^{\mu\rho}_{\mu}$), and these dynamical fields are incorporated in $\mathcal{L}$ by constructing the invariant matter Lagrangian of the full Kibble-Sciama theory in the usual way, replacing the tetrad h^{μ}_{a} by $\delta^{\mu}_{a} + \xi^{\mu}_{a}$, and then discarding second order terms in ξ^{μ}_{a} and $\lambda^{\mu\nu}_{\rho}$. The theory given by eq. (6) can now be considered simply as an ordinary Lorentz-covariant, flat space-time theory of a massless spin-2 boson in interaction with matter. From the corresponding reformulation of Kibble-Sciama theory it follows that the theory given by (6) is fully equivalent to the one given by the field equations

$$\Box\chi_{\nu\mu} - \partial^{\rho}\,(\partial_{\nu}\chi_{\mu\rho} + \partial_{\mu}\chi_{\nu\rho} - \eta_{\mu\nu}\partial^{\sigma}\,\chi_{\rho\sigma}) = -\,2\kappa B_{\mu\nu} + O\,(\kappa^{2}). \tag{7}$$

where

$$\chi_{\mu\nu} = \xi_{\mu\nu} + \xi_{\nu\mu} - \eta_{\mu\nu}\,\xi \tag{8}$$

Of course, the left hand side of eq. (7) is just the linearised Einstein tensor. The theory given by eq. (6) possesses invariance under a gauge group (a residual manifestation of the general covariance of the full Kibble-Sciama theory) which allows us to impose a Lorentz-like gauge condition on the fields:

$$\partial_\nu \chi^{\mu\nu} = 0 \tag{9}$$

(actually a linearised version of the harmonic co-ordinate condition). With this subsidiary condition, the theory given by eq. (6) is simply the conventional massless spin-2 theory with the Belinfante tensor as its source, to first order in the coupling constant:

$$\Box\chi_{\mu\nu} = 2\kappa B_{\mu\nu} + O(\kappa^2). \tag{10}$$

The improved energy-momentum tensor $\theta_{\mu\nu}$ occurring in eq. (1) exists whenever the Lagrangian is such that a tensor $\sigma_{\mu\nu}$ satisfying

$$\partial_\nu \sigma^{\mu\nu} = l^\mu + S^\mu \qquad (S^\mu = S_\rho^{\rho\mu}) \tag{11}$$

can be found. Explicitly,

$$\begin{aligned} \theta_{\mu\nu} &= B_{\mu\nu} + \tfrac{1}{2}\,\partial_\lambda\partial_\rho X_{\mu\nu}^{\lambda\rho}, \\ X_{\mu\nu}^{\lambda\rho} &= \eta^{\lambda\rho}\,\overset{+}{\sigma}_{\mu\nu} - \delta_\mu^\lambda\,\overset{+}{\sigma}{}_\nu^\rho - \eta_\nu^\lambda\,\overset{+}{\sigma}{}^\rho_\mu + \eta_{\mu\nu}\overset{+}{\sigma} + \tfrac{1}{3}\,(\delta_\mu^\lambda\,\delta_\nu^\rho - \eta_{\mu\nu}\eta^{\lambda\rho})\,\sigma, \\ \overset{+}{\sigma}_{\mu\nu} &= \tfrac{1}{2}\,(\sigma_{\mu\nu} + \sigma_{\nu\mu}), \qquad \sigma = \sigma_\mu^\mu \end{aligned} \tag{12}$$

(see de Alfaro *et al* 1973). In order for $\theta_{\mu\nu}$, rather than $B_{\mu\nu}$, to be the source of the spin-2 field, we have to add an additional coupling term

$$(1/2\kappa)\,M^{\mu\nu}\,[\Box\zeta_{\mu\nu} - \partial^\rho\,(\partial_\nu\zeta_{\mu\rho} + \delta_\mu\zeta_{\mu\rho}) + \partial_\mu\partial_\nu\zeta] \tag{13}$$

where

$$\zeta_{\mu\nu} = \tfrac{1}{2}\,(\xi_{\mu\nu} + \xi_{\nu\mu}) \tag{14}$$

and the matter tensor $M^{\mu\nu}$ is

$$M^{\mu\nu} = \overset{+}{\sigma}{}^{\mu\nu} - \tfrac{1}{6}\,\eta^{\mu\nu}\,\sigma \tag{15}$$

By applying Gauss's theorem the derivatives in eq. (13) can be shifted from the $\zeta_{\mu\nu}$ to $M_{\mu\nu}$, and the matter terms coupled to $\zeta_{\mu\nu}$ in eq. (13) are just those occurring in eq. (12). It follows immediately that the new spin-2 theory is given by

$$\partial_\nu\chi^{\mu\nu} = 0, \qquad \Box\chi_{\mu\nu} = 2\kappa\theta_{\mu\nu} + O(\kappa^2). \tag{16}$$

The improved energy-momentum tensor appears as the source of the spin-2 field. The coupling term (13) has a very remarkable form. It is the term that would arise in the linearisation of a *generally covariant* theory with an additional coupling between matter and gravitation, of the form

$$(1/2\kappa)\,(-g)^{\frac{1}{2}}\,M^{\mu\nu}\,R_{\mu\nu}, \tag{17}$$

where

$$M^{\mu\nu} = \overset{+}{\sigma}{}^{\mu\nu} - \tfrac{1}{6}\,g^{\mu\nu}\,\sigma, \qquad \sigma^{\mu\nu}_{\ ;\nu} = l^\mu + S^\mu \tag{18}$$

[Note that in (17) the Ricci tensor is of course the usual one constructed from the metric and its Christoffel symbols, not the asymmetric one occurring in the Kibble-Sciama Lagrangian].

Thus, we have obtained a modified version of gravitational theory, in which the stress-energy tensor occurring in Einstein's equations is the improved energy momentum tensor of Callan, Coleman and Jackiw, if terms of second order in the coupling constant are ignored. That the coupling (17) gives rise to the 'improvement terms' in the energy-momentum tensor, was pointed out by Panday (1973).

In the context of particle physics, it seems more natural to interpret the spin-2 field as the field of an f-meson, rather than the gravitational field. We can insert a mass term for the spin-2 field in eq. (6) without making any difference to our arguments concerning the linear theory. However, with a mass term included, the possibility of interpreting the theory as the linear limit of a curved space-time theory no longer exists, because a covariant mass term (*i.e.* a 'cosmological' term) in Einstein's theory or Kibble-Sciama theory makes linearisation on a *flat* background no longer possible. The background would have to be a de Sitter space with a radius of curvature of the same order of magnitude as the Compton wavelength associated with the mass (Lord *et al* 1974). Moreover, if κ were a strong coupling constant the $O(\kappa^2)$ terms in eq. (16) [which have the form of a spin-spin coupling (Kibble 1963)], rather than being negligible, would actually be much larger than the $\kappa\theta_{\mu\nu}$ term.

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