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Some results in the theory of interacting electron systems

A K RAJAGOPAL*
Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560012

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Abstract. The expressions for the longitudinal dielectric function, spin and orbital susceptibilities in the static, long wavelength limit are evaluated by solving the corresponding linearized vertex functions exactly in this limit. The plasma dispersion relation to leading order in the long wave limit is similarly obtained. These are compared with the corresponding results obtained previoulsy by us by a variational solution to the same vertex equations. It is established that the variational method gives the exact results in the static, zero wave vector limit, involving the proper renormalizations. The plasma dispersion relation is found to be the same as in the exact calculation whereas the coefficient of q^2 in the static density correlation function has an important additional contribution to the variational result. Applications of these results are briefly discussed.

Keywords. Longitudinal and transverse dielectric functions : Plasma dispersion relations; static density correlation function, interacting electron systems.

1. Introduction

Correlations among electrons are of paramount importance in determining the physical properties of a solid. The mechanical, magnetic, and electrical properties of varieties of solids found in nature are all attributable to the correlated motions of electrons due to both their interactions among themselves and with the lattice vibrations. To understand these various aspects of correlations, the model of an electron gas with a neutralizing positive charge background has been found to be useful for describing at least the behaviour of electrons in simple metals. This model captures in essence most of the aspects of electron interactions even though it leaves out entirely the important crystal structure effects, which may not be of too much consequence in discussing simple metals such as light alkali metals. Even after dilineating the interaction effects from the specific solid state effects, the problem of electron interactions cannot be solved in any complete way. A variety of approximation methods have been put forward in treating this system. At present, the most elegant formulation is in terms of the field theoretical formalism. For a description of this with special emphasis on such solid state problems, one may refer to the book of Pines and Noziérés (1966). Henceforth we will focus our attention on this model. Besides being one of the most important many body problems, this model has become also a basis for a

* On leave from Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, U.S.A.

meaningful discussion of inhomogeneous electron systems such as real solids, solids with a surface, etc. (Lang 1973).

The effect of correlations are succinctly described by examining the frequency and wave vector dependent longitudinal and transverse dielectric functions of this system. With a knowledge of these two functions, one may examine the spatial and temporal correlations among particle densities and the current densities. The magnetic susceptibility of the system is related to a combination of the longitudinal and transverse dielectric functions. These functions are usually required to compute the neutron scattering cross section of this system, which is perhaps one of the best experimental tools in probing these properties of this system. On the theoretical side, one may examine the mechanical and magnetic instabilities of this system by studying the behaviour of these functions in the static, long wavelength limit (Pines-Nozierés 1966) as a function of density of the system. From the longitudinal dielectric function, one may deduce the dispersion of the plasma mode in the system at least in the long wavelength limit. The effect of interaction is to modify the dispersion relation one obtains for it in a non-interacting system, where only the correlations due to Fermi statistics exist. Similarly, the compressibility of the system may be deduced when the effects of interactions are incorporated in its computation, and hence the mechanical instability of the system can be discussed. By studying the spin contribution to the magnetic susceptibility of the system (which also has an orbital contribution) one may study the paramagnetic instability towards ferromagnetism by examining the static long wavelength limit whereas the spin density wave type instabilty may be discussed by studying it for the zero frequency, finite wave vectors. The above discussion is very general, and meant only to give the range of interesting properties one can learn from studying these correlation functions.

The effects of interactions may be neatly described by expressing the above correlation functions in terms of appropriate "vertex functions" (Rajagopal and Jain 1972). These vertex functions obey complicated (non-linear) integral equations in their most general forms. One is thus forced to make approximations so as to make these equations manageable. Even after this step is taken, often the resulting simpler equations are not capable of exact solutions in all regions of the frequency and wave vector of physical interest. In this paper, we develop a method of solving exactly the linearized vertex equation in the random phase approximation in the zero frequency, long wavelength limit. The linearized form of the vertex equation is equivalent to the time dependent Hartree-Fock approximation with statically screened potential in the exchange processes. In principle, the solution in a power series of the vertex function in the wave vector can be developed. Rajagopal and Jain (1972) used a variational method introduced earlier by the author (Rajagopal 1966*) for obtaining the correlation functions of interest. We will compare the expressions obtained by this method in the appropriate limits with the exact results derived here.

To bring out the effects of correlations on the system properties we will here define the quantities of interest and express them in suitable forms. Their deri-

* The variational method given here has been rediscovered by Langreth D C (1969) *Phys. Rev.* **181** 753, **187** 768E and Jha S S, Gupta K K and Woo J. W. F (1971) *Phys. Rev.* **B1** 1005.

vation will not be given since they may be found elsewhere. For a simple derivation of the equations, one may refer to Rajagopal *et al* (1973) and Rajagopal and Grest (1974). Except for minor differences, the notations are as in the paper by Rajagopal and Jain (1972). The longitudinal and transverse dielectric functions are given by

$$\epsilon_L(q) = 1 + \frac{8\pi e^2}{q^2} \tilde{K}_{\rho\rho}(q) \tag{1}$$

$$\epsilon_T(q) = 1 + \left(\frac{e^2}{m^2 q_0^2}\right) \tilde{K}_{xx}(q) \left[1 - \frac{e^2 \tilde{K}_{xx}(q)}{m^2 (q_0^2 - c^2 q^2)}\right]^{-1} \tag{2}$$

where $q = (\boldsymbol{q}, q_0)$ represents in a single notation both the wave vector, $\boldsymbol{q}$, and frequency, q_0. $\tilde{K}_{\rho\rho}$ and $\tilde{K}_{xx}$ are the irreducible longitudinal and transverse polarization functions (which now include both spin and particle current contributions) which will be specified on more detail presently. The magnetic susceptibility, which includes both spin and orbital parts, is given by the relationship, first deduced by Lindhard,

$$\chi(q)[1 + \chi(q)]^{-1} = \frac{q_0^2}{c^2 q^2} [\epsilon_T(q) - \epsilon_L(q)]. \tag{3}$$

