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Phenomenology of S and P pion-pion partial wave amplitudes

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Abstract. A scheme of parametrization of $\pi\pi$ S and P partial wave amplitudes is proposed. Analytic, crossing and unitarity properties are rigorously satisfied. The analyticity property is made explicit by the use of a conformally mapped variable. Our expressions fit excellently with both phase shift and inelastic curves for centre of mass energy up to 1·4 GeV.

Keywords. S and P partial waves; crossing; unitarity; analyticity; conformally mapped variable.

1. Introduction

Recently Basdevant *et al* (1973) have applied Roy's equations (Roy 1971) to $\pi\pi$ phenomenology. They have constructed the $\pi\pi$ amplitudes by using the conformal mapping technique (Cutkosky 1969) and have obtained a good fit for the high statistics data (Protopopescu *et al* 1973) for the phase shifts and inelasticities. However, their parametrization suffers from one serious drawback. The inelasticity, presumably weak, starts (Martin 1969) from the threshold of the 4π channel. But in constructing the mapped variable for expansion they have completely neglected the part of cut $16m_\pi^2 \leqslant s \leqslant 4m_K^2$. The aim is to account for the fact that η_0^0 remains close to unity for $16m_\pi^2 \leqslant s \leqslant 4m_K^2$. Even with this specific assumption with regards to inelasticities they need quite a large number of parameters. For $S = 0$, $I = 0$ wave the total number of parameters, including fixed, driving and variable parameters, is 12.

We propose to improve on their phenomenology by choosing a suitably mapped plane variable, which takes into account the effect of the 4π cut and also shows a dip in the inelasticity, η_0^0, at the $K\bar{K}$ threshold. Essentially, we map the cut plane in such a way that the cuts map onto the boundary of a strip, enclosing the entire plane of analyticity with the $K\bar{K}$ threshold, $s = 4m_K^2$, pushed to infinity. The reason will be made clear in section 2. Recently such strip mapping has been used to advantage by Chao (1973) for π-N scattering. The partial waves are then expanded as polynomials in mapped variable and the parameters are fixed by a least χ^2 fit using the phase shifts, inelasticities and crossing constraints as inputs. It turns out that we need a lot less number of parameters. A CDD pole (Castillejo *et al* 1956) has been introduced in the $I = 0$ S-wave to take into account the shooting up of δ_0^0 through 180^0 near $K\bar{K}$ threshold. Such an addition, after a polynomial expansion of the partial waves in mapped variable seems to be permissible (Atkinson *et al* 1972, Pisut 1970), at least for the S-wave. It has been shown by Atkinson *et al* (1972) that it does not affect the convergence and analytic properties of the expansion. Addition of a CDD pole to $I = 2$, S-wave, though possible, is not used here as $I = 2$ is, perhaps, truely exotic. Results obtained agree well with the experimental data up to 1·4 GeV. An important result of our analysis is that the partial wave amplitudes constructed by us satisfy the unphysical region constraints besides the unitarity constraint of the physical region. So we obtain definite values of scattering lengths and also compute the position of zeroes of the $I = 0$ and $I = 2$ S-waves in the unphysical region. Thus, a satisfactory parametrization is obtained for $0 \leqslant s < 2\ \mathrm{GeV}^2$.

The plan of the paper is as follows. In section 2 we discuss the scheme of parametrization. Section 3 contains our search procedure. In section 4 we give our results and discussion on the same. A list of theoretical constraints is also added at the end in the form of an appendix along with our calculated values.

2. Schemes for parametrization

Since the N/D method (Chew and Mandelstam 1960, Uretsky 1961) linearises the nonlinear integral equation derived from unitarity and the partial wave dispersion relation with a finite number of subtraction, we choose the representation

$$f_l^I(s) = \frac{k^{2l} N_l^I(s)}{1 + N_l^I(s)\, D_l^I(s)} \tag{1}$$

where $N_l^I(s)$ and $D_l^I(s)$ have only the left hand cut and right hand cut, respectively. To ensure elastic unitarity we define

$$D_l^I(s) = H_l^I(s) + k^{2l}\, h(s) \tag{2}$$

where $h(s)$ is the usual Chew-Mandelstam (1960) function

$$h(s) = \frac{k}{2\sqrt{s}}\left[\frac{2}{\pi}\log(\sqrt{s} + k) - i\right]; \quad s > 1 \tag{3}$$

$$h(s) = \frac{k}{\pi\sqrt{s}}\,\mathrm{Artg}\,\frac{\sqrt{s}}{k}; \quad 0 \leqslant s \leqslant 1 \tag{4}$$

in a unit in which $s = k^2 + 1$ and as such is a real analytic function whose only singularity is a cut from $s = 1$ to $s = \infty$, with discontinuity $\rho(s) = -k/2\sqrt{s}$. $H_l^I(s)$ is a meromorphic function except for the right hand inelastic cuts. To construct $H_l^I(s)$ and $N_l^I(s)$ we take the help of the optimized polynomial technique due to Cutkosky and Deo (1968 *a*, *b*). In actuality, the technique is to select, from functions of a given class, a rapidly convergent sequence of functions for use in fitting experimental data. The sequence is derived by subjecting the cut s-plane to a conformal mapping, which maps the domain of analyticity of f_l onto the domain of convergence of the polynomial sequence. Cutkosky (1973) and Ciulli (1972) have also shown that the stability of the derived parameters against errors in the data depends directly on the rate of convergence. Therefore, the partial wave thus constructed will converge optimally because no region of analyticity will be left out of the region of convergence by the mapping.

2.1. *Parametrization of* H_0^0

In parametrizing H_0^0 we take explicit account of the fact that the inelastic channel sets in at $s = 4$. However η_0^0 becomes observably less than unity only after the threshold for the opening up of the $K\bar{K}$ channel. This effective right hand inelastic cut structure for $I = 0$ S-wave is shown in figure 1 *a*.

As a first step to use the optimized polynomial technique we make the simple mapping,

$$W_1 = \frac{s + s_{K\bar{K}} - 8}{s_{K\bar{K}} - s} \tag{5}$$

where $s_{K\bar{K}}$, though behaves as a free parameter, is still the s value in the vicinity of the $K\bar{K}$ threshold. This symmetrizes the cuts from $s = 4$ to $s = s_{K\bar{K}}$ and from $s = s_{K\bar{K}}$ to $s = \infty$ on the real axis of the complex W_1 plane, (figure 1 *b*). As,

$$\eta_0^0 \approx 1 \cdot 0; \quad s \leqslant s_{K\bar{K}} \tag{6}$$

it shows that the imaginary part of $[f_0^0]^{-1}$ for $s \leqslant s_{K\bar{K}}$ is very nearly zero except for the elastic unitarity term,

$$I_m \, [f_0^0]^{-1} = \rho(s) \tag{7}$$

This suggests the strip mapping,

$${}_1Z_+ = \sin^{-1} W_1 \tag{8}$$

This maps the analytic cut W_1 plane into the interior of a strip (figure 1 *c*), in the complex, ${}_1Z_+$ plane. Since the domain of convergence of the Hermite polynomial is a strip we construct the function H_0^0 in terms of the Hermite polynomials, H_n. Assuming $D(s)$ to go to ∞ as s, we write,

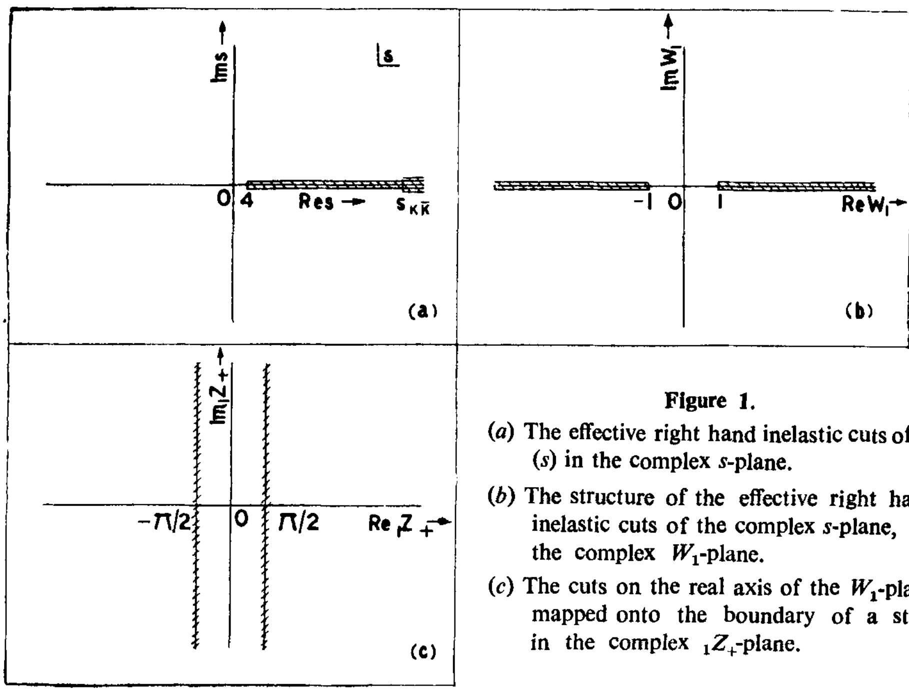

Figure 1.
(*a*) The effective right hand inelastic cuts of f_0^0 (*s*) in the complex *s*-plane.
(*b*) The structure of the effective right hand inelastic cuts of the complex *s*-plane, in the complex W_1-plane.
(*c*) The cuts on the real axis of the W_1-plane mapped onto the boundary of a strip in the complex ${}_1Z_+$-plane.

$$H_0^0 = \sum_{n=0}^{\infty} b_{0,n}^0 H_n + \sum_{n=0}^{\infty} s\,\delta_{0,n} C_{0,n}^0 H_n + \frac{\lambda}{s - s_{K\bar{K}}} \tag{9}$$

The third term is a CDD pole term with position $s_{K\bar{K}}$ and strength λ.

To incorporate the constraint, eq. (6), it is found convenient to determine the coefficients occurring in eq. (9) by writing H_0^0 in a way that the imaginary parts cancel for $4 \leqslant s \leqslant 4m_K^2$. Retaining three terms in the first series, a relation between $b_{0,1}^0$ and $b_{0,2}^0$ can be deduced for the above cancellation. If future accurate experiments show small inelasticities this relation will be weakly violated. Thus we rewrite H_0^0 in the following way,

$$H_0^0 = b_{0,0}^0 H_0 + b_{0,1}^0 H_1 - \frac{b_{0,1}^0}{\pi} H_2 + \ldots + C_{0,0}^0 s + \frac{\lambda}{s - s_{K\bar{K}}} \tag{10}$$

Note that $b_{0,2}^0 = - b_{0,1}^0/\pi$.

Addition of more terms did not improve χ^2.

2.2. *Parametrization of H_0^2, H_1^1 and N_l^I*

Since the $K\bar{K}$ threshold does not produce any appreciable effect in other channels

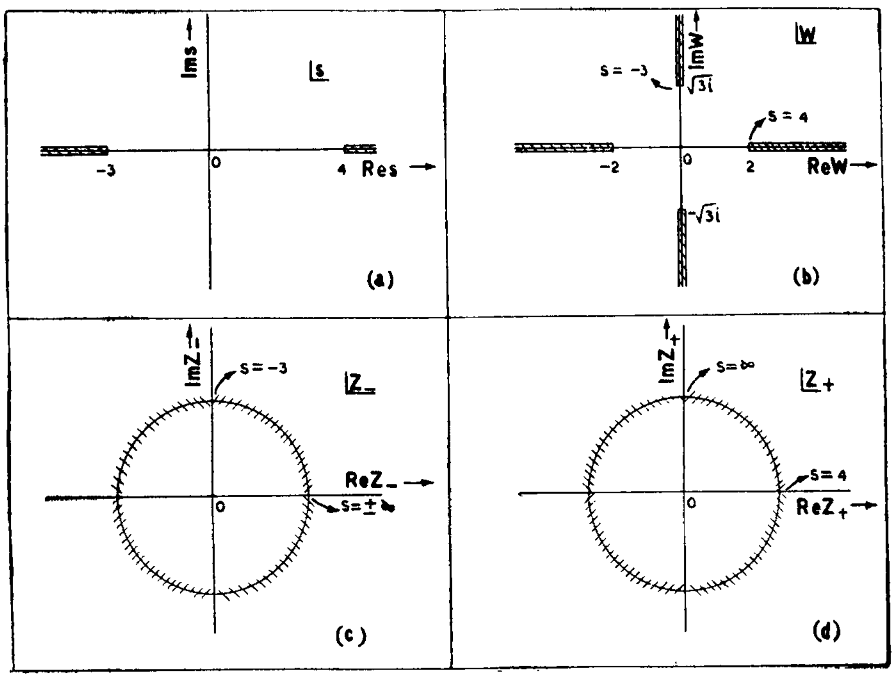

Figure 2. (*a*) The effective inelastic cuts on the complex *s*-plane of $f_0^2(s)$ and $f_1^1(s)$. (*b*) The structure of the inelastic cuts of the *s*-plane in the complex *W*-plane. (*c*) The cuts on the imaginary axis of the *W*-plane mapped onto the boundary of a circle of radius $(2+\sqrt{3})$ in the Z_--plane. (*d*) The cuts on the real axis of the *W*-plane mapped onto the boundary of a circle of radius $(2+\sqrt{3})$ in the Z_+-plane.

except $I = 0$ (Basdevant *et al* 1973) it is enough to take into account the right hand inelastic cut from $s = 4$ to $s = \infty$, for constructing H_0^2 and H_1^1. We make the mapping

$$W = \sqrt{s} \tag{11}$$

which symmetrizes separately the left hand inelastic cut from $s = -3$ to $s = -\infty$ and the cut from $s = 4$ to $s = \infty$, (figure 2 *a*), respectively, on the imaginary and real axes of the complex W plane on both sheets (figure 2 *b*).