The appearance of q_0^2 in eqs (2) and (3) is due to a proper treatment of the transverse correlations employing Maxwell's electromagnetic equations. If we suppress this frequency dependence in comparison to cq in the above expressions, one obtains a simple expression for $\chi(q)$:

$$\chi(q) \cong \frac{e^2}{m^2 c^2 q^2} \tilde{K}_{xx}(q) \tag{4}$$

Only in this approximation, we may deduce that the frequency and wave vector dependent susceptibility as a *sum* of the orbital and spin susceptibilities of the system, because $\tilde{K}_{xx}$ is sum of these two contributions. This, however, obtains quite rigorously from (3) in the static limit. In this paper we will be interested in $\epsilon_L(q)$ both in the static limit and in the dynamic limit where $(mq_0/qk_F) \gg 1$; and in $\chi(q)$ only in the static, long wavelength limit. One may then write the expressions of interest in a neat form using dimensionless variables. (all momenta are scaled by Fermi momentum, k_F, and all energies by Fermi energy, $k_F^2/2m$ and the integration variable $\boldsymbol{k}$ is then written as $k_F x$, $|\boldsymbol{k}|^2 = k_F^2 x$).

$$\epsilon_L(q) = 1 + \left(\frac{4\alpha r_s}{\pi}\right) \frac{1}{y^2} \tilde{K}\rho\rho(y)$$

with

$$\tilde{K}\rho\rho\,(y) = \int_0^\infty x^{\frac{1}{2}}\, dx \int \frac{d\hat{x}}{4\pi} \mathscr{F}(x, y)\, \Gamma_\rho(x; y) \tag{5}$$

$$\chi(\boldsymbol{q}; q_0 = 0) = \chi_{sp}(\boldsymbol{q}; q_0 = 0) + \chi_{orb}(\boldsymbol{q}; q_0 = 0) \tag{6}$$

with

$$\chi_{sp}(\boldsymbol{q}; 0) = \left(\frac{e^2 k_F}{4\pi^2 mc^2}\right) \tilde{K}_{spin}(y) \tag{6 a}$$

$$\chi_{\rm orb}(\boldsymbol{q};0) = -\left(\frac{e^2k_F}{3\pi^2mc^2}\right)\frac{1}{y^2}\tilde{K}_{\rm orbit}(y) \tag{6 b}$$

where

$$\tilde{K}_{\rm spin}(y) = \int_0^\infty x^{\frac{1}{2}}\,dx \int \frac{d\hat{x}}{4\pi}\,\mathcal{F}(x,y)\,\Gamma_s(x;y) \tag{7 a}$$

$$\tilde{K}_{\rm orbit}(y) = 1 + 3\int_0^\infty x dx \int \frac{d\hat{x}}{4\pi}\sin\theta_x\cos\phi_x\,\mathcal{F}(x,y)\;\Gamma_0(x,y) \tag{7 b}$$

r_s in the above is the usual electron gas parameter which represents the effective radius in units of Bohr radius of the electron in the interacting system and it is inversely proportional to the cube root of the density of the electrons. Γ_ρ, Γ_s, and Γ_0 are the irreducible vertex functions which are respectively 1, 1, and $-x^{\frac{1}{2}}\sin\theta_x\cos\phi_x$ in the non-interacting case. These functions obey linear integral equations in the random phase approximation including exchange processes and they may all be represented by a single equation in the form:

$$\Gamma_A(x;y) = \gamma_A(x;y) + \left(\frac{2\alpha r_s}{\pi}\right)\int x_1^{\frac{1}{2}}\,dx_1 \int \frac{d\hat{x}_1}{4\pi}$$

$$V_s(|\boldsymbol{x}-\boldsymbol{x}_1|)\,\mathcal{F}(x_1,y)\left[\Gamma_A(x_1;y) - \left(\frac{\nu - 2x_1^{\frac{1}{2}}\,y\cos\theta_{x_1}}{\nu - 2x^{\frac{1}{2}}\,y\cos\theta_x}\right)\Gamma_A(x;y)\right] \tag{8}$$

where $\nu = q_0/E_F$ is the dimensionless frequency variable.

$$\gamma_A(x;y) = \begin{cases} 1 & A=\rho, \\ 1 & A=s, \\ -x^{\frac{1}{2}}\sin\theta_x\cos\phi_x & A=0 \end{cases} \tag{9}$$

In all the above,

$$\mathcal{F}(x,y) = [f_0(\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{y}) - f_0(\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{y})]/[(\nu - 2x^{\frac{1}{2}}\,y\cos\theta\, x] \tag{10}$$

$f_0(x)$ is the usual Fermi distribution function.

A variational procedure for solving these equations was given earlier (Rajagopal and Jain 1972). The object of the present paper is to solve eq. (8) exactly in some physically interesting limits. The effects of electron interactions are contained in the second part of eq. (8). For the most part, we find that the exact expressions for the $\tilde{K}$'s are almost the same as those obtained by us using the variational method, including the leading terms of the plasma dispersion relation. A similar exact analysis was used some time ago (Rajagopal 1966) to determine the spin wave dispersion in the long wavelength limit.

In section 2 we will examine the solutions in the static long wavelength limit. In section 3 we develop the solution to Γ_ρ equation in the limit $(mq_0/qk_F) \gg 1$ to leading order and deduce therefrom the plasma dispersion relation. In section 4 we briefly collect the results obtained in the variational scheme and compare them with the exact solution. In the last section, we give a brief summary of the results and make a few concluding remarks, including those concerning the physical processes that are left out in an equation such as eq. (8).

2. Solution to the irreducible vertex equation, static long wavelength limit

2.1 Orbital susceptibility

Set $v = 0$ in (8) and choose $A = 0$. We are interested in computing

$$\chi_{\text{orb}}(0;0) = -\left(\frac{e^2 k_F}{3\pi^2 mc^2}\right) \lim_{y\to 0} \left[\frac{1}{y^2} \tilde{K}_{\text{orbit}}(y)\right] \tag{11}$$

In the limit of $y \to 0$, we observe (at absolute zero of temperature)

$$\mathscr{F}(x, y) \cong \delta(x-1) + \tfrac{1}{4} y^2 [\delta'(x-1) + \tfrac{2}{3} x \cos^2\theta_x \, \delta''(x-1)] \tag{12}$$

where primes denote differentiations with respect to x. We now expand $\Gamma_0(x; y)$ in the form:

$$\Gamma_0(x; y) = \sum_{l,n} [\Gamma^{(0)}_{0lm}(x) + y^2 \Gamma^{(2)}_{0lm}(x) + \ldots] y_{lm}(\hat{x}) \tag{13}$$

This follows from the symmetry of the problem.

Substitute (12) and (13) in both sides of the equation for Γ_0 and equate like powers of y:

$$\Gamma_0(x; y) = -x^{\frac{1}{2}} \sin\theta_x \cos\phi_x + \left(\frac{2\alpha r_s}{\pi}\right) \int_0^\infty x_1^{\frac{1}{2}}\, dx_1 \int \frac{d\hat{x}_1}{4\pi}$$

$$\times V_s(|\boldsymbol{x} - \boldsymbol{x}_1|)\, \mathscr{F}(x_1, y) \left[\Gamma_0(x_1; y) - \left(\frac{x_1}{x}\right)^{\frac{1}{2}} \frac{\cos\theta_{x1}}{\cos\theta_x} \Gamma_0(x; y)\right] \tag{14}$$

Use the standard expressions and relationships for spherical harmonics and

$$V_s(|\boldsymbol{x} - \boldsymbol{x}_1|) = \sum_{lm} \frac{4\pi}{2l+1} y_{lm}(x)\, V_s^{(l)}(x; x_1)\, y^*_{lm}(\hat{x}_1) \tag{15}$$

where

$$\frac{V_s^{(l)}(x; x_1)}{2l+1} = \tfrac{1}{2} \int_{-1}^{+1} d\mu\, P_l(\mu)\, V_s(|\boldsymbol{x} - \boldsymbol{x}_1|)$$

and for the Yukawa potential

$$\frac{V_s^{(l)}(x; x_1)}{2l+1} = \frac{1}{2(xx_1)^{\frac{1}{2}}} Q_l\left(\frac{x + x_1 + \xi^2}{2(xx_1)^{\frac{1}{2}}}\right) \tag{16}$$

where

$$V_{\text{yukawa}}(|\boldsymbol{x} - \boldsymbol{x}_1|) = 1/[(\boldsymbol{x} - \boldsymbol{x}_1)^2 + \xi^2]$$

Then pick out the coefficients $\Gamma^{(i)}_{0lm}(x)$ by multiplying through by $y^*_{lm}(x)$ and integrating over all $\hat{x}$. Now

$$\Gamma^{(0)}_{0lm}(x) = \frac{1}{2} x^{\frac{1}{2}} \left(\frac{8\pi}{3}\right)^{\frac{1}{2}} \delta_{l,1}(\delta_{m,1} - \delta_{m,-1}) +$$

$$+ \left(\frac{2\alpha r_s}{\pi}\right) \left[\frac{V_s^{(l)}(x; 1)}{2l+1} \Gamma^0_{0lm}(1) - x^{-\frac{1}{2}} \frac{V_s^{(1)}(x; 1)}{3} \Gamma^{(0)}_{0lm}(x)\right] \tag{17}$$

From this we obtain at once the exact result:

$$\Gamma_{0lm}^{(0)}(x) = \tfrac{1}{2}\, x^{\frac{1}{2}} \left(\frac{8\pi}{3}\right)^{\frac{1}{2}} \delta_{l,1}\,(\delta_{m,1} - \delta_{m,-1}) \tag{18}$$

Observe that this is the same as the non-interecting vertex! Let us go back to the expression for $\chi_{\rm orb}$ and examine what kinds of vertex components are required. Thus, from (7 *b*), (12), and (13), we obtain after some algebra and using only (18) as the input at this stage,

$$\chi_{\rm orb}(0;0) = -\left(\frac{e^2 k_F}{12m\pi^2 e^2}\right)\left[1 - 4\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \{\Gamma_{01,1}^{(2)}(1) - \Gamma_{01,-1}^{(2)}(1)\}\right] \tag{19}$$