The symmetrical cuts appearing on imaginary axis of the W plane are mapped onto the boundary of a circle of radius $2 + \sqrt{3}$ by the mapping (figure 2 *c*).

$$Z_- = (\sqrt{3 + W^2} - \sqrt{3})(2 + \sqrt{3})/W \tag{12}$$

Next the cuts on the real axis of the W plane are mapped onto the circumference of a circle of the same radius, $(2 + \sqrt{3})$, by the mapping (figure 2 *d*),

$$Z_+ = (2 + i\sqrt{W^2 - 4})(2 + \sqrt{3})/W \tag{13}$$

In both the cases the radii, $(2 + \sqrt{3})$, of the circles are so chosen that the unphysical region (0, 1) of the s-plane is now mapped onto the interval (0, 1) of the mapped plane.

The functions N_l^I and H_l^I (for $I = 1$ and $I = 2$) can be constructed as expansion in polynomials of Z_- and Z_+ respectively. H_l^I (for $I = 1$ and $I = 2$) is written as,

$$H_l^I = \sum_{n=0}^{\infty} b_{l,n}^I Z_+^n + \sum_{n=0}^{\infty} s\delta_{0,n} C_{l,n}^I Z_+^n \tag{14}$$

The second term shows that we have again used one subtraction*. N_l^I can be parametrized as,

$$N_l^I = \sum_{n=0}^{\infty} C_{l,n}^{I'} Z_-^n (1 - \delta_{n,1}) \tag{15}$$

The multiplicative factor $(1 - \delta_{n,1})$ has been introduced because the analytic constraints near $s = 0$, are,

$$\begin{vmatrix} \operatorname{Im} f_0^0 \\ \operatorname{Im} f_0^2 \\ \operatorname{Im} f_1^1 \end{vmatrix} = \tfrac{2}{3}(-s)^{3/2} \begin{vmatrix} -\frac{1}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & -\frac{5}{6} \end{vmatrix} \begin{vmatrix} |f_0^0(1)|^2 \\ \\ |f_0^2(1)|^2 \end{vmatrix} \tag{16}$$

They suggest that any term containing Z_- should be absent from the expansion (15). We found it convenient to determine the coefficients by writing N_l^I as a series of the following form in Z_-.

$$N_l^I = C_{l,0}^{I'} + C_{l,2}^{I'} Z_-^I + C_{l,3}^{I'} Z_-^2 (1 - Z_-) + \dots \tag{17}$$

3. Search Procedure

The parameters were fixed by a least χ^2 fit using a random search programme,

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* One can add a CDD pole to this expansion as in the case of $I = 0$, S-wave. However, such an addition for the P-wave will affect convergence (Atkinson 1972) of the expansion. On the other hand, by adding a CDD pole to $I = 2$, S-wave it was observed that it leads to serious troubles in so far as making a simultaneous fit to the phase shifts and inelasticities is concerned.

where the data points are the experimental phase shifts, inelasticities and also the theoretical constraints in the unphysical region, $0 \leqslant s \leqslant 1$. To minimise errors in truncating our optimized polynomial expansion of the partial waves, we use the recipe of Cutkosky's CTF (Cutkosky 1972). Thus, one is safe in taking conservatively large number of parameters without sacrificing the uniqueness and stability of the fit. This means that even though one takes a large number of parameters in the expansion the number of degrees of freedom remains practically the same. In the literature one can already find many fruitful application of this with excellent results (Chao 1970, 1971; Cutkosky and Shih 1971; Miller *et al* 1972).

After some random searches it was soon realised that primarily N_l^I controls the shapes of the partial waves in the unphysical region where as D_l^I becomes the main contributing factor in the physical region. The constraints in the region $0 \leqslant s \leqslant 1$ were easily satisfied to a good degree of accuracy by the first three terms in N_l^I as written in eq. (17). The first two parameters give essentially the position and strength of the zeroes of the partial waves for $0 \leqslant s \leqslant 1$. The third parameter is needed to satisfy the constraints (16).

Since the phase shifts and inelasticities will primarily fix the parameters of D_l^I and since N_l^I is a real function for $s \geqslant 0$ one has to take at least two terms in H_l^I, apart from the subtraction term, so that one can hope to take account of any inelasticity, however small, present for $s > 16m_\pi^2$. Over and above this one more term in H_0^0, as given in eq. (10), is taken to ensure the experimental constraint, eq. (6). It was seen, with completely arbitrary starting parameters, that the contribution from higher terms in H_0^0 to the fit was negligible and thus one can safely truncate the expansion for H_0^0 at the third term as given in eq. (10). As regards H_0^2 and H_1^1, it was observed that higher terms in the expansion of H_l^I have to be taken so that one can keep η_0^2 and $\eta_1^1 \leqslant 1$. Searches for $s_{K\bar{K}}$ is the vicinity of $K\bar{K}$ threshold with completely arbitrary λ were done.

In our analysis the χ^2/NDF ratio for the S-wave is only 0·78 and for the P-wave is 1·1. Only the final results are reported here.

4. Results and Discussion

4.1. *Results in the region* $0 \leqslant s \leqslant 1$

(i) Defining the S-wave neutral π–π amplitude as

$$f_0 = \tfrac{1}{3}(f_0^0 + 2f_0^2) \tag{18}$$

we make a plot of the same along with those obtained by Krinsky, Kr, (1970) Kang and Lee, KL, (1971) and Bonnier and Gauron, BG, (1972) in figure 3. Our curve shows a minimum at $s = 0{\cdot}364$, where the curves of Kr, KL and BG show minimum at $s = 0{\cdot}41$, $s = 0{\cdot}407$, $s = 0{\cdot}40$ respectively. It is also seen from figure 3 that in our case $f_0(1) > f_0(s)$ for $0 \leqslant s < 1$. Thus Martin's (1967) inequalities (M–4), (M–5) and (M–6), (See appendix), are automatically satisfied. The value of the other three Martin's inequalities are given in table 1.