This shows that we only need $\Gamma_{01,\pm 1}^{(2)}(1)$. Even though the most general form for $\Gamma_{0lm}^{\prime(2)}(x)$ may be written down, we will here be content to exhibit only quantity $[\Gamma_{01,1}^{(2)}(1) - \Gamma_{01,-1}^{(2)}(1)]$ as it is the only relevant part needed here. After a fair amount of algebra, one obtains a simple result

$$\Gamma_{01,1}^{(2)}(1) - \Gamma_{01,-1}^{(2)}(1) = \frac{2}{15}\left(\frac{\alpha r_s}{\pi}\right)\left(\frac{8\pi}{3}\right)^{\frac{1}{2}} \left\{\frac{d^2}{dx^2}\left[x^2\left(\frac{V_s^{(3)}(1;x)}{7} - \frac{V_s^{(1)}(1;x)}{3}\right)\right]\right\}_{x=1} \tag{20}$$

Hence, we obtain finally the required result:

$$\chi_{\rm orb}(0;0) = -\left(\frac{e^2 k_F}{12m\pi^2 c^2}\right)\left\{1 - \frac{8}{15}\left(\frac{\alpha r_s}{\pi}\right)\frac{d^2}{dx^2}\left[x^2\left(\frac{V_s^{(3)}(1;x)}{7} - \frac{V_s^{(1)}(1;x)}{3}\right)\right]\right\}_{x=1} \tag{21}$$

2.2 *Longitudinal dielectric function and spin susceptibility*

From the structure of the Γ_p, Γ_s equations we observe that it is enough if we compute the integral

$$I(y) = \int_0^\infty x^{\frac{1}{2}}\,dx \int \frac{d\hat{x}}{4\pi}\,\mathscr{F}(x,y)\,\Gamma(x;y) \tag{22}$$

where $\Gamma(x,y)$ obeys the equation

$$\Gamma(x;y) = 1 + \left(\frac{2\alpha r_s}{\pi}\right)\int_0^\infty x_1^{\frac{1}{2}}\,dx_1 \int \frac{d\hat{x}_1}{4\pi}$$

$$V_s(|\mathbf{x} - \mathbf{x}_1|)\,\mathscr{F}(x_1,y)\left[\Gamma(x_1;y) - \left(\frac{x_1}{x}\right)^{\frac{1}{2}}\frac{\cos\theta_{x1}}{\cos\theta_x}\,\Gamma(x;y)\right] \tag{23}$$

We then obtain

$$\epsilon_L(q;0) = 1 + \left(\frac{4\alpha r_s}{\pi}\right)\frac{1}{y^2}\,I(y),$$

$$\chi_{\rm sp}(q;0) = \left(\frac{e^2 k_F}{4\pi^2 mc^2}\right) I(y) \tag{24}$$

We compute now $I(y)$ to second power in y. Here again, we write

$$\Gamma(x;y) = \sum_{l,m} [\Gamma_{lm}^{(0)}(x) + y^2 \Gamma_{lm}^{(2)}(x) + \ldots] y_{lm}(\hat{x}) \tag{25}$$

and proceed as before. Thus, we obtain for $I(y)$ to y^2 order,

$$\begin{aligned} I(y) = \frac{1}{4\pi} [& (4\pi)^{\frac{1}{2}} (\Gamma_{00}^{(0)}(1) + y^2 \Gamma_{00}^{(2)}(1)) + \\ & + (4\pi)^{\frac{1}{2}} \frac{1}{4} y^2 \left(-\frac{\mathrm{d}}{\mathrm{d}x} (x^{\frac{1}{2}} \Gamma_{00}^{(0)}(x)) + \frac{2}{9} \frac{\mathrm{d}^2}{\mathrm{d}x^2} (x^{3/2} \Gamma_{00}^{(0)}(x)) \right)_{x=1} + \\ & + \left(\frac{16\pi}{5}\right)^{\frac{1}{2}} \frac{1}{18} y^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} (x^{3/2} \Gamma_{20}^{(0)}(x)) \right)_{x=1}] \end{aligned} \tag{26}$$

The derivation of the equations for $\Gamma_{lm}^{(i)}(x)$ and their solution proceeds as before. Now,

$$\begin{aligned} \Gamma_{lm}^{(0)}(x) = & (4\pi)^{\frac{1}{2}} \delta_{l,0} \delta_{m,0} + \\ & + \left(\frac{2\alpha r_s}{\pi}\right) \left[\frac{V_s^{(l)}(x;1)}{2l+1} \Gamma_{lm}^{(0)}(1) - x^{-\frac{1}{2}} \frac{V_s^{(1)}(x;1)}{3} \Gamma_{lm}^{(0)}(x) \right]. \end{aligned} \tag{27}$$

This shows at once that

$$\Gamma_{lm}^{(0)}(x) = 0 \quad \text{for} \quad l \neq 0, \quad m \neq 0. \tag{28}$$

Introduce the vertex renormalisation constant Γ as:

$$\Gamma = \left[1 - \left(\frac{2\alpha r_s}{\pi}\right) \left(V_s^{(0)}(1;1) - \frac{V_s^{(1)}(1;1)}{3} \right) \right]^{-1}. \tag{29}$$

Then the solution may be written as

$$\Gamma_{lm}^{(0)}(x) = (4\pi)^{\frac{1}{2}} \tilde{\Gamma}(x) \Gamma \delta_{l,0} \delta_{m,0} \tag{30}$$

where

$$\begin{aligned} \tilde{\Gamma}(x) = & \left[1 - \left(\frac{2\alpha r_s}{\pi}\right) \left(V_s^{(0)}(1;1) - V_s^{(0)}(1;x) \right) + \left(\frac{2\alpha r_s}{\pi}\right) \frac{V_s^{(1)}(1;1)}{3} \right] \times \\ & \times \left[1 + \left(\frac{2\alpha r_s}{\pi}\right) x^{-\frac{1}{2}} \frac{V_s^{(1)}(1;x)}{3} \right]^{-1} \end{aligned} \tag{31}$$