(ii) The results of the two sides of Roskies' (1969) sum rules are given in table 2. The first four sum rules are satisfied within an error of 0·5 per cent.

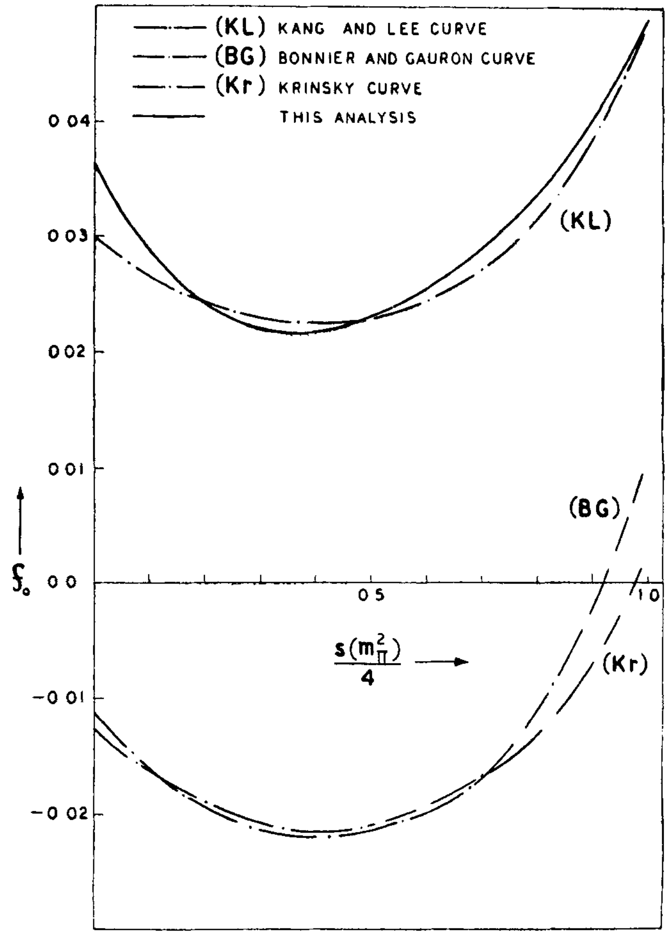

Figure 3. Behaviour of the neutral amplitude $f_0(s)$ as obtained from this analysis in the unphysical region $0 \leqslant s \leqslant 1$ is compared with those obtained by Kang and Lee (1971), Bonnier and Gauron (1972) and Krinsky (1970).

Table 1. Numerical values of Martin's inequalities, M–1, M–2 and M–3

Martin's Inequality	L.H.S.	R.H.S.
M–1	0·03409	0·03160
M–2	0·03160	0·03130
M–3	0·03643	0·03335

Table 2. Numerical values of Roskies' sum rules

Roskies' sum rule	L.H.S.	R.H.S.
R–1	$0\cdot2202\times10^{-1}$	$0\cdot2198\times10^{-1}$
R–2	$0\cdot1844\times10^{-1}$	$0\cdot1846\times10^{-1}$
R–3	$0\cdot27714\times10^{-1}$	$0\cdot27716\times10^{-1}$
R–4	$0\cdot6092\times10^{-2}$	$0\cdot6090\times10^{-2}$
R–5	$0\cdot4504\times10^{-3}$	$0\cdot3609\times10^{-3}$

Table 3 A. Numerical values of the left hand side of Pennington's inequalities from P-1 to P-7

P-1	$0\cdot26858\times10^{-3}$	P-4	$0\cdot73486\times10^{-4}$
P-2	$0\cdot43034\times10^{-3}$	P-5	$0\cdot13952\times10^{-4}$
P-3	$0\cdot10691\times10^{-3}$	P-6	$0\cdot10289\times10^{-4}$
P-7	$0\cdot17616\times10^{-4}$		

Table 3 B. Numerical values of the left hand side of Pennington's inequalities from P-8 to P-13

ϵ	P-8	P-9	P-10	P-11	P-12	P-13
0·0	$0\cdot3332\times10^{-4}$	$0\cdot8344\times10^{-3}$	$0\cdot9638\times10^{-4}$	$0\cdot7024\times10^{-4}$	$0\cdot3568\times10^{-3}$	$0\cdot2710\times10^{-3}$
0·1	$0\cdot1733\times10^{-4}$	$0\cdot5266\times10^{-3}$	$0\cdot4441\times10^{-4}$	$0\cdot4226\times10^{-4}$	$0\cdot2761\times10^{-3}$	$0\cdot2029\times10^{-3}$
0·2	$0\cdot1208\times10^{-4}$	$0\cdot3362\times10^{-3}$	$0\cdot3548\times10^{-4}$	$0\cdot2495\times10^{-4}$	$0\cdot2062\times10^{-3}$	$0\cdot1454\times10^{-3}$
0·3	$0\cdot1758\times10^{-4}$	$0\cdot2634\times10^{-3}$	$0\cdot6958\times10^{-4}$	$0\cdot1833\times10^{-4}$	$0\cdot1469\times10^{-3}$	$0\cdot9865\times10^{-4}$
0·4	$0\cdot3382\times10^{-4}$	$0\cdot3080\times10^{-3}$	$0\cdot1467\times10^{-3}$	$0\cdot2239\times10^{-4}$	$0\cdot9852\times10^{-4}$	$0\cdot6255\times10^{-4}$
0·5	$0\cdot6080\times10^{-4}$	$0\cdot4701\times10^{-3}$	$0\cdot2668\times10^{-3}$	$0\cdot3713\times10^{-4}$	$0\cdot6080\times10^{-4}$	$0\cdot3713\times10^{-4}$
0·6	$0\cdot9852\times10^{-4}$	$0\cdot7498\times10^{-3}$	$0\cdot4300\times10^{-3}$	$0\cdot6255\times10^{-4}$	$0\cdot3382\times10^{-4}$	$0\cdot2239\times10^{-4}$
0·7	$0\cdot1469\times10^{-3}$	$0\cdot1146\times10^{-2}$	$0\cdot6363\times10^{-3}$	$0\cdot9865\times10^{-4}$	$0\cdot1758\times10^{-4}$	$0\cdot1833\times10^{-4}$
0·8	$0\cdot2062\times10^{-3}$	$0\cdot1661\times10^{-2}$	$0\cdot8856\times10^{-3}$	$0\cdot1454\times10^{-3}$	$0\cdot1208\times10^{-4}$	$0\cdot2495\times10^{-4}$
0·9	$0\cdot2761\times10^{-3}$	$0\cdot2293\times10^{-2}$	$0\cdot1177\times10^{-2}$	$0\cdot2029\times10^{-3}$	$0\cdot1733\times10^{-4}$	$0\cdot4226\times10^{-4}$
1·0	$0\cdot3568\times10^{-3}$	$0\cdot3043\times10^{-2}$	$0\cdot1513\times10^{-2}$	$0\cdot2710\times10^{-3}$	$0\cdot3332\times10^{-4}$	$0\cdot7024\times10^{-4}$

The fifth sum rule is valid to within 20 per cent which is much better than those obtained from other models (Krinsky 1970, Bonnier and Gauron 1970).

(iii) The two inequalities referred in Kang's work (Kang *et al* 1971) are satisfied in the following way by our neutral $\pi\pi$ S-wave.

$$f_0\,(0\cdot07342) > f_0\,(0\cdot60565)\,; \qquad f_0\,(0\cdot84717) > f_0\,(0\cdot12007)$$
$$0\cdot03015 > 0\cdot02570; \qquad 0\cdot03671 > 0\cdot02734$$

(iv) The values of the large number of Pennington's (1971) inequalities as obtained from our analysis are given in table 3.

4.2. *Results in the region* $s > 1$

(i) The $I = 0$ S-wave phase shift as obtained from our fit is plotted in figure 4. It agrees well with the latest experimental data (Protopopescu *et al* 1973). The best fit analysis gives the position of CDD pole at $s = 12\cdot57$ with a strength $\lambda = 0\cdot39$.

(ii) Our values for $I = 2$ S-wave phase shifts are plotted in figure 5. They remain small and repulsive.

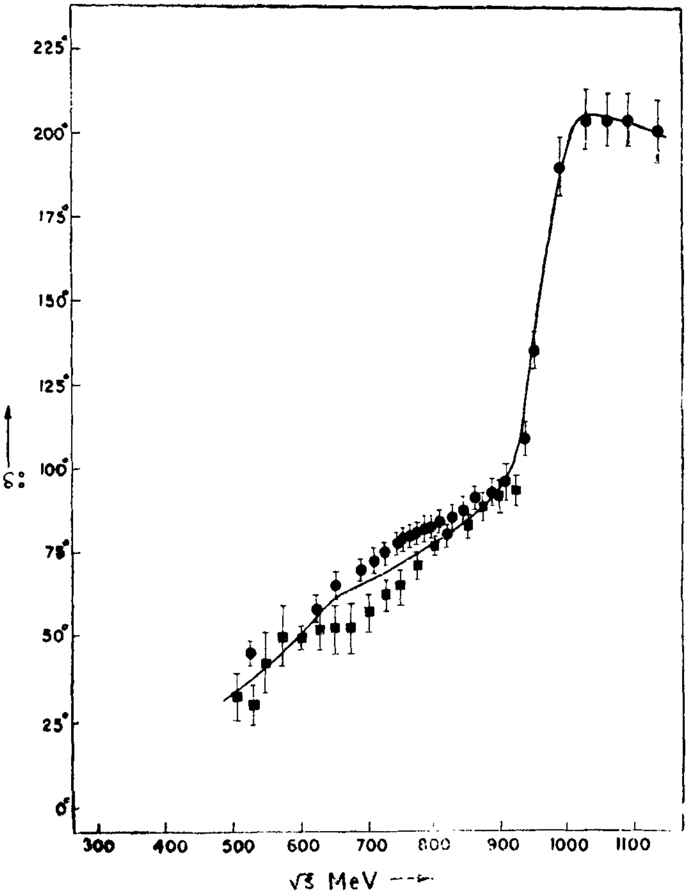

Figure 4. The δ_0^0 phase shifts up to 1150 MeV., ⧫ data points of Baton *et al* (1970), ⧫ data points of Protopopescu *et al* (1973).

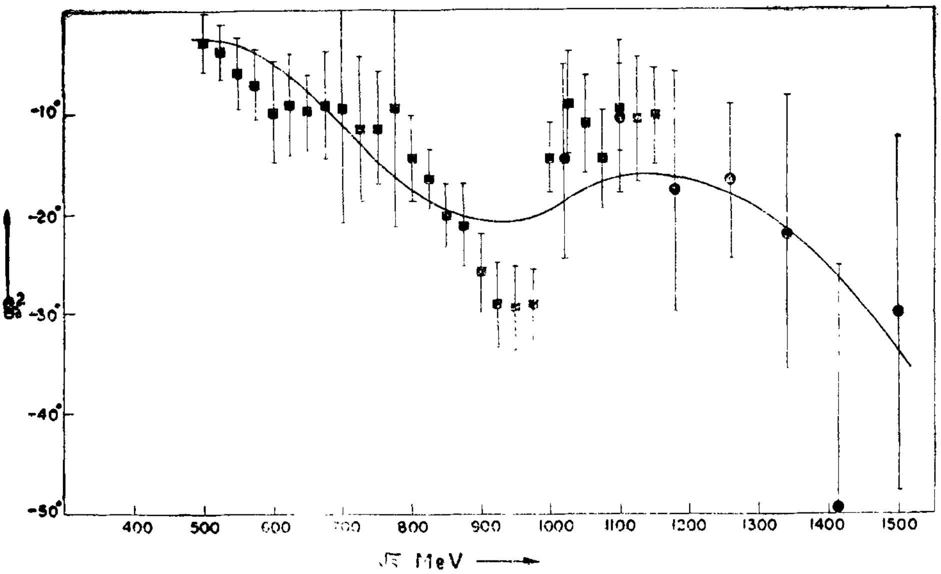

Figure 5. The δ_0^2 phase shifts up to 1400 MeV, data points of Baton *et al* (1970), data points of Carroll *et al* (1972).

(iii) The $I = 1$ P-wave phase shifts are plotted in figure 6, and show good agreement with the experimental data (Carrol *et al* 1972). It shows the ρ resonance at 776 MeV.

(iv) We get $(\delta_0^0(m_K) - \delta_0^2(m_K)) = 35°$, which is within the error limits of the experimental value (Gutay *et al* 1969) $50° \pm 20°$.

(v) Defining η_l^I as

$$\eta_l^I = | 1 + 2i\rho(s) f_l^I(s) | \tag{19}$$

we plot η_0^0 in figure 7 and our curve fits well with the experimental data (Protopopescu *et al* 1973). η_0^2 and η_1^1 are plotted in figures 8 and 9 respectively, and they remain close to unity.