In view of this, we observe that the last term in eq. (26) drops out. The equation obeyed by $\Gamma_{lm}^{(2)}(x)$ is found to be

$$\begin{aligned} \Gamma_{lm}^{(2)}(x) = & \left(\frac{2\alpha r_s}{\pi}\right) \left[\Gamma_{lm}^{(2)}(1) \frac{V_s^{(l)}(1;x)}{2l+1} - x^{-\frac{1}{2}} \Gamma_{ml}^{(2)}(x) \frac{V_s^{(1)}(1;x)}{3} \right] \\ & + \left(\frac{2\alpha r_s}{\pi}\right) \left[-\frac{1}{4} \frac{\mathrm{d}}{\mathrm{d}x_1} \left\{ x_1^{\frac{1}{2}} \left(\Gamma_{00}^{(0)}(x_1) V_s^{(0)}(x;x_1) - \left(\frac{x_1}{x}\right)^{\frac{1}{2}} \Gamma_{00}^{(0)}(x) \right.\right.\right. \\ & \left.\left. \times \frac{V_s^{(1)}(x;x_1)}{3} \right) \right\} \delta_{l,0} \delta_{m,0} + \frac{1}{6} \frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \left\{ x_1^{3/2} \left(\Gamma_{00}^{(0)}(x_1) \frac{V_s^{(2)}(x;x_1)}{5} \right.\right. \\ & \left.\left. - \left(\frac{x_1}{x}\right)^{\frac{1}{2}} \Gamma_{00}^{(0)}(x) \frac{V_s^{(3)}(x;x_1)}{7} \right) \right\} \frac{1}{3} \left(\frac{4}{5}\right)^{\frac{1}{2}} \delta_{l,2} \delta_{m,0} + \frac{1}{18} \frac{\mathrm{d}^2}{\mathrm{d}x_1^2} \times \end{aligned}$$

$$\times \left\{x_1^{3/2}\left(\Gamma_{00}^{(0)}(x_1)\, V_s^{(0)}(x;x_1) - \left(\frac{x_1}{x}\right)^{\frac{1}{2}} \Gamma_{00}^{(0)}(x)\right.\right.$$

$$\left.\left.\times\left(-\frac{4}{5}\frac{V_s^{(3)}(x;x_1)}{7} + \frac{9}{5}\frac{V_s^{(1)}(x;x_1)}{3}\right)\right)\right\} \delta_{l,0}\delta_{m,0}\Bigg]_{x_1=1} \qquad (32)$$

From (26), we observe that we need only $\Gamma_{00}^{(2)}(1)$. Putting together the various expressions, we finally obtain,

$$I(y) = \Gamma\Bigg[1 + \frac{1}{4}y^2\Bigg\{\Bigg[-\frac{d}{dx}(x^{\frac{1}{2}}\tilde{\Gamma}(x)) + \frac{2}{9}\frac{d^2}{dx^2}(x^{3/2}\tilde{\Gamma}(x))\Bigg]$$

$$+\left(\frac{2\alpha r_s}{\pi}\right)\Gamma\Bigg[-\frac{d}{dx}\left(x^{\frac{1}{2}}\tilde{\Gamma}(x)\, V_s^{(0)}(1;x) - x\frac{V_s^{(1)}(1;x)}{3}\right)$$

$$+\frac{2}{9}\frac{d^2}{dx^2}\Bigg(x^{3/2}\tilde{\Gamma}(x)\, V_s^{(0)}(1;x) + \frac{4}{5}x^2\frac{V_s^{(3)}(1;x)}{7}$$

$$-\frac{9}{5}x^2\frac{V_s^{(1)}(1;x)}{3}\Bigg)\Bigg]\Bigg\}_{x=1}\Bigg] \qquad (33$$

Using the explicit form for $\tilde{\Gamma}(x)$ this may be simplified further. We observe the following properties:

$$\tilde{\Gamma}(x=1) = 1, \qquad (34\,a)$$

$$\left(\frac{d\tilde{\Gamma}(x)}{dx}\right)_{x=1} = \left(\frac{2\alpha r_s}{\pi}\right) Z_1\left[\frac{d}{dx}\left(V_s^{(0)}(1;x) - x^{-\frac{1}{2}}\frac{V_s^{(1)}(1;x)}{3}\right)\right]_{x=1} \qquad (34\,b$$

$$\left(\frac{d^2\tilde{\Gamma}(x)}{dx^2}\right)_{x=1} = \left(\frac{2\alpha r_s}{\pi}\right) Z_1\left[\frac{d^2}{dx^2}\left(V_s^{(0)}(1;x) - x^{-\frac{1}{2}}\frac{V_s^{(1)}(1;x)}{3}\right)\right]_{x=1}$$

$$-2\left(\frac{2\alpha r_s}{\pi}\right)^2 Z_1^2\left[\frac{d}{dx}\left(V_s^{(0)}(1;x) - x^{-\frac{1}{2}}\frac{V_s^{(1)}(1;x)}{3}\right)\right]_{x=1}$$

$$\left[\frac{d}{dx}\left(x^{-\frac{1}{2}}\frac{V_s^{(1)}(1;x)}{3}\right)\right]_{x=1} \qquad (34\,c)$$

Here Z_1 is defined by

$$Z_1 = 1\Big/\left[1 + \left(\frac{2\alpha r_s}{\pi}\right)\frac{V_s^{(1)}(1;1)}{3}\right] \qquad (35)$$

Using the definition of Γ in combination with Z_1 defined here, we find the following relationship useful in clearing the expressions finally:

$$\Gamma = Z_1 + Z_1\Gamma\left(\frac{2\alpha r_s}{\pi}\right) V_s^{(0)}(1;1) \qquad (36)$$

After discussing the dynamical case in the limit $(mq_0/qk_F) \gg 1$ in the next section, we will simplify the above expressions further in the light of the expressions obtained using variational technique.