4.3. *Computed results*

(i) Scattering length is defined as

$$a_l^I = \underset{s \to 1}{\mathcal{L}t} \left[\frac{k^{2l+1}}{2\sqrt{s}} \cot \delta_l^I \right]^{-1} \tag{20}$$

Crossing symmetry puts between the various partial wave amplitudes a very strong correlation which manifests itself, in particular, through the "universal curve" of figure 10, relating the two S-wave scattering lengths. Our values of $a_0^0 = 0{\cdot}22$ and $a_0^2 = -0{\cdot}0366$ lie on this curve. This gives a value of 6·01 for $-a_0^0/a_0^2$, which is very much in agreement with those obtained by others (Morgan 1972; Weinberg 1966; Brandt *et al* 1972; Pennington and Protopopescu 1973; Krinsky 1970; Kang *et al* 1971). The P-wave scattering length obtained by us is 0·054. Olsson's (1967) value is 0·061, whereas Morgan (1972) gives a value 0·036 for the same.

(ii) The partial wave amplitudes f_0^0, f_0^2, f_1^1 are plotted in figure 11. f_0^0 has

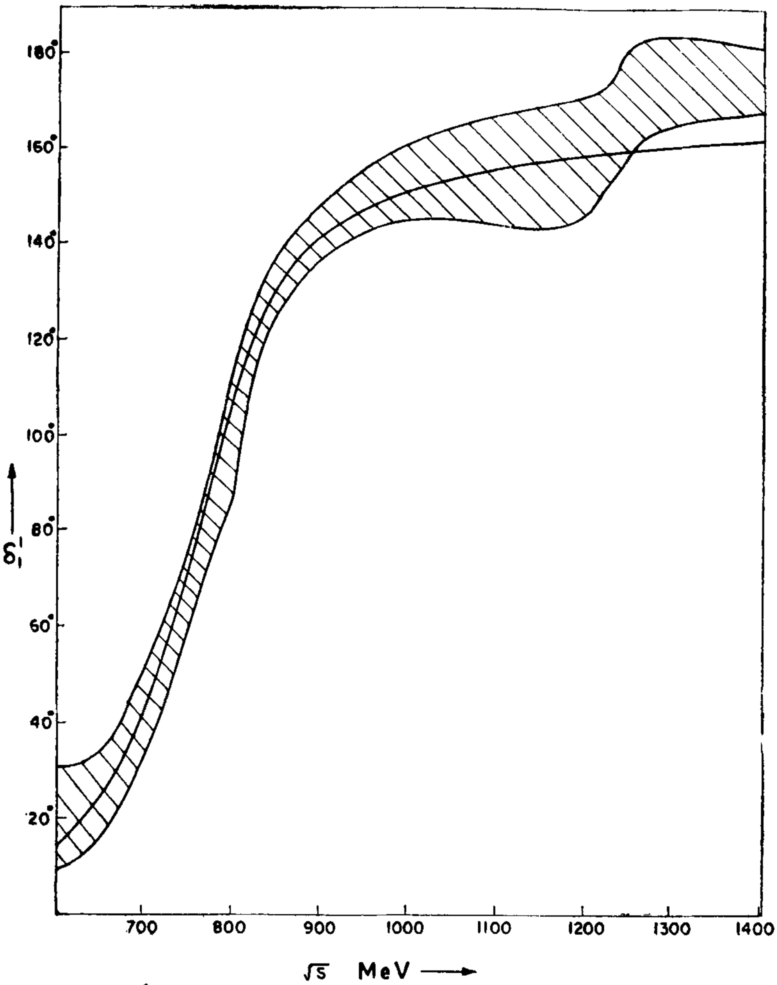

Figure 6. The δ_1^1 phase shifts up to 1400 MeV, ▥ data band of Carroll *et al.* (1972).

a zero at $s = 0 \cdot 13$. The zero of f_0^2 is at $0 \cdot 452$ and f_1^1 has the trivial zero at $s = 1$. Current algebra predictions are also in the same range.

4.4. *Discussion*

In conclusion we wish to stress that by using the strip mapping for S-wave we have not only taken account of the 4π cut but also have been able to keep η_0^0 close to unity up to $K\bar{K}$ threshold. We have also succeeded in reducing considerably the number of parameters of f_0^0 to 7, which is a definite improvement over the fit of Basdevant *et al* (1973). Further the parametrisation of Atkinson *et al* (1973) as well as those of earlier workers are quite inadequate to explain the data of δ_0^2 beyond 900 MeV. In our case the number of parameters for f_0^2 is 7, which, though large, is still less than the parameters required by Basdevant *et al* (1973). We need 7 parameters mainly because of the peculiar behaviour of $I = 2$ S-wave phase shifts in the centre of mass energy between 900 MeV and 1400 MeV [data of Baton *et al* (1970) and Caroll *et al* (1972)]. With the non-availability of reliable data our curve of figure 8 for η_0^2 is a prediction, which can be verified by future experiments.

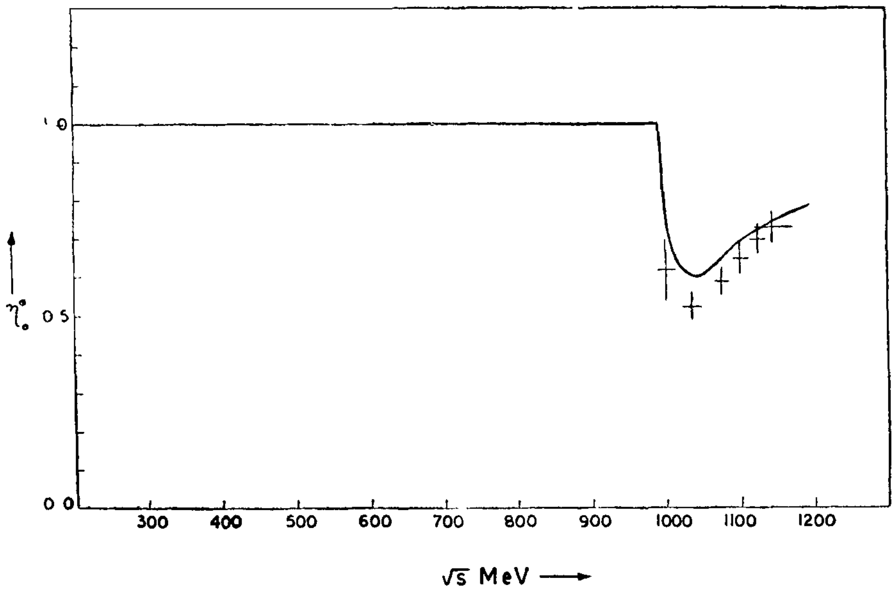

Figure 7. The inelasticity η_0^0 up to 1200 MeV.