3. Plasma dispersion—$(mq_0/qk_F) \gg 1$ limit

The vertex equation to be considered now is the dynamical version of eq. (23):

$$\Gamma(x;y) = 1 + \left(\frac{2ar_s}{\pi}\right) \int_0^\infty x_1^{\frac{1}{2}}\, dx_1 \int \frac{d\hat{x}_1}{4\pi} V_s(|\mathbf{x} - \mathbf{x}_1|)$$

$$\left[\frac{f_0(x_1 + \frac{1}{2}y) - f_0(x_1 - \frac{1}{2}y)}{\nu - 2x_1^{\frac{1}{2}} y \cos\theta_{x_1}}\right]\left[\Gamma(x_1;y) - \left(\frac{\nu - 2yx_1^{\frac{1}{2}} \cos\theta x_1}{\nu - 2yx^{\frac{1}{2}}\cos\theta_x}\right)\Gamma(x;y)\right] \tag{37}$$

Now we expand as follows:

$$f_0(x_1 + \tfrac{1}{2}y) - f_0(x_1 - \tfrac{1}{2}y) \cong (-2yx_1^{\frac{1}{2}}\cos\theta_{x_1})\{\delta(x_1 - 1) + \tfrac{1}{4}y^2 \times [\delta'(x_1 - 1) + \tfrac{2}{3}x_1\cos^2\theta_{x_1}\,\delta''(x_1 - 1)]\} \tag{38 a}$$

$$1/(\nu - 2yx_1^{\frac{1}{2}}\cos\theta_{x_1}) \cong \frac{1}{\nu}\left[1 + \frac{2yx_1^{\frac{1}{2}}\cos\theta_{x_1}}{\nu} + \dots\right] \tag{38 b}$$

We work to order $(2y/\nu)^2$ only, as this is sufficient to determine the plasma dispersion relation in the long wavelength region. From (37) it is clear that

$$\Gamma(x; y = 0, \nu) = 1. \tag{39}$$

We could write most generally the expansion

$$\Gamma(x;y) = 1 + \sum_{l,m}\left[\left(\frac{2y}{\nu}\right)\Gamma_{lm}^{(1,0)}(x) + \left(\frac{2y}{\nu}\right)^2\Gamma_{lm}^{(2,0)}(x) + y^2\left(\frac{2y}{\nu}\right)\Gamma_{lm}^{(1,2)}(x) + y^2\left(\frac{2y}{\nu}\right)^2\Gamma_{lm}^{(2,2)}(x) + \dots\right] y_{lm}(\hat{x}).$$

By studying the right side of eq. (37) a bit carefully we find that

$$\Gamma_{lm}^{(1,0)}(x) = 0, \qquad \Gamma_{lm}^{(1,2)}(x) = 0 \quad \text{for all } l, m \tag{40}$$

so that we may write

$$\Gamma(x;y) = 1 + \sum_{l,m}\left[\left(\frac{2y}{\nu}\right)^2\Gamma_{lm}^{(2,0)}(x) + y^2\left(\frac{2y}{\nu}\right)^2\Gamma_{lm}^{(2,2)}(x) + \dots\right] y_{lm}(\hat{x}) \tag{41}$$

Calculating $I(q)$ to order $(2y/\nu)^2$ we obtain $\epsilon_L(q)$:

$$\epsilon_L(q) \cong 1 - \left(\frac{16ar_s}{3\pi}\right)\frac{1}{\nu^2}\left\{1 + \frac{12}{5}\left(\frac{y}{\nu}\right)^2 + \frac{1}{\pi}\left(\frac{y}{\nu}\right)^2\left[\left(\frac{16\pi}{5}\right)^{\frac{1}{2}}\Gamma_{20}^{(2,0)}(1) + (4\pi)^{\frac{1}{2}}\Gamma_{00}^{(2,0)}(1)\right]\right\} \tag{42}$$

We observe therefore that $\Gamma_{lm}^{(2,\,2)}$ is not required. A simple calculation shows that $\Gamma_{lm}^{(2,\,0)}(x)$ is given by

$$\Gamma_{lm}^{(2,\,0)}(x) = \left(\frac{2\alpha r_s}{\pi}\right)\left\{\frac{V_s^{(1)}(1\,;1)}{3}\,x^{\frac{1}{2}}\left[\left(\frac{16\pi}{5}\right)^{\frac{1}{2}}\delta_{l,\,2}\delta_{m,\,0} + (4\pi)^{\frac{1}{2}}\,\delta_{l,\,0}\delta_{m,\,0}\right]\right.$$
$$\left. - \frac{V_s^{(2)}(1\,;x)}{5}\,\frac{1}{3}\left(\frac{16\pi}{5}\right)^{\frac{1}{2}}\delta_{l,\,2}\delta_{m,\,0} - V_s^{(0)}(1\,;x)\,\frac{(4\pi)^{\frac{1}{2}}}{3}\,\delta_{l,\,0}\delta_{m,\,0}\right\} \tag{43}$$

The plasma frequency in the units employed here is given by

$$\nu_{Pl}^2 = \frac{16}{3}\left(\frac{\alpha r_s}{\pi}\right) \tag{44}$$

Thus the plasma dispersion is obtained by setting $\epsilon_L(q) = 0$ and thus we obtain

$$\nu^2(q) \cong \nu_{Pl}^2\left\{1 + \frac{9}{20}\,\frac{\pi}{\alpha r_s}\,y^2 + \right.$$
$$\left. + \frac{9}{2}\,y^2\left[\frac{1}{5}\,\frac{V_s^{(1)}(1\,;1)}{3} - \frac{4}{45}\,\frac{V_s^{(2)}(1\,;1)}{5} - \frac{1}{9}\,V_s^{(0)}(1\,;1)\right]\right\} \tag{45}$$

In the next section we will rewrite the variational answers obtained before in the new notation so as to facilitate comparison.

4. Results based on variational method

4.1 *Orbital susceptibility*

This case is particularly simple to discuss because the exact result, eq. (21), coincides with the variational answer precisely. This derivation is given (Rajagopal and Jain 1972) and we shall not repeat it here.

4.2. *Longitudinal dielectric function and spin susceptibility* (*Static case*)

As in the last section, we consider $I(q)$ only. The variational answer is given by

$$I_{\rm var}(q) = I_0^2(y)/[I_0(y) - J(y)] \tag{46 a}$$

Here

$$I_0(y) = \int_0^\infty x^{\frac{1}{2}}\,dx \int \frac{d\hat{x}}{4\pi}\,\mathcal{F}(x, y) \tag{46 b}$$

and

$$J(y) = \left(\frac{2\alpha r_s}{\pi}\right)\int_0^\infty x^{\frac{1}{2}}\,dx \int \frac{d\hat{x}}{4\pi}\int_0^\infty x_1^{\frac{1}{2}}\,dx_1 \int \frac{d\hat{x}_1}{4\pi}$$
$$\mathcal{F}(x, y)\;V_s(\,|\,\mathbf{x} - \mathbf{x}_1\,|\,)\,\mathcal{F}(x_1, y)\left[1 - \left(\frac{x_1}{x}\right)^{\frac{1}{2}}\frac{\cos\theta_{x1}}{\cos\theta_x}\right] \tag{46 c}$$

We evaluate $I_{\rm var}(q)$ to second power in q as before,

$$I_0(y) \cong I_0^{(0)} + y^2 I_0^{(2)},\; J(y) = J_0 + y^2 J_2$$

so that to that order we have

$$I_{\rm var}(q) \cong \frac{1}{4\pi} \frac{I_0^{(0)}}{[1 - J_0/I_0^{(0)}]} \left\{1 + y^2 \left(2 \frac{I_0^{(2)}}{I_0^{(0)}} - \frac{[I_0^{(2)} - J_2]}{[I_0^{(0)} - J_0]}\right)\right\} \tag{47}$$

The integrals $I_0^{(0)}$, J_0 have been evaluated before (Rajagopal *et al* 1966, Rajagopal and Grest 1974). The others will be evaluated here

$$I_0^{(0)} = 4\pi;\; J_0 = 4\pi \left(\frac{2\alpha r_s}{\pi}\right) \left[V_s^{(0)}(1;1) - \frac{V_s^{(1)}(1;1)}{3}\right],$$

$$I_0^{(2)} = 4\pi(-\ 1/12) \tag{48}$$

We observe that

$$\Gamma \equiv 1/[1 - J_0/I_0^{(0)}], \tag{49}$$

and

$$I_{\rm var}(q) \cong \Gamma \left\{1 + y^2 \left[-\frac{1}{6} + \Gamma\left(\frac{1}{12} + \frac{J_2}{4\pi}\right)\right]\right\} \tag{50}$$

Now

$$J_2 = \frac{1}{4}\left(\frac{2\alpha r_s}{\pi}\right) \int_0^\infty x^{\frac{1}{2}}\,{\rm d}x \int_0^\infty x_1^{\frac{1}{2}}{\rm d}x_1 \int {\rm d}\hat{x} \int {\rm d}\hat{x}_1 \sum_{l,m} \frac{V_s^{(l_1)}(x;x_1)}{2l_1+1}$$

$$y_{l_1m_1}(\hat{x})\, y_{l_1m_1}^{*}(\hat{x}_1) \left[2 - \left(\frac{x_1}{x}\right)^{\frac{1}{2}} \frac{\cos\theta_{x_1}}{\cos\theta_x} - \left(\frac{x}{x_1}\right)^{\frac{1}{2}} \frac{\cos\theta_x}{\cos\theta_{x_1}}\right] \delta(x_1 - 1)$$

$$[\delta'(x-1) + \tfrac{2}{3}\, x \cos^2\theta\, \delta''(x-1)]$$

The integrals can all be done in a straightforward way and the result is

$$J_2 = 4\pi\left(\frac{2\alpha r_s}{\pi}\right)\frac{1}{4}\left\{-\frac{\rm d}{{\rm d}x}\left[2x^{\frac{1}{2}} V_s^{(0)}(1;x) - (x+1)\frac{V_s^{(1)}(1;x)}{3}\right]\right.$$

$$+ \frac{2}{9}\frac{{\rm d}^2}{{\rm d}x^2}\left[2x^{3/2} V_s^{(0)}(1;x) - \frac{9}{5}x^2 \frac{V_s^{(1)}(1;x)}{3} - x\frac{V_s^{(1)}(1;x)}{3}\right.$$

$$\left.\left. + \frac{4}{5}x^2 \frac{V_s^{(3)}(1;x)}{7}\right]\right\}_{x=1} \tag{51}$$