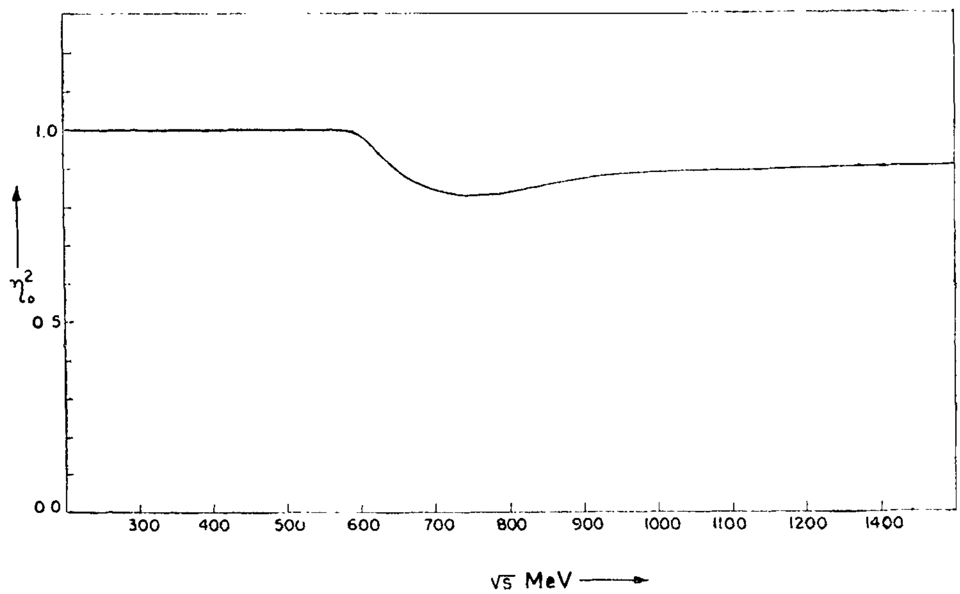

Figure 8. The inelasticity η_0^2 up to 1400 MeV.

A rough picture of the shape of the neutral *S*-wave amplitude is known through the inequalities of Martin and the relations

$$f_0(s) < f_0(1); \qquad 0 \leqslant s < 1 \tag{21}$$

$$\frac{d^2 f_0(s)}{ds^2} > 0; \qquad 0 \leqslant s \leqslant 0{\cdot}42 \tag{22}$$

Current algebra and PCAC suggest the existence of zeros of f_0^0 and f_0^2. How

P—2

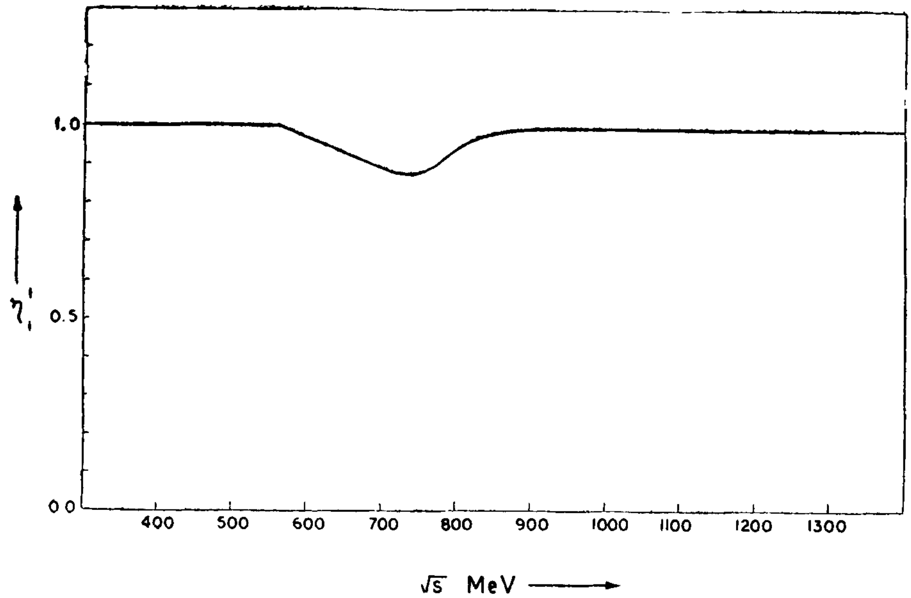

Figure 9. The inelasticity η_1^1 up to 1400 MeV.

ever, to extract the isospin components is the difficult problem. In our analysis, we have tried to use both physical region and unphysical region information in a very effective manner exploiting the analytic extrapolation properties to our advantage. The values of scattering lengths deduced, by us are, therefore, expected to be the best, obtained so far.

Acknowledgement

We are thankful to the staff of the Computer Centre of Utkal University for their assistance in the calculations.

Appendix

1. Roskie's sum rules:

$$\int_0^1 (1-s)\ f_0^0(s)\,ds = 5/2\int_0^1 (1-s)\ f_0^2(s)\,ds \tag{R-1}$$

$$\int_0^1 (1-s)(3s-1)\ f_0^0(s)\,ds = -2\int_0^1 (1-s)(3s-1)\ f_0^2(s)\,ds \tag{R-2}$$

$$\int_0^1 s(1-s)\left(2\ f_0^0(s) - 5\ f_0^2(s)\right)ds = -3\int_0^1 (1-s)^2\ f_1^1(s)\,ds \tag{R-3}$$

$$\int_0^1 s(1-s)^2\left(2 f_0^0(s) - 5\ f_0^2(s)\right)ds = -3\int_0^1 s(1-s)^2\ f_1^1(s)\,ds \tag{R-4}$$

$$\int_0^1 s(1-s)^3\left(2 f_0^0(s) - 5 f_0^2(s)\right)ds = -3\int_0^1 s(1-s)^2(3s-1) f_1^1(s)\,ds \tag{R-5}$$