We finally obtain the result

$$I_{\rm var}(q) \cong \Gamma\left\{1 + \frac{1}{4}y^2\left[-\frac{2}{3} + \Gamma\left(\frac{1}{3} +\right.\right.\right.$$

$$+\left(\frac{2\alpha r_s}{\pi}\right)\left\{-\frac{\rm d}{{\rm d}x}\left(2x^{\frac{1}{2}} V_s^{(0)}(1;x) - (x+1)\frac{V_s^{(1)}(1;x)}{3}\right) +\right.$$

$$+\frac{2}{9}\frac{{\rm d}^2}{{\rm d}x^2}\left(2x^{3/2} V_s^{(0)}(1;x) - \frac{9}{5}x^2\frac{V_s^{(1)}(1;x)}{3} - x\frac{V_s^{(1)}(1;x)}{3} +\right.$$

$$\left.\left.\left.\left.\left. + \frac{4}{5}x^2\frac{V_s^{(3)}(1;x)}{7}\right)\right\}_{x=1}\right)\right]\right\} \tag{52}$$

We will now combine the exact answer given by eq. (33) and this result after using the various relationships eqs (34 *a*, *b*, *c*) and (36) to clear up the terms to bring them into a neat form. The trick is to directly subtract $I_{\rm var}(q)$ given by eq. (52) from $I(q)$ given by eq. (33) and then simplify. After a great deal of cancellations one obtains,

$$I(q) = I_{\rm var}(q) + y^2 \frac{\Gamma^2}{9} \left(\frac{2\alpha r_s}{\pi}\right)^2 Z_1 \left[\frac{d}{dx}\left(V_s^{(0)}(1;x) - x^{-\frac{1}{2}} \frac{V_s^{(1)}(1;x)}{3}\right)\right]^2_{x=1} \tag{53}$$

This is indeed a remarkable result in that it exhibits the relation of the exact answer with the variational one and spells out clearly the correction term. We will discuss this in some detail in the next section.

4.3 *Plasma dispersion*

The plasma dispersion relation is deduced by using the generalized form of $I_{\rm var}(q)$ which is given by

$$I_{\rm var}(q) = I_0^2(q, \nu)/[I_0(q, \nu) - J(q, \nu)] \tag{54}$$

The calculation of $I_0(y, \nu)$ and $J(y, \nu)$ up to $(y/\nu)^2$ order are given earlier (Rajagopal *et al* 1973, Rajagopal and Grest 1974). We shall therefore not repeat the calculation here. It is found that the exact answer for the plasma dispersion relation eq. (45) is also obtained in the variational calculation.

5. Summary and discussion of the results

The expressions for $\chi_{\rm orb}(0\,;\,0)$, eq. (19); for $I(y)$, eq. (26); and for $\epsilon_L(q, \nu)$, eq. (42), are exact in the respective regions of interest and are new. They are expressed in terms of respective vertex functions which contain all the effects of interactions in their most general forms. In this paper we have confined our investigation to mutual electron interactions only and one may incorporate in the formalism the electron-phonon interactions as is done for example by Joshi and Rajagopal (1968). Even here, the exact equations obeyed by the vertex functions are not known in their most general form. The most common approximation employed in the literature is the random phase approximation including exchange processes. This approximation leads to a linear integral equation for the vertex function. In this paper we have solved this equation exactly and the various physical quantities are computed in some physically interesting regions of the frequency wave vector space. The striking feature of this solution is its close relationship with the longitudinal dielectric function and the spin susceptibility are determined by the same vertex function in the random phase approximation which includes contributions from exchange processes in the lowest order. Since these two functions represent the spin singlet and triplet correlations in this system, the above observation clearly indicates that the antiparallel correlations are entirely neglected in this approximation. In the next scheme, known as the ladder-bubble approximation, which incorporates some of the antiparallel spin correlation, the vertex equations associated with the longitudinal dielectric function and the spin susceptibility become non-linear. Only recently these equations have been investigated using a new variational technique. A preliminary account of the results of such a calculation is reported in a short communication (Rau and Rajagopal 1975 *a*) and we plan to present an expository account of this method in a longer paper (Rau and Rajagopal 1975 *b*). This calculation shows clearly that the two correlation functions are indeed different in their structure and in particular in the static long wavelength limit, they exhibit a dependence on the density of the system in markedly different forms.

Within the linearized theory for the vertex function, a new scheme for computing the correlation functions considered above has been introduced recently (Shastry *et al* 1974). The method consists in assuming the Yukawa form eq. (16), for the effective interaction potential. The screening constant ξ is then determined self-consistently using a Ward identity. The results of such a calculation, even though interesting, do not lead to a proper treatment of the important antiparallel correlations in the system. We are presently investigating the possibility of extending this attractive scheme to cover this aspect as well. Another feature that is missed by the above self-consistent scheme is that in the extreme low density limit, the screening does not go to zero, as is expected physically. In reality, the screened exchange potential is dependent on the wave vector in a more complicated form than the Yukawa form.

The application of the results obtained for I (q) to the inhomogeneous electron system has been made recently (Rajagopal and Ray 1975). We may conclude that within the RPA scheme including exchange processes, we have been able to make precise calculations of quantities of physical interest. Such a scheme, even though neglects the antiparallel spin correlations, may be of some use in the high density regime, at least. An important result that emerged out of this is that the gradient term vanishes for both the very high and the very low densities, unlike the previous calculations which were correct on one or the other limit only.

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