2. Martin's inequalities:

$$f_0(0{\cdot}80125) > f_0(0{\cdot}05335) \tag{M-1}$$

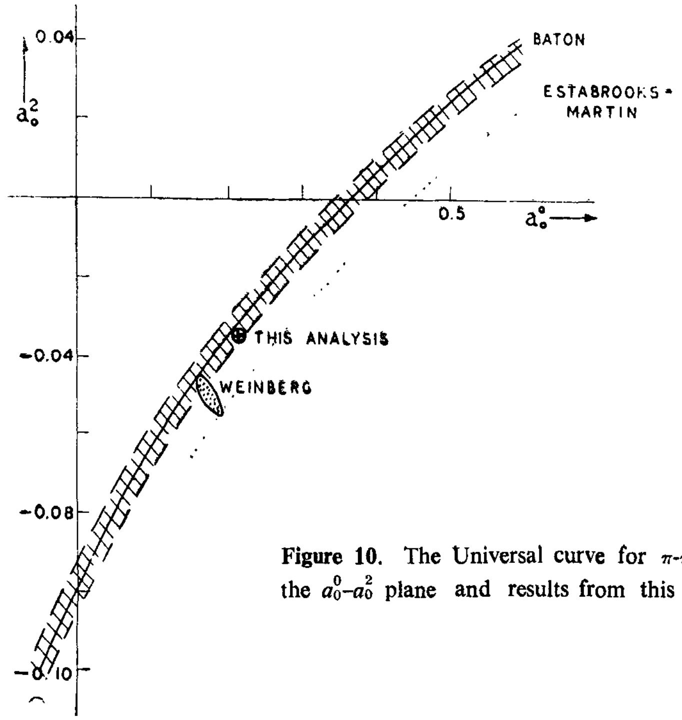

Figure 10. The Universal curve for π-π scattering in the a_0^0–a_0^2 plane and results from this analysis.

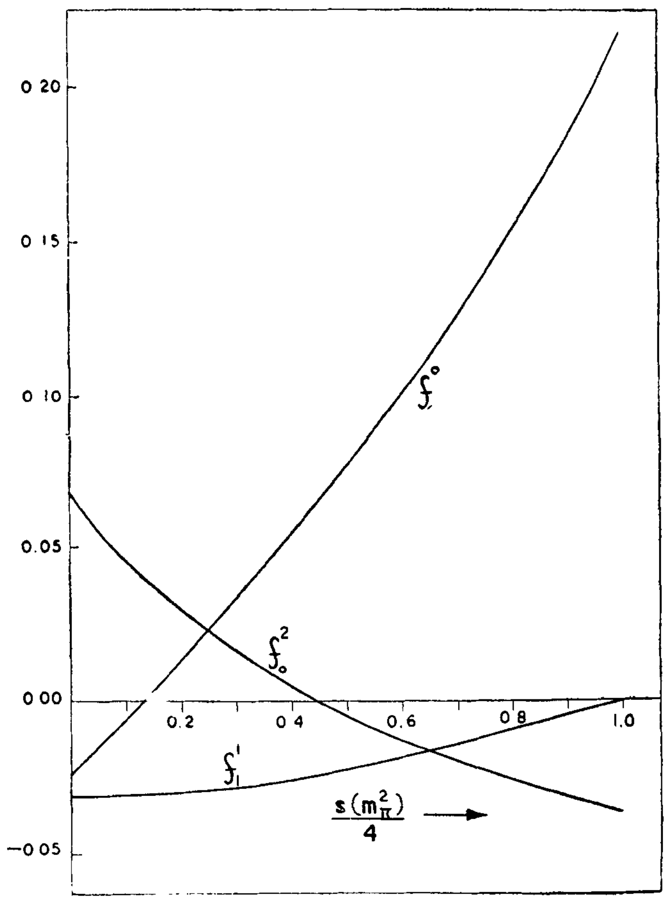

Figure 11. Shapes of partial wave amplitudes f_0^0, f_0^2 and f_1^1 in the unphysical region $0 \leqslant s \leqslant 1$.

$$f_0(0{\cdot}05335) > f_0(0{\cdot}746575) \qquad \text{(M-2)}$$

$$f_0(0) \geqslant f_0(0{\cdot}7875) \qquad \text{(M-3)}$$

$$f_0(s) < f_0(1); \quad 0 \leqslant s < 1. \qquad \text{(M-4)}$$

$$\mathrm{d}/\mathrm{d}s\,(f_0(s)) < 0; \quad 0 \leqslant s \leqslant 0{\cdot}3225 \qquad \text{(M-5)}$$

$$\mathrm{d}/\mathrm{d}s\,(f_0(s)) > 0; \quad s \geqslant 0{\cdot}44 \qquad \text{(M-6)}$$

3. Pennington's inequalities:

$$\int_0^1 \mathrm{d}s\,(1-s)\,(2s^2-s)\; f_0(s) \geqslant 0 \qquad \text{(P-1)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,(5s^3-3s^2)\; f_0(s) \geqslant 0 \qquad \text{(P-2)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,(-5s^3+7s^2-2s)\; f_0(s) \geqslant 0 \qquad \text{(P-3)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,(-s^4-s^3+3s^2-s)\; f_0(s) \geqslant 0 \qquad \text{(P-4)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,(18s^4-32s^3+18s^2-3s)\; f_0(s) \geqslant 0 \qquad \text{(P-5)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,(21s^5-15s^4-20s^3+20s^2-4s)\; f_0(s) \geqslant 0 \qquad \text{(P-6)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,(-21s^5+51s^4-44s^3+16s^2-2s)\; f_0(s) \geqslant 0 \qquad \text{(P-7)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,[(s^4-4s^3+4s^2-s)-2\epsilon(-5s^3+7s^2-2s) + 2\epsilon^2(2s^2-s)]\; f_0(s) \geqslant 0; \; 0 \leqslant \epsilon \leqslant 1 \qquad \text{(P-8)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,[(49s^5+130s^4-395s^3+285s^2-57s) - 110\,\epsilon\,(s^4-4s^3)(4s^2-s) + 55\epsilon^2(-5s^3+7s^2-2s)]\, f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \text{(P-9)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,[(7s^5-15s^4+5s^2-2s+5s^3) - 10\epsilon(-s^4-s^3+3s^2-s) + 5\epsilon^2(5s^3-3s^2)]\; f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \text{(P-10)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,[(-7s^5+20s^4-25s^3+15s^2-3s) - 10\epsilon(s^4-4s^3+4s^2-s) + 5\epsilon^2(-5s^3+7s^2-2s)]\; f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \text{(P-11)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,[(s^4+6s^3-6s^2+s)-2\epsilon(5s^3-3s^2) + 2\epsilon^2(2s^2-s)]\; f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \qquad \text{(P-12)}$$

$$\int_0^1 \mathrm{d}s\,(1-s)\,[(-7s^5+10s^4-10s^3+10s^2-3s) - 10\epsilon\,(-s^4-s^3+3s^2-s) + 5\epsilon^2\,(-5s^3+7s^2-2s)]\; f_0(s) \geqslant 0; \quad 0 \leqslant \epsilon \leqslant 1 \tag{P-13}$$

